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EXTREMAL MAHLER MEASURES AND L_s NORMS OF POLYNOMIALS RELATED TO BARKER SEQUENCES

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Abstract. In the present paper, we study the class $\mathcal{L}P_n$ which consists of Laurent polynomials

$$P(z) = (n+1) + \sum_{\substack{k=1 \\ k\,-\,\text{odd}}}^{n} c_k(z^k + z^{-k}),$$

with all coefficients c_k equal to -1 or 1. Such polynomials arise in the study of Barker sequences of even length — binary sequences with minimal possible autocorrelations. By using an elementary (but not trivial) analytic argument, we prove that polynomials $R_n(z)$ with all coefficients $c_k = 1$ have minimal Mahler measures in the class $\mathcal{L}P_n$. In conjunction with an estimate $M(R_n) > n - 2/\pi \log n + O(1)$ proved in an earlier paper, we deduce that polynomials whose coefficients form a Barker sequence would possess unlikely large Mahler measures. A generalization of this result to L_s norms is also given.

1. Introduction and motivation of the problem

Let $a_0, a_1, \ldots, a_n$ be the sequence of complex numbers, all of modulus $|a_j| = 1$. For this sequence, the *aperiodic autocorrelation* coefficients $c_k, -n \le k \le n$, are defined by

$$c_k := \sum_{j=0}^{n-k} a_j \overline{a}_{k+j} \quad \text{for} \quad 0 \le k \le n \quad \text{and} \quad c_{-k} := c_k.$$

The sequences a_k having small autocorrelation values $|c_k|$ have a long history in signal processing; see [1], [11] and [13]. In particular, the sequences which have all numbers a_k equal to 1 or -1 and $|c_k| \le 1$ for $k \ne 0$ are called *Barker* sequences. Since we enumerate the terms in the sequence starting with 0, the *length* of the sequence is equal to $n+1$ — we shall keep this convention in mind through the paper.

Clearly, all the restrictions to which Barker sequences are subjected seem to be quite special and hard to satisfy. Because of this it is widely believed that only finitely many Barker sequences exist. Turyn and Storer [28] proved that no Barker

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sequences of *odd length* exist for $n \geq 13$. Thus, by the result of Turyn and Storer [28], all the Barker sequences which are longer than 12 should have *even length*. However, all the known Barker sequences of even length are very short:

$$n = 1:$$
$$++, \qquad --, \qquad +-, \qquad -+,$$
$$n = 3:$$
$$++-+, \qquad --+-, \qquad +---, \qquad -+++,$$
$$+++-, \qquad ---+, \qquad +-++, \qquad -+--.$$

Furthermore, it is possible to obtain all the 12 Barker sequences above from two of them, say, $++$, $+++-$. This can be achieved by using three simple operations: inverting the signs of all numbers of the sequence at once, inverting every second term of the sequence, or by rewriting the sequence backwards.

In view of this situation, one would expect to find a simple and concise proof that there exist no Barker sequences of even length greater than 4. However, the proof of nonexistence of long Barker sequences still remains elusive despite a substantial amount of research in the last 45 years. This problem has been attacked by combinatorial and number theoretical methods. We remark that various restrictions on the possible values of n were derived by Eliahou, Kervaire, Saffari [7], [8], Jedwab and Lloyd [14], Leung and Schmidt [21] and Turyn [25], [26], [27]. These restrictions were used to check the nonexistence of Barker sequences on computers for very large values of n. The current computation record belongs to Mossinghoff [18], who showed that if a Barker sequence of even length exists, then either $n = 1892604680010344415227667816 04$ or $n > 2 \cdot 10^{30}$. All the mentioned results provide strong evidence in support of the nonexistence conjecture.

In recent literature [5] and [23], the question of the existence of long Barker sequences was tied to the existence of polynomials with small integer coefficients $\{-1, 1\}$ having remarkable analytic properties. In particular, the polynomials constructed by means of long Barker sequences are thought to have extremely large Mahler measures and L_s norms on the unit circle, which seems to be unlikely. The questions about the existence of such extremal polynomials go back to Littlewood [29], [30], Mahler [16] and Erdős [10] and have been open for half a century now.

