LENGTH INEQUALITIES FOR RIEMANN SURFACES

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Abstract. We establish inequalities between the lengths of certain closed loops in the triply punctured sphere and in the twice-punctured disc.

1. Introduction

Let $\mathbb{H}$ be the hyperbolic plane $\{x + iy : y > 0\}$ with the hyperbolic metric $|dz|/y$ and hyperbolic distance $\rho$. The Uniformisation Theorem states that each hyperbolic Riemann surface $\mathcal{R}$ is conformally equivalent to a quotient space $\mathbb{H}/G$, where $G$ is a discrete group of Möbius transformations (that is, a Fuchsian group) that acts on $\mathbb{H}$. The group $G$ is isomorphic to the fundamental group of $\mathcal{R}$, no element of $G$ (except the identity map) has a fixed point in $\mathbb{H}$, and the hyperbolic metric in $\mathbb{H}$ projects to give a unique metric (the hyperbolic metric) of curvature $-1$ on $\mathcal{R}$.

Let $\mathcal{R}$ be a hyperbolic Riemann surface which we write as $\mathbb{H}/G$. Let $\zeta$ be a point on $\mathcal{R}$, and let $\alpha$ be a closed loop on $\mathcal{R}$ that starts and ends at $\zeta$. Then $\alpha$ lifts to a curve in $\mathbb{H}$ with endpoints, say, $w$ and $g(w)$, where $g \in G$, and the hyperbolic length $\ell(\alpha)$ of $\alpha$ satisfies $\ell(\alpha) \geq \rho(w, g(w))$, with equality when $\alpha$ is the projection of the geodesic segment from $w$ to $g(w)$. Moreover, $\alpha$ is homotopically trivial on $\mathcal{R}$ if and only if $g$ is the identity map in $G$. If $\beta$ is another loop on $\mathcal{R}$, also starting and ending at $\zeta$, then, in a similar way, $\ell(\beta) \geq \rho(w, h(w))$ for some $h$ in $G$. The author has proved [2, p. 198] that if $g$ and $h$ generate a non-cyclic group $\langle g, h \rangle$, then, for all $z \in \mathbb{H}$,

$$\sinh\left(\frac{1}{2}\rho(z, g(z))\right) \sinh\left(\frac{1}{2}\rho(z, h(z))\right) \geq 1.$$ 

This is valid for every hyperbolic $\mathcal{R}$ and every $z \in \mathbb{H}$, and it is stronger than what is customarily known as the Collar Lemma. It shows, for example, that providing $\langle g, h \rangle$ is not cyclic,

$$\sinh\left(\frac{1}{2}\ell(\alpha)\right) \sinh\left(\frac{1}{2}\ell(\beta)\right) \geq 1.$$ 

In this paper we establish stronger inequalities of a similar nature when $\mathcal{R}$ is a triply punctured sphere and a twice-punctured disc.

First, we consider $\mathcal{R}$ to be topologically a sphere punctured at the three points $p_1$, $p_2$ and $p_3$. Now suppose that $\zeta \in \mathcal{R}$ and, for $i = 1, 2, 3$, let $\gamma_i$ be a simple closed curve that starts and ends at $\zeta$ and that separates $p_i$ from the other two punctures.
Now \( \gamma_s \) and \( \gamma_t \), applied to each pair \( \gamma_s \) and \( \gamma_t \), yields
\[
\sinh\left(\frac{1}{2} \ell_1\right) \sinh\left(\frac{1}{2} \ell_2\right) \sinh\left(\frac{1}{2} \ell_3\right) \geq 1,
\]
where \( \ell_s \) is the length of \( \gamma_s \). Our first result strengthens this inequality.

**Theorem 1.1.** Let \( \mathcal{R} \) be a sphere punctured at the three points \( p_1, p_2 \) and \( p_3 \). Let \( \zeta \) be any point of \( \mathcal{R} \) and, for \( j = 1, 2, 3 \), let \( \ell_j \) be the length of the shortest closed curve that starts and ends at \( \zeta \) and that separates \( p_j \) from the other two punctures. Then
\[
(2) \quad \sinh\left(\frac{1}{2} \ell_1\right) \sinh\left(\frac{1}{2} \ell_2\right) \sinh\left(\frac{1}{2} \ell_3\right) \geq \left(\frac{2}{\sqrt{3}}\right)^3 = 1.5396 \cdots.
\]
This bound is attained if and only if \( \zeta \) is at the common intersection of the three geodesics \( \sigma_j \), where \( \sigma_j \) leaves \( p_j \) and is orthogonal to the geodesic joining the other two punctures.

