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EMBEDDING OF THE DUNCE HAT

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ABSTRACT. In this note we show that the famous Borsuk contractible non-collapsible 2-polyhedron, generally known as the *dunce hat*, does not embed in any product of two curves but quasi-embeds in the "three-page book".

All *spaces* discussed in this paper are assumed to be metrizable and all *mappings* (also called *maps*) are continuous. By a *compactum* we mean a compact (metric) space, by a *continuum* we mean a non-void connected compactum, and by a *curve* we mean a 1-dimensional continuum. All polyhedra are compact.

By μ we denote the Menger curve. It is well known that if a compactum X quasi-embeds in μ, then X embeds in μ. (A metric space X is said to *quasi-embed* in Y if for each $\varepsilon > 0$ there is an ε-mapping[1] $f : X \to Y$.) In [3] the authors asked the following question about a possible extension of this result to finite products of copies of the Menger curve:

Suppose X quasi-embeds in μ^n. Does X embed in μ^n?

To our surprise, this question has been answered in the negative in a recent joint paper by S. A. Melikhov and J. Zając [6]. Actually, they proved that the Sklyarenko absolute retract [8] quasi-embeds in a product of two dendrites but does not embed in any product of two curves. The problem was left open for cases where X is less complicated, for instance, for X being a polyhedron. (Sklyarenko's example is not a polyhedron.)

The purpose of this note is to show that even for polyhedra the answer is negative. In fact, we are going to prove that the famous Borsuk contractible non-collapsible 2-dimensional polyhedron [1] (generally known as the *dunce hat* (cf. [9]); also called the *Borsuk tube* by Polish topologists (cf. [2])) is such a counterexample. In other words, we shall prove the following.

Theorem. *The dunce hat does not embed in any product of two curves but quasi-embeds in the "three-page book" $T \times I$.*

By T we denote the simple triod, and by I we denote the unit interval, $I = [0, 1]$.

First we recall a description of the Borsuk tube, slightly modifying the original description of Borsuk [1]. Let $\mathbb{B}^2$ denote the unit disc in the complex plane $\mathbb{C}$, and let $\mathbb{S}^1$ denote its boundary, the unit circle in $\mathbb{C}$. We define the Borsuk tube to be

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[1]A mapping $f : X \to Y$ is said to be an ε-*mapping* if $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in f(X)$.

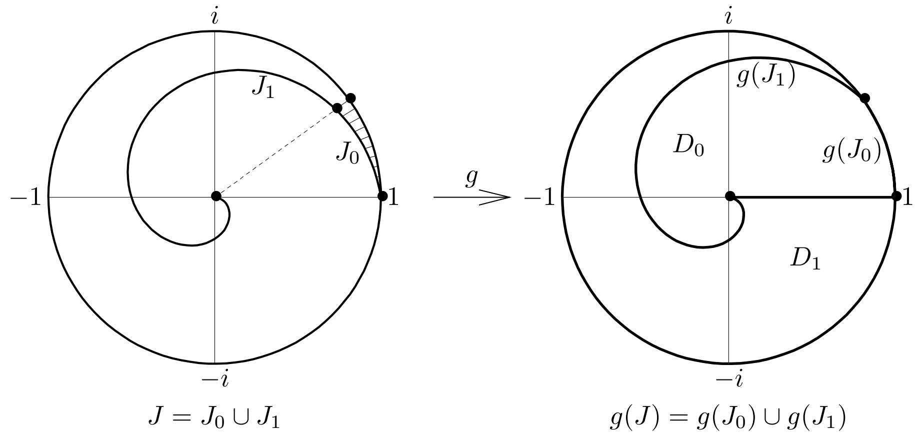

FIGURE 1

the quotient $B = \mathbb{B}^2/\sim$, where $(1-t)\exp(2\pi i t) \sim \exp(2\pi i t)$ for each $t \in I$. (The interval I is a subset of $\mathbb{B}^2$ because the reals are regarded to be a subset of $\mathbb{C}$.) Thus B is the quotient of $\mathbb{B}^2$ obtained by identifying the arc $J = \{(1-t)\exp(2\pi i t) : t \in I\}$ with $\mathbb{S}^1$; see Figure 1. (In the original description, Borsuk identifies $t \in I$ with $\exp(2\pi i t) \in \mathbb{S}^1$ for each $t \in I$.) By its basic property B is a contractible but not collapsible 2-polyhedron ([1]; cf. [9]). One easily verifies that B is a quasi-2-manifold.[2] Hence by the Structure Theorem in [4] we infer that B does not embed in any product of two curves; see [4], Corollary 5.4. This proves the first assertion of our theorem. To complete the proof it remains to establish the second one.