In the present paper we will also focus on the polynomial setting. In Section 2, we will explain in detail the relation between the Barker conjecture and the conjectures on extremal Mahler measures and L_s norms of polynomials. We will state *two problems* whose solution implies the Barker conjecture. Finally (and most importantly), we will solve *one* of these two problems.

2. Barker sequences and norms of polynomials on the unit circle

Let $P(z) = a_n(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n) \in \mathbb{C}[z]$ be a complex polynomial. The *Mahler measure* of $P(z)$ is defined by

$$M(P) = |a| \prod_{j=1}^{n} \max\{1, |\alpha_j|\}.$$

In view of Jensen's formula [17], one has

$$\log M(P) = \frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{it})| dt.$$

More generally, for a real number $s > 0$, the integral L_s norm of a complex polynomial is defined by the formula

$$\|P\|_s = \left(\frac{1}{2\pi}\int_0^{2\pi} |P(e^{it})|^s dt\right)^{1/s}.$$

We should make a note here that $\|\cdot\|_s$ is a *norm* in $\mathbb{C}[z]$ (as a vector space over $\mathbb{C}$) only if $s \geq 1$. L_s norms have the *continuity property*: if the polynomial $P(z)$ is fixed, then the norm $||P||_s$ is a monotonically increasing and continuous function of s. Moreover,

$$\lim_{s\to 0^+} ||P||_s = M(P), \qquad \lim_{s\to\infty} ||P||_s = ||P||_\infty = \sup_{|z|=1} |P(z)|.$$

Hence the Mahler measure of $P(z)$ is sometimes called L_0 norm of $P(z)$. Here is one more important observation. If $Q(z) \in \mathbb{C}[z, 1/z]$ is a nonzero *Laurent polynomial*, then there exists a unique polynomial $P(z) \in \mathbb{C}[z]$, $P(0) \neq 0$, such that $Q(z)$ may be rewritten as $Q(z) = z^{-k}P(z)$ for some integer power k. We define $M(Q) := M(P)$, $||Q||_s := ||P||_s$. Since $P(z)$ and $Q(z)$ have the same absolute values on the unit circle, one can use the integral formulas with $Q(z)$ instead of $P(z)$ to calculate $M(Q)$ and $||Q||_s$. This notation allows a considerable simplification of the formulas.

We remark that, in general, the computation of Mahler measures or L_s norms of an arbitrary complex polynomial is a hard problem. For practical computations, the most useful is the L_2 norm. By Parseval's identity, the L_2 norm of the polynomial

$$P(z) = a_n z^n + \cdots + a_1 z + a_0$$

is given explicitly by

$$||P||_2 = \left(|a_n|^2 + \cdots + |a_1|^2 + |a_0|^2\right)^{1/2}.$$

As for the values of $M(P)$ and $||P||_s$, $s \neq 2$, some rough estimates in terms of coefficients of the polynomial are more useful; see [3] and [22]. Most often, these estimates use the easily computable L_2 norm as a reference point. For instance, $M(P) < ||P||_2$ by the monotonicity property. Hence it is natural to ask the following question:

Question 2.1. Given a polynomial $P(z) \in \mathbb{C}[z]$, how large is the Mahler measure $M(P)$ of this polynomial in relation to its L_2 norm?

Mahler [16] investigated the maximum of $M(P)$ for polynomials with bounded coefficients. Mahler proved that $M(P)$ is maximized if one takes polynomials P with complex coefficients of equal modulus. Subsequently, Fielding [12], Beller and Newman [2] proved that for such polynomials, the maximum of $M(P)/||P||_2$ tends to 1 as the degree n increases to infinity.

Recall that a polynomial $p(z) \in \mathbb{Z}[z]$ is called a Littlewood polynomial if all coefficients of $p(z)$ are equal to 1 or -1. The set of Littlewood polynomials is denoted by

$$\mathcal{L}_n := \{p(z) = a_n z^n + \cdots + a_0 : a_j = 1 \text{ or } a_j = -1,\ 0 \leq j \leq n\}.$$

Obviously, the L_2 norm of a Littlewood polynomial $p \in \mathcal{L}_n$ is $||p||_2 = \sqrt{n+1}$. In contrast with a general complex case $P(z) \in \mathbb{C}[z]$, it is not known whether Mahler measures of $p \in \mathcal{L}_n$ can be arbitrarily close to or are bounded away from L_2 norm, i.e. if there exists a constant $c > 0$ such that $M(p)/\sqrt{n+1} < c < 1$ for all $p \in \mathcal{L}_n$. Here we formulate a weaker conjecture:

Conjecture 2.2. *For any polynomial $p(z) \in \mathcal{L}_n, n \geqslant 1$, there exists an absolute constant $c > 0$ such that*

$$M(p) < \sqrt{n+1} - c.$$

Even in this weak form, Conjecture 2.2 is is still open. Newman [19], [20] and Littlewood [29], [30] asked similar questions for the L_1 norm instead of the Mahler measure. By the monotonicity property of L_s norms, proving Conjecture 2.2 for the L_1 norm also implies the same result for the Mahler measure. The best known result towards Conjecture 2.2 is

$$M(p) < ||p||_1 < \sqrt{n+0.91}.$$

This inequality has been obtained in [5] by optimizing a previous estimate of Newman [19]. Recently, Erdélyi made significant progress in this direction by proving the L_1 version of Conjecture 2.2 for cyclotomic Littlewood polynomials (see [9]).

Littlewood polynomials are very convenient in signal analysis to study properties of binary sequences and Barker sequences in particular. Recall that the polynomial $p(z) \in \mathcal{L}_n$ is called a *Barker polynomial* if the coefficients $a_0, a_1, \ldots, a_n$ form a Barker sequence of length $n+1$. Since the autocorrelations of the coefficients do not change in magnitude by replacing the polynomial $p(z)$ with $p(-z)$ or $-p(-z)$, one can normalize Barker polynomials using the conditions $a_n = a_{n-1} = 1$. Also, if $p(z)$ is a Barker polynomial, then the *reciprocal* polynomial $p^*(z) = z^n p(1/z)$ is also a Barker polynomial. With our convention in mind, the Barker sequences of *even* length $n+1$ correspond to the Barker polynomials of *odd* degree n. So there are essentially two interesting Barker polynomials of odd degree, namely $z+1$ and z^3+z^2+z-1. Only according to the conjecture can all odd Barker polynomials be obtained from the two basic ones.

In [23], Saffari proved that Barker polynomials of large degree would possess the property of *flatness*, namely,

$$c_1\sqrt{n+1} < |p(z)| < c_2\sqrt{n+1},$$

on the complex unit circle $|z| = 1$ for some positive constants c_1 and c_2. The very existence of such flat polynomials is yet another old question which dates back to Littlewood [29], [30] and Erdős [10]. By using the estimate of Saffari, Borwein and Mossinghoff showed in [5] that if an infinite sequence of Barker polynomials would exist, they should have extremely large Mahler measures: $M(p) > \sqrt{n+1} - 1$ by Theorem 4.1 in [5]. In view of the last estimate, we formulate the second conjecture on Mahler measures of Barker polynomials.

Conjecture 2.3. *Suppose that $p_{n_k}(z)$ is an infinite sequence of Barker polynomials of increasing degree n_k. Then one has*

$$\lim_{k\to\infty} \left(M(p_{n_k}) - \sqrt{n_k+1}\,\right) = 0.$$

We note that if both Conjectures 2.2 and 2.3 are true, then there are only finitely many Barker sequences. In Section 3 we shall completely solve Conjecture 2.3. Although Conjecture 2.2 is still open, recent results [9] provide some hope that Conjecture 2.2 can be settled in the affirmative.

3. Main results

We use the notation introduced in the previous paper [4]. Let $n \in \mathbb{N}$ be odd. We define $\mathcal{L}P_n$ to be the class of Laurent polynomials $P(z)$ of the form

$$P(z) = (n+1) + \sum_{\substack{k=1 \\ k\text{ – odd}}}^{n} c_k(z^k + z^{-k}),$$

with all the coefficients $c_k \in \{-1, 1\}$. Such auxiliary polynomials arise in a natural way in connection with Barker polynomials of odd degree. Indeed, if $p(z)$ is a Barker polynomial of degree n, then the product $p(z)p(1/z)$ is a Laurent polynomial which belongs to the class $\mathcal{L}P_n$; see [4]. Mahler measure and L_s norms of polynomials $P \in \mathcal{L}P_n$ are defined by the respective integral formulas. One should note that, while Barker polynomials of large degree are only hypothetical, the class $\mathcal{L}P_n$ exists and has some very peculiar extremal properties. As in [4], the notation $R_n(z)$ shall be reserved for the polynomials which have all coefficients $c_k = 1$:

$$R_n(z) = (n+1) + \sum_{\substack{k=-n \\ k\text{ – odd}}}^{n} z^k.$$

After some computer experimentation, we have conjectured in [4] that the polynomials $R_n(z)$ and $R_n(-z)$ have smallest possible Mahler measures in $\mathcal{L}P_n$. Now we are able to prove this result.