Next, we assume that \( \mathcal{R} \) is a twice-punctured disc. Here we may assume that \( \mathcal{R} = \mathbb{D} \setminus \{p_1, p_2\} \), where \( \mathbb{D} \) is the open unit disc in \( \mathbb{C} \) and \( p_1 \) and \( p_2 \) are in \( \mathbb{D} \). Select any point \( \zeta \) in \( \mathcal{R} \), and let \( \ell_j, j = 1, 2, \) be the length of the shortest closed curve in \( \mathcal{R} \) that starts and ends at \( \zeta \), and that separates the puncture \( p_j \) from the other puncture, and from \( \partial \mathbb{D} \). Also, let \( \ell \) be the length of the shortest closed geodesic that separates both punctures from \( \partial \mathbb{D} \).

**Theorem 1.2.** For the twice-punctured disc described above,
\[
(3) \quad \sinh\left(\frac{1}{2} \ell_1\right) \sinh\left(\frac{1}{2} \ell_2\right) \geq \cosh^2\left(\frac{1}{4} \ell\right) \cosh \rho(\zeta, \sigma),
\]
where \( \sigma \) is the geodesic joining the two punctures.

### 2. Some hyperbolic geometry

The conformal isometries of \( \mathbb{H} \) are the maps \( f(z) = (az + b)/(cz + d) \), where \( a, b, c, d \in \mathbb{R} \) and \( ad - bc = 1 \). Any such \( f \) may be represented by the pair of matrices
\[
\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
in \( \text{SL}(2, \mathbb{R}) \), and \( \text{tr}^2(f) \) is uniquely defined as \( (a + d)^2 \). Next, the displacement function for \( f \) is \( \rho(z, f(z)) \), and this is given by
\[
(4) \quad \sinh\frac{1}{2} \rho(z, f(z)) = \frac{|z - f(z)|}{2\sqrt{\text{Im}[z]} \sqrt{\text{Im}[f(z)]}},
\]
where \( z = x + iy \) (see [2, p. 130]).

Now suppose that \( f \) has two distinct fixed points on \( \mathbb{R} \cup \{\infty\} \). Then \( f \) is conjugate to a map \( F(z) = k z \), where \( k > 1 \). Then it is easy to see (and well-known) that the displacement function of \( F \) attains its minimum value \( T_F \) on the positive imaginary axis and that \( T_F \), which is the translation length of \( F \), is given by \( \text{tr}^2(F) = 4 \cosh^2\left(\frac{1}{2} T_F\right) \). Since all terms here are invariant under conjugation, we see that
\[
(5) \quad \text{tr}^2(f) = 4 \cosh^2\left(\frac{1}{2} T_f\right).
\]

For more details on these and related facts, see [2].
3. The proof of Theorem 1.1

As any triply punctured sphere is conformally equivalent to $\mathbb{C}_\infty \setminus \{0, 1, \infty\}$, we may assume that $\mathcal{R} = \mathbb{C}_\infty \setminus \{0, 1, \infty\}$. Then (see [11, pp. 277–282]) $\mathcal{R} = \mathbb{H}/\Gamma(2)$, where $\Gamma(2)$, which is a congruence subgroup of the Modular group, is generated by $z \mapsto z + 2$ and $z \mapsto z/(2z + 1)$. Further, the hyperbolic quadrilateral in $\mathbb{H}$ with vertices $-1, 0, 1$ and $\infty$ is a fundamental polygon for $G$. The quotient map $\lambda : \mathbb{H} \to \mathcal{R}$ is the elliptic modular function, and, using the construction of $\lambda$ given in [11], we may take (with the appropriate interpretation) $p_1 = 0 = \lambda(\infty)$, $p_2 = 1 = \lambda(0)$ and $p_3 = \infty = \lambda(1)$. Finally, the stabilizers of $\infty$, 0 and 1 in $\Gamma(2)$ are generated by $g_1$, $g_2$ and $g_3$, respectively, where

$$g_1(z) = z + 2, \quad g_2(z) = \frac{z}{2z + 1}, \quad g_3(z) = g_2^{-1}(z) = \frac{3z - 2}{2z - 1}.$$