From the definition it follows that B is a compact metrizable space; let d denote a fixed metric on B. Fix a number $\epsilon > 0$. Hence it remains to construct an ϵ-mapping $f : B \to T \times I$. Let $q : \mathbb{B}^2 \to B$ denote the quotient mapping. Observe that there is a number $a > 0$ such that for all $z, z' \in \mathbb{B}^2$ we have

(1) $|z - z'| < a \Rightarrow d(q(z), q(z')) < \epsilon$.We may also assume that $a < 1$. Then it is easy to define an equicontinuous family of mappings $g_t : (I, 0, 1) \to (I, 0, 1)$, $t \in I$, satisfying the conditions

(2) $g_0 = g_1 = id_I$,

(3) g_t *is a relative homeomorphism* $(I, [1-t, 1]) \to (I, \{1\})$ *for each* $0 \le t \le a$,

(4) g_t *is a homeomorphism for each* $a < t \le 1$.[3]

These mappings define a mapping $g : \mathbb{B}^2 \to \mathbb{B}^2$ given by the formula

$$g(s \cdot \exp(2\pi i t)) = g_t(s) \cdot \exp(2\pi i t)$$

for each $s, t \in I$.

[2]A 2-dimensional compactum is called a *quasi-2-manifold* if each point $x \in X$ admits an open neighborhood U such that every closed set F separating X between x and $X \setminus U$ admits an essential map into $\mathbb{S}^1$.

[3]For instance, let us define: (i) if $t \in [0, a]$ put $g_t(s) = \frac{s}{1-t}$ for $s \le 1-t$ and $g_t(s) = 1$ for $s \ge 1-t$; (ii) if $t \in (a, 1]$ put $g_t(s) = (\frac{1-t}{(1-a)^2} + \frac{t-a}{1-a})s$ for $s \le 1-a$ and $g_t(s) = \frac{t-a}{1-a}s + \frac{1-t}{1-a}$ for $s > 1-a$. Notice that each g_t is composed of two linear maps defined on adjacent (possibly degenerate) subintervals of I.

Then one can define another quotient space B' of $\mathbb{B}^2$ analogous to B, identifying the arc $g(J)$ with $\mathbb{S}^1$; precisely, $B' = \mathbb{B}^2/\sim'$, where $g((1-t)\exp(2\pi it)) \sim' \exp(2\pi it)$ for each $t \in I$. Let $q' : \mathbb{B}^2 \to B'$ denote the quotient mapping. Since g preserves the identifications, there exists a mapping $g' : B \to B'$ such that $q' \circ g = g' \circ q$; that is, the following diagram commutes:

$$\begin{tikzcd} \mathbb{B}^2 \arrow[r, "g"] \arrow[d, "q"'] & \mathbb{B}^2 \arrow[d, "q'"] \\ B \arrow[r, "g'"'] & B' \end{tikzcd} .$$

Then

(5) *g' is an ϵ-mapping.*

Indeed, suppose g' sends two different points $q(z)$, $q(z')$ of B to the same point of B'. Then z, z' do not lie in the same fiber of q. Since $g(z)$, $g(z')$ lie in the same fiber of q', it follows that both z, z' lie in the same fiber of g, i.e. $z, z' \in [1-t, 1]\exp(2\pi it)$ for some $t \in [0, a]$. Therefore, $|z - z'| < a$; hence $d(q(z), q(z')) < \epsilon$ by (1). This proves (5).