Theorem 3.1. *If a polynomial $P \in \mathcal{L}P_n$, then $M(P) \geqslant M(R_n)$.*

One should note that precise extremal results such as Theorem 3.1 are quite rare. In general, it is hard to establish nontrivial lower bounds for Mahler measures of polynomials. See, for example, a nice survey of Smyth [24]. Surprisingly, Theorem 3.1 admits a a simple (but not trivial) analytical proof.

In the previous paper [4], we also proved the estimate

$$M(R_n) > n - \frac{2}{\pi}\log n + O(1).$$

Now, let $p(z)$ be a Barker polynomial. Then $P(z) = p(z)p(1/z) \in \mathcal{L}P_n$. Then

$$M(p)^2 = M(p(z))M(p(1/z)) = M(p(z)p(1/z)) = M(P).$$

Theorem 3.1 implies the following corollary:

Corollary 3.2. *For any Barker polynomial $p(z)$ of degree n, we have*

$$M(p) \geq \left(n - \frac{2}{\pi}\log n + O(1)\right)^{1/2}.$$

This improves Theorem 4.1 in [5]. In addition to this,

$$\left|\sqrt{n+1} - \left(n - \frac{2}{\pi}\log n + O(1)\right)^{1/2}\right| \sim \frac{\log n}{\pi\sqrt{n+1}} \to 0$$

as $n \to \infty$. Hence Conjecture 2.3 is proved.

In fact, one can extend the argument of the proof of Theorem 3.1 to prove a more general result on the extremal L_s norms of the polynomials in the class $\mathcal{L}P_n$.

Theorem 3.3. *For $s < 1$, the polynomials $R_n(\pm z)$ have minimal L_s norms in the class $\mathcal{L}P_n$. On the other hand, $R_n(z)$ have maximal L_s norms in $\mathcal{L}P_n$ for $s \in [2j-1, 2j]$, $j \in \mathbb{N}$, and also for all s which are sufficiently large: $s > s_0(n)$.*

Theorem 3.3 is of less significance for studying Barker polynomials. Nevertheless, we include this result to demonstrate that the integral norms in the set $\mathcal{L}P_n$ are bounded in a predictable way, which is not the case for general complex polynomials in $\mathbb{C}[z]$.

4. Proofs

We give full proofs of Theorem 3.1 and Theorem 3.3.

We need some notation. Set $N := n+1$. Let

$$T(z) = \sum_{\substack{k=-n \\ k\,-\,\text{odd}}}^{n} c_k z^k \qquad \text{and} \qquad U_n(z) = \sum_{\substack{k=-n \\ k\,-\,\text{odd}}}^{n} z^k.$$

Then $P(z) = N + T(z)$ and $R_n(z) = N + U_n(z)$. From the proof of Theorem 2.3 in [4], for any real number $t \in [0, 2\pi)$, one has

$$T(e^{it}) = 2 \sum_{\substack{k=1 \\ k\,-\,\text{odd}}}^{n} c_k \cos kt \qquad \text{and} \qquad U_n(e^{it}) = 2 \sum_{\substack{k=1 \\ k\,-\,\text{odd}}}^{n} \cos kt = \frac{\sin Nt}{\sin t}.$$

We start with the simple observation.

Lemma 4.1. *For any $m \in \mathbb{Z}$, $m \geqslant 0$,*

$$\frac{1}{2\pi}\int_0^{2\pi} T^m(e^{it})dt \leqslant \frac{1}{2\pi}\int_0^{2\pi} U_n^m(e^{it})dt.$$

Moreover, the equality only holds for $T(z) = U_n(z)$. For odd n, we have

$$\frac{1}{2\pi}\int_0^{2\pi} T^m(e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} U_n^m(e^{it})dt = 0 \tag{4.1}$$

for any odd m.

