ON A THEOREM OF HAZRAT AND HOOBLER

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Abstract. We use cycle complexes with coefficients in an Azumaya algebra, as developed by Kahn and Levine, to compare the \( G \)-theory of an Azumaya algebra to the \( G \)-theory of the base scheme. We obtain a sharper version of a theorem of Hazrat and Hoobler in certain cases.

1. Introduction

Let \( K_\ast(X;A) \) be the \( K \)-theory of left \( A \)-modules which are locally free and finite rank coherent \( \mathcal{O}_X \)-modules and let \( G_\ast(X;A) \) be the \( K \)-theory of left \( A \)-modules which are coherent \( \mathcal{O}_X \)-modules.

We prove the following theorem.

Theorem 1.1. Let \( X \) be a \( d \)-dimensional scheme of finite type over a field \( k \), and let \( A \) be an Azumaya algebra on \( X \) of constant degree \( n \). Let \( B_A : G_i(X) \to G_i(X;A) \) and \( B_K^A : K_i(X) \to K_i(X;A) \) be the homomorphisms induced by the functor \( F \mapsto A \otimes \mathcal{O}_X F \). Then,

1. the kernel and cokernel of \( B_A : G_i(X) \to G_i(X;A) \) are torsion groups of exponents dividing \( n^{2d+2} \);
2. the kernel and cokernel of \( B_K^A : K_i(X) \to K_i(X;A) \) are torsion groups of exponents dividing \( n^{2d+2} \) if \( X \) is regular.

Corollary 1.2. If \( A \) is an Azumaya algebra of constant degree \( n \) over a scheme \( X \) of finite type over a field \( k \), then the base extension homomorphism

\[
B_A : G_\ast(X) \otimes \mathbb{Z} \left[ \frac{1}{n} \right] \to G_\ast(X;A) \otimes \mathbb{Z} \left[ \frac{1}{n} \right]
\]

is an isomorphism.

The theorem above should be compared to the following two theorems, which motivated us in the first place.

Theorem 1.3 (Hazrat-Millar [9]). If \( A \) is an Azumaya algebra of constant degree \( n \) which is free over a noetherian affine scheme \( X \), then

\[
B_K^A : K_i(X) \to K_i(X;A)
\]

has torsion kernel and cokernel of exponents at most \( n^4 \).
Theorem 1.4 (Hazrat-Hoobler [8]). Let $X$ be a $d$-dimensional noetherian scheme, and let $\mathcal{A}$ be an Azumaya algebra on $X$ of constant degree $n$. Then:

1. the kernel of $B_{\mathcal{A}}: G_i(X) \to G_i(X; \mathcal{A})$ is torsion of exponent dividing $n^{2d(d+1)+2}$, and the cokernel is torsion of exponent dividing $n^{4d+2}$;
2. the kernel of $B^K_{\mathcal{A}}: K_i(X) \to K_i(X; \mathcal{A})$ is torsion of exponent dividing $n^{2d(d+1)+2}$ if $X$ is regular, and the cokernel is torsion of exponent dividing $n^{4d+2}$ in this case;
3. the kernel and cokernel of $B^K_{\mathcal{A}}: K_i(X) \to K_i(X; \mathcal{A})$ are torsion groups of exponent dividing $n^{2d+2}$ if $X$ has an ample line bundle.

Since a degree $n$ Azumaya algebra is locally split by degree $n$ extensions, it is expected that the base extension map

$$B^K_{\mathcal{A}}: K_*(X) \otimes_{\mathbb{Z}} \mathbb{Z} \left[\frac{1}{n}\right] \to K_*(X; \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z} \left[\frac{1}{n}\right]$$

should be an isomorphism.

Here is a partial history of results and techniques in this direction. Wedderburn’s theorem [10] easily implies that $K_0(k) \to K_0(\mathcal{A})$ is injective with cokernel isomorphic to $\mathbb{Z}/m$, where $\mathcal{A} \cong M_m(D)$ for a central $k$-division algebra $D$.

Green-Handelman-Roberts [5] proved that the map $B^K_{\mathcal{A}}$ in equation (1) is an isomorphism when $\mathcal{A}$ is a central simple algebra of degree $n$ over a field. They used the Skolem-Noether theorem. That case has also been proven by Hazrat [7] using the fact that $\mathcal{A}$ is étale locally a matrix algebra.

The theorem of Hazrat-Millar quoted above uses the opposite algebra. The theorem of Hazrat-Hoobler uses Bass-style stable range arguments and Zariski descent for $G$-theory.

Our result uses twisted versions of Bloch’s cycle complexes. These twisted cycle complexes and the twisted motivic spectral sequence that relates them to $G$-theory are due to Kahn and Levine [11]. It is possible that our result could be extended to essentially smooth schemes over Dedekind rings by a combination of the work of Kahn and Levine [11] and Geisser [4].

The following is an interesting corollary of our approach: there are natural filtrations of length $d$ on $G_i(X)$ and $G_i(X; \mathcal{A})$ coming from [11]. The map $B_{\mathcal{A}}: G_i(X) \to G_i(X; \mathcal{A})$ respects the filtrations. We show that the induced maps on each of the $d+1$ slices have kernel and cokernel groups of exponent at most $n^2$.

It is worth mentioning two related functors on Azumaya algebras with values in abelian groups where the base extension maps are isomorphisms. Dwyer and Friedlander [3] 2.4, 3.1] showed that

$$K^\text{ét}_*(R; \mathbb{Z}/m) \to K^\text{ét}_*(R; \mathcal{A}/\mathbb{Z}/m)$$

is an isomorphism in some cases (all of which are Azumaya algebras over a noetherian ring), where $K^\text{ét}_*$ denotes étale $K$-theory, as, for instance, in Thomason [12].

In this direction, it is possible to show (for instance, in the setting of Antieau [11]) that $K^\text{ét}_*(X; \mathcal{A})$ is an invertible object (in the sense of the Picard group) over $K^\text{ét}_*(X)$ in the category of étale sheaves of $K^\text{ét}_*$-module spectra on a scheme $X$.

Finally, Cortiñas and Weibel [2] proved that the base extension maps induce isomorphisms in Hochschild homology over a field $k$. 

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2. Twisted higher Chow groups and twisted $G$-theory

Let $X$ in $\text{Sch}/k$ be an integral $k$-scheme of finite type, and let $\mathcal{A}$ be a sheaf of Azumaya algebras on $X$ of rank $n^2$. The degree of $\mathcal{A}$ is defined to be the integer $n$. Let $\mathcal{E}$ be a left $\mathcal{A}$-module which is locally free and finite rank $na$ as an $\mathcal{O}_X$-module. For generalities on Azumaya algebras, which as $\mathcal{O}_X$-modules are always locally free and of finite rank, see [6].

As in Kahn-Levine [11], define the cycle complex of $X$ with coefficients in $\mathcal{A}$ as follows. Let $S^X_{(s)}(t)$ denote the set of closed subsets $W \subset X \times_k \Delta^t$ such that

$$\dim_k W \cap X \times_k F \leq s + \dim_k F$$

for all faces $F$ of $\Delta^n$. Taking inverse images, $S^X_{(s)}(*)$ becomes a simplicial set. Let $X_s(t)$ denote