Notice that J is the union of two subarcs $J_0 = \{(1-t)\exp(2\pi it) : t \in [0, a]\}$ and $J_1 = \{(1-t)\exp(2\pi it) : t \in [a, 1]\}$. Likewise, $\mathbb{S}^1$ is the union of two arcs $S_0 = \{\exp(2\pi it) : t \in [0, a]\}$ and $S_1 = \{\exp(2\pi it) : t \in [a, 1]\}$. Moreover, we have: $g(J)$ is the union of two subarcs $g(J_0)$ and $g(J_1)$, $g(J_0) = S_0$ and $g(J_1)$ meets S_1 at point $\exp(2\pi ia)$ which is a common endpoint of these arcs, and off that point $g(J_1)$ lies in the interior of $\mathbb{B}^2$. It follows that B' is the quotient of $\mathbb{B}^2$ obtained by identifying the arcs $g(J_1)$ and S_1. Therefore, B' is a collapsible 2-polyhedron and $q'(S_0)$ is a free face of B' (i.e. each interior point x of $q'(S_0)$ admits a neighborhood that is a closed 2-disc with x lying on its boundary). To complete the proof it suffices to show that B' embeds in $T \times I$.

To this end, present B' as a union of two sets $q'(D_0)$ and $q'(D_1)$, where D_0 is the disc bounded by the arcs I, S_0 and $g(J_1)$, and D_1 is the disc bounded by I, S_1 and $g(J_1)$. Clearly, $q'(D_1)$ is a disc and the circle $q'(I)$ is its boundary. On the other hand, $q'(D_0)$ meets $q'(D_1)$ along the union $q'(g(J_1) \cup I)$. Since the arc $q'(g(J_1))$ off its endpoint $q'(0)$ lies entirely in the interior of the disc $q'(D_1)$ and $q'(I)$ is the boundary of that disc, one easily sees that B' embeds in $T \times I$.

Lemma. *The suspension of a starlike compact set X lying in $\mathbb{R}^n$ embeds in the product $X \times I$.*

To prove this lemma we may assume that X is starlike with respect to the point $\mathbf{0} \in \mathbb{R}^n$ and that $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. Identify X with $X \times \{0\}$. Then the suspension is the union of two cones over X with vertices $(\mathbf{0}, -1)$ and $(\mathbf{0}, 1)$. In this setting, the suspension is a subset of $X \times [-1, 1]$.

Applying our theorem and this lemma we get the following.

Corollary. *For each $n \geq 2$ there is an n-dimensional contractible polyhedron P not embeddable in any product of n curves but quasi-embeddable in a product of n triods. Moreover, if $n = 2k$, then P quasi-embeds in $T^k \times I^k$, and if $n = 2k+1$, then P quasi-embeds in $T^k \times I^{k+1}$.*

Proof. Actually, we shall show that for P we can take either the product $B^k = B \times \ldots \times B$ if $n = 2k$, or the suspension $\sum B^k$ if $n = 2k+1$. In fact, the product B^k of k copies of the Borsuk tube is a quasi-$2k$-manifold that is a product of quasi-2-manifolds; see [4], Corollary 2.3. Since B^k is contractible, by the Structure Theorem 5.1 of [4], it does not embed in any product of $2k$ curves. Since B quasi-embeds in $T \times I$, B^k quasi-embeds in $(T \times I)^k = T^k \times I^k$. On the other hand, by Corollary 2.3 of [4], $\sum B^k$ is a quasi-$(2k+1)$-manifold off two points (cf. the proof of Theorem 1.3 in [5]). Notice that $\sum B^k$ is contractible. Hence, by the Second Structure Theorem 6.1 of [5], $\sum B^k$ does not embed in any product of $2k+1$ curves. Moreover, $\sum B^k$ quasi-embeds in $\sum (T \times I)^k$, hence quasi-embeds in $(T \times I)^k \times I = T^k \times I^{k+1}$, by the above lemma and the starlikeness of $(T \times I)^k$ in $\mathbb{R}^{3k}$. □

Problems

Problem 1. Characterize 2-dimensional polyhedra quasi-embeddable in a product of two curves.

Problem 2. Does the "Bing house" quasi-embed in a product of two curves?

The "Bing house" is another example of a contractible non-collapsible 2-polyhedron [7]. By the same argument as in the case of the Borsuk tube, it does not embed in any product of two curves.

Problem 3. Does the suspension $\Sigma^n B$, $n > 1$, embed in a product of $n+2$ curves?

Notice that by the Structure Lemma 4.8 in [4], $\Sigma^n B$ does not embed in any product of $n+2$ graphs. Therefore, a positive solution of the problem from [5] implies a negative solution to Problem 3 above. Also notice that by our lemma, $\Sigma^n B$ quasi-embeds in $T \times I^{n+1}$ because B quasi-embeds in $T \times I$.

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