Proof. Write

$$T^m(z) = \left(\sum_{\substack{k=-n, \\ n\,-\,\text{odd}}}^{n} z^k \right)^m = \sum_{k=-mn}^{mn} A_k z^k,$$

where the coefficients A_k are defined by

$$A_k := \sum_{\substack{k_1+\cdots+k_m=k, \\ -n\leqslant k_j\leqslant n, \\ k_j\,-\,\text{odd}}} c_{k_1}\cdots c_{k_m}.$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ikt}dt = \begin{cases} 0, & \text{if } k \neq 0, \\ 1, & \text{if } k = 0, \end{cases}$$

this yields

$$\frac{1}{2\pi}\int_0^{2\pi} T^m(e^{it})dt = \sum_{k=-mn}^{mn} \frac{A_k}{2\pi}\int_0^{2\pi} e^{ikt}dt = A_0.$$

Observe that A_0 achieves maximal value if and only if all products $c_{k_1}\cdots c_{k_m}$ in the sum are equal to 1. This is possible if all the coefficients $c_k = 1$, i.e., $T(z) = U_n(z)$. This proves the result.

To prove the last assertion, since all m and k_j are odd, then

$$k_1 + \cdots + k_m \equiv 1 + \cdots + 1 \equiv m \equiv 1 \not\equiv 0 \pmod 2,$$

and hence $A_0 = 0$. □

Proof of Theorem 3.1. From now on, we assume that n and $N = n+1$ are fixed. Let $P(z) \in \mathcal{L}P_n$. If $P(z) = R_n(\pm z)$, then the proof is obvious. Assume that $P(z) \neq R_n(z)$. Recall that

$$-\log(1-u) = u + \frac{u^2}{2} + \cdots + \frac{u^m}{m} + \ldots \tag{4.2}$$

holds for any real number $u \in [-1, 1)$. Moreover, the infinite series converges absolutely if $|u| < 1$. If $P(z) \neq R_n(\pm z)$, then the polynomial $T(z)$ of $P(z)$ satisfies $|T(e^{it})| \leqslant \delta N$, $t \in [0, 2\pi)$, where

$$\delta(T) := \max_{t\in[0,2\pi)} \frac{|T(e^{it})|}{N} < 1.$$

To see this, note that the only polynomials $T(z)$ which achieve the value $-N$ or N for some $z = e^{it}$, $t \in [0, 2\pi)$, have all coefficients $c_k = 1$ or all $c_k = -1$ at $t = 0$ or $t = \pi$, respectively, so $T(z) = \pm U_n(z)$ and $P(z) = \pm R_n(z)$.

Hence, for $u = T(e^{it})/N$ in (4.2), the Weierstrass M–criterion implies that the series converges uniformly for all $t \in [0, 2\pi)$ if $T(z) \neq \pm U_n(z)$. Since the convergence is uniform with respect to t, we can integrate and exchange the integration and summation and get

$$-\int_0^{2\pi} \log\left(1 - \frac{T(e^{it})}{N}\right) dt = \sum_{m=1}^{\infty} \int_0^{2\pi} \frac{T^m(e^{it})}{mN^m} dt.$$

The application of Lemma 4.1 gives

$$\sum_{m=1}^{\infty} \int_0^{2\pi} \frac{T^m(e^{it})}{mN^m} dt \leqslant \sum_{m=1}^{\infty} \int_0^{2\pi} \frac{U_n^m(e^{it})}{mN^m} dt.$$

The next step is to show that

$$\sum_{m=1}^{\infty} \int_0^{2\pi} \frac{U_n^m(e^{it})}{mN^m} dt = -\int_0^{2\pi} \log\left(1 - \frac{U_n(e^{it})}{N}\right) dt. \tag{4.3}$$