Now suppose that $w$ in $\mathbb{H}$ projects to $\zeta$ in $\mathcal{R}$. Then, for $j = 1, 2, 3$, $\ell_j = \sigma(w, g_j(w))$, so that (2) is equivalent to the assertion that

$$\inf_{z \in \mathbb{H}} \left( \sinh \frac{1}{2} \rho(z, g_1(z)) \sinh \frac{1}{2} \rho(z, g_2(z)) \sinh \frac{1}{2} \rho(z, g_3(z)) \right) = \left(\frac{2}{\sqrt{3}}\right)^3. \tag{6}$$

We shall now establish (6). First, (4) shows that

$$\sinh \frac{1}{2} \rho(z, g_1(z)) \sinh \frac{1}{2} \rho(z, g_2(z)) \sinh \frac{1}{2} \rho(z, g_3(z)) = \frac{1}{y} \frac{|z|^2}{y} \frac{|z - 1|^2}{y},$$

so that (6) is equivalent to

$$\inf_{z \in \mathbb{H}} \frac{|z(z - 1)|^2}{y^3} = \left(\frac{2}{\sqrt{3}}\right)^3. \tag{6}$$

If we change the variable to $u + iv = z - \frac{1}{2}$ and write

$$\Phi(u, v) = \frac{(u^2 + v^2 - \frac{1}{4})^2 + v^2}{v^3},$$

we see that (6) is equivalent to

$$\Phi(u, v) \geq \Phi(0, \frac{1}{2}\sqrt{3}) = \left(\frac{2}{\sqrt{3}}\right)^3. \tag{7}$$

We now establish the inequality in (7) (the equality is true). If $v \leq \frac{1}{2}$, then $\Phi(u, v) \geq v^2/v^3 \geq 2 > (2/\sqrt{3})^3$. If not, then $v > \frac{1}{2}$ so that

$$\Phi(u, v) \geq \Phi(0, v) = \frac{(v^2 - \frac{1}{4})^2 + v^2}{v^3} = \frac{(v^2 + \frac{1}{4})^2}{v^3} = \varphi(v),$$

say, with equality if and only if $u = 0$. By considering $\varphi'(v)$ we see that $\varphi(v)$ attains its minimum when $4v^2 = 3$, and (6) is proved. This argument shows that equality occurs in (2) if and only if $u = 0$ and $v = \sqrt{3}/2$ or, equivalently, when $z = (1 + i\sqrt{3})/2$, which is at the intersection of the altitudes of the hyperbolic triangle in $\mathbb{H}$ with vertices 0, 1 and $\infty$. 

4. The proof of Theorem 1.2

We may assume that \( \mathcal{R} = \mathbb{D} \setminus \{a, -a\} \) for some \( a \) in \((0, 1)\), and we may take \( G \) to be generated by

\[
g(z) = z + t, \quad h(z) = \frac{z}{z + 1},
\]

where \( t > 4 \). The region \( \Sigma \) bounded by the two lines \( x = \pm t/2 \) and the two circles \( |z \pm 1| = 1 \) is a fundamental region for \( G \) and the sides of \( \Sigma \) are paired by \( g \) and \( h \). An examination of the geometry of the actions of \( g \) and \( h \) shows that \( \ell \) is the translation length of \( h^{-1}g \), and if we compute \( h^{-1}g \) and use (5) we find that

\[
t = 4 \cosh^2(\ell/4).
\]

Now choose any \( z \) in \( \mathbb{H} \) with \( z = x + iy \). Then, from (4) and [2, p. 162],

\[
\sinh \frac{1}{2} \rho(z, g(z)) \sinh \frac{1}{2} \rho(z, h(z)) = \left( \frac{t}{2y} \right) \left( \frac{|z|^2}{2y} \right) = \frac{1}{4} t \cosh \rho(z, I),
\]

where \( I \) is the positive imaginary axis, and this gives (3).

References


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