To see this, first note that the normalized Dirichlet kernel

$$g(t) = \frac{U_n(e^{it})}{N} = \frac{\sin(Nt)}{N \sin t}$$

takes the value of maximal modulus 1 in the interval $[0, 2\pi]$ only at $t = 0$ or $t = 2\pi$. Hence, $|g(t)| < \varrho$ for some $\varrho < 1$ in the interval $I = [\pi/N, 2\pi - \pi/N]$. Using (4.2) with $u = g(t)$ one obtains

$$-\log(1 - g(t)) = \sum_{m=1}^{\infty} \frac{g^m(t)}{m}, \tag{4.4}$$

and the convergence is uniform in the interval I. Thus one can integrate (4.4) over I and exchange the integration and summation. Next, observe that in the

complement interval $J = [0, 2\pi]/I$, the Dirichlet kernel $g(t)$ is positive. Hence the series on the right hand side of (4.4) monotonically converges to the function on the right hand side pointwise in the interval J, with the exception of the points $t = 0$ and $t = 2\pi$. By the Lebesgue monotone convergence theorem, the series are integrable in J, and one can exchange integration and summation. It follows that (4.4) is integrable in $I \cup J = [0, 2\pi]$, and the exchange of integration and summation proves (4.3). Thus we have proved that

$$-\int_0^{2\pi} \log\left(1 - \frac{T(e^{it})}{N}\right) dt \leqslant -\int_0^{2\pi} \log\left(1 - \frac{U_n(e^{it})}{N}\right) dt.$$

It remains to observe that the integral on the right hand side of the above identity is equal to $2\pi(\log M(P(-z)) - \log N)$ and that the integrand on the right hand side is $2\pi(\log M(R_n(-z)) - \log N)$ by Jensen's formula. Multiplying the last inequality by -1 leads to the desired result. □

Proof of Theorem 3.3. We use the integral L_s norm formula instead of Jensen's formula and the binomial formula instead of $-\log(1-u)$:

$$(1+u)^s = \sum_{m=0}^{\infty} \binom{s}{m} u^m, \qquad \text{for } |u| < 1.$$

We have

$$\begin{aligned} \|P/N\|_s^s &= \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{P(e^{it})}{N} \right|^s dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(1 + \frac{T(e^{it})}{N}\right)^s dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=0}^{\infty} \binom{s}{m} \left(\frac{T(e^{it})}{N}\right)^m dt. \end{aligned}$$

The integration and summation can be exchanged using the uniform convergence argument if $T(z) \neq \pm U_n(z)$. By the second part of Lemma 4.1, one has

$$\frac{1}{2\pi} \int_0^{2\pi} T^m(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} U_n^m(e^{it}) dt = 0, \qquad \text{if } m \text{ is odd.}$$

Hence

$$\|P/N\|_s^s = \frac{1}{2\pi} \sum_{m=0}^{\infty} \binom{s}{2m} \int_0^{2\pi} \left(\frac{T(e^{it})}{N}\right)^{2m} dt.$$

The binomial coefficients are given by

$$\binom{s}{0} = 1, \quad \binom{s}{m} = \frac{s(s-1)\cdots(s-m+1)}{m!} \quad \text{for } m = 1, 2, 3, \ldots.$$

If $0 < s < 1$, the coefficients $\binom{s}{2m}$ are negative for $m \geqslant 1$, whereas $\binom{s}{2m+1}$ are positive for $m \geqslant 0$. By the first part of Lemma 4.1,

$$\begin{aligned}
-\|P/N\|_s^s &= \frac{1}{2\pi}\sum_{m=0}^{\infty} -\binom{s}{2m}\int_0^{2\pi}\left(\frac{T(e^{it})}{N}\right)^{2m} dt \\
&\leqslant \frac{1}{2\pi}\sum_{m=0}^{\infty} -\binom{s}{2m}\int_0^{2\pi}\left(\frac{U_n(e^{it})}{N}\right)^{2m} dt \\
&= -\frac{1}{2\pi}\int_0^{2\pi}\left(1+\frac{U_n(e^{it})}{N}\right)^{s} dt \\
&= -\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{R_n(e^{it})}{N}\right|^{s} dt = -\|R_n/N\|_s^s,
\end{aligned}$$

since the constant term $m = 0$ in the binomial power series is the same on both sides. The exchange of the integration and summation in the binomial series is possible by the uniform convergence in the interval $I = [\pi/N, 2\pi - \pi/N]$ and the Lebesgue monotone convergence argument in the interval $J = [0, 2\pi] \setminus I$. The usage of the monotone convergence is a bit more tricky, however. To accomplish this, consider the even and odd power parts

$$\sum_{m=0}^{\infty}\binom{s}{2m}\left(\frac{U_n(e^{it})}{N}\right)^{2m}, \qquad \sum_{m=0}^{\infty}\binom{s}{2m+1}\left(\frac{U_n(e^{it})}{N}\right)^{2m+1}$$

in the binomial power series of $(1 + U_n(e^{it})/N)^s$. The first series is monotonically decreasing, while the second series is monotonically increasing to the limit function value pointwise in the complement interval J, with the exception of two points $t = 0$ and 2π. Hence, the even and odd power parts and their sum are integrable term by term in the interval J. This completes the proof of $\|P\|_s \geqslant \|R_n\|_s$ for $s \in (0, 1)$.

Now suppose that $s \geqslant 1$. The binomial coefficients $\binom{s}{m}$ are positive for $m < s+1$. If s is an integer, the binomial coefficient $\binom{s}{m} = 0$ for $m \geqslant s+1$. If $s \notin \mathbb{N}$, binomial coefficients alternate in sign for $m > s+1$, and the first negative coefficient occurs at $m = [s] + 2$ (here $[s]$ denotes the integer part of a real number). Thus

$$\binom{s}{2m} \geqslant 0 \text{ for } m = 0, 1, 2, \ldots \qquad \text{if } s \in (2j-1, 2j), j \in \mathbb{N}.$$

Hence

$$\begin{aligned}
\|P/N\|_s^s &= \frac{1}{2\pi}\sum_{m=0}^{\infty}\binom{s}{2m}\int_0^{2\pi}\left(\frac{T(e^{it})}{N}\right)^{2m} dt \\
&\leqslant \frac{1}{2\pi}\sum_{m=0}^{\infty}\binom{s}{2m}\int_0^{2\pi}\left(\frac{U_n(e^{it})}{N}\right)^{2m} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{R_n(e^{it})}{N}\right|^{s} dt = \|R_n/N\|_s^s.
\end{aligned}$$

This proves the result. Also, observe that the same argument fails if $s \in (2j, 2j+1)$, since $\binom{s}{2m} > 0$ for $2m < s+1$, while $\binom{s}{2m} < 0$ for $2m > s+1$.

It remains to prove the last statement of Theorem 3.3. Recall that for a fixed polynomial $P(z)$, the norm $\|P\|_s$ is a continuous, monotonically increasing function of s and $\lim_{s\to\infty}\|P\|_s = \|P\|_\infty$. Since the polynomials $R_n(z)$ have maximal infinity norms $\|R_n\|_\infty = 2(n+1)$ in the class $\mathcal{L}P_n$, it follows that $R_n(z)$ have maximal norms for all s sufficiently large. □

References

[1] R.H. Barker, *Group synchronizing of binary digital systems,* Communication Theory, 273–287, Butterworths Sci. Pub., London, 1953.

[2] E. Beller, D.J. Newman, *An extremal problem for the geometric mean of polynomials*, Proc. Amer. Math. Soc. **39** (1973), 313–317. MR0316686 (47:5233)

[3] P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer-Verlag, New York, 1995. MR1367960 (97e:41001)

[4] P. Borwein, S. Choi, J. Jankauskas, *On a class of polynomials related to Barker sequences*, Proc. Amer. Math. Soc. **140** (2012), no. 8, 2613–2625. MR2910749

[5] P. Borwein, M.J. Mossinghoff, *Barker sequences and flat polynomials*, Number Theory and Polynomials (Bristol, U.K., 2006), J. McKee and C. Smyth, eds., 71–88, London Math. Soc. Lecture Note Ser. **352**, Cambridge Univ. Press, 2008. MR2428516 (2009k:11040)

[6] P. Borwein, M.J. Mossinghoff, J. Vaaler, *Generalizations of Gonçalves' inequality*, Proc. Amer. Math. Soc. **135** (2007), no. 1, 253–261. MR2280202 (2007h:30005)

[7] S. Eliahou, M. Kervaire, *Barker sequences and difference sets,* Enseign. Math. (2) **38** (1992), no. 3-4, 345–382. MR1189012 (93i:11018)

[8] S. Eliahou, M. Kervaire, B. Saffari, *A new restriction on the lengths of Golay complementary sequences,* J. Combin. Theory Ser. A **55** (1990), no. 1, 49–59. MR1070014 (91i:11020)

[9] T. Erdélyi, *On the L_q norm of cyclotomic Littlewood polynomials on the unit circle*, Math. Proc. Cambridge Philos. Soc. **151** (2011), no. 2, 373–384. MR2823142 (2012j:30010)

[10] P. Erdős, *An inequality for the maximum of trigonometric polynomials*, Annales Polonica Math. **12** (1962), 151–154. MR0141933 (25:5330)

[11] P. Fan, M. Darnell, *Sequence design for communications applications,* Research Studies Press, Somerset, England, 1996.

[12] G.T. Fielding, *The expected value of the integral around the unit circle of a certain class of polynomials*, Bull. London Math. Soc. **2** (1970), 301–306. MR0280689 (43:6408)

[13] J. Jedwab, *A survey of the merit factor problem for binary sequences,* Sequences and Their Applications, Proceedings of SETA 2004, Lecture Notes in Computer Science **3486**, 30–55, Springer Verlag, Berlin, 2005.

[14] J. Jedwab, S. Lloyd, *A note on the nonexistence of Barker sequences,* Des. Codes Cryptogr. **2** (1992), no. 1, 93–97. MR1157481 (93e:11032)

[15] D.H. Lehmer, *Factorizations of certain cyclotomic functions*, Ann. of Math. **34** (1933), 461–479. MR1503118

[16] K. Mahler, *On two extremum properties of polynomials*, Illinois J. Math. **7** (1963), 681–701. MR0156950 (28:192)

[17] K. Mahler, *An application of Jensen's formula to polynomials*, Mathematika **7** (1960), 98–100. MR0124467 (23:A1779)

[18] M.J. Mossinghoff, *Wieferich Pairs and Barker Sequences*, Des. Codes Cryptogr. **53** (2009), no. 3, 149–163. MR2545689 (2011c:11039)

[19] D.J. Newman, *Norms of polynomials,* Amer. Math. Monthly **67** (1960), 778–779. MR0125205 (23:A2510)

[20] D.J. Newman, *An L_1 extremal problem for polynomials,* Proc. Amer. Math. Soc. **16** (1965), 1287–1290. MR0185119 (32:2589)

[21] K.H. Leung, B. Schmidt, *The field descent method,* Des. Codes Cryptogr. **36** (2005), no. 2, 171–188. MR2211106 (2007g:05023)

[22] Q. I. Rahman, G. Schmeisser, *Analytic Theory of Polynomials*, Clarendon Press, Oxford, 2002. MR1954841 (2004b:30015)

[23] B. Saffari, *Barker sequences and Littlewood two-sided conjectures on polynomials with ± 1 coefficients,* Séminaire d'Analyse Harmonique, Année 1989/90, 139–151, Univ. Paris XI, Orsay, 1990. MR1104693 (92i:11032)

[24] C. Smyth, *Mahler measure of algebraic numbers: A survey,* Number Theory and Polynomials, 322–349, London Math. Soc. Lecture Note Ser., **352**, Cambridge Univ. Press, Cambridge, 2008. MR2428530 (2009j:11172)

[25] R. Turyn, *On Barker codes of even length,* IEEE Trans. Inform. Theory **51** (1963), no. 9, 1256.

[26] R. Turyn, *Character sums and difference sets,* Pacific J. Math. **15** (1965), 319–346. MR0179098 (31:3349)

[27] R. Turyn, *Sequences with small correlation,* Error Correcting Codes (Proc. Sympos. Math. Res. Center, Madison, Wis.), 195–228, John Wiley, New York, 1968. MR0242566 (39:3897)

[28] R. Turyn, J. Storer, *On binary sequences,* Proc. Amer. Math. Soc. **12** (1961), 394–399. MR0125026 (23:A2333)

[29] J.E. Littlewood, *On polynomials* $\sum^n \pm z^m, \sum^n e^{\alpha_m i} z^m, z = e^{\theta i}$, J. London Math. Soc. **41** (1966), 367–376. MR0196043 (33:4237)

[30] J.E. Littlewood, *Some problems in real and complex analysis,* D.C. Heath and Co., Lexington, Mass., 1968. MR0244463 (39:5777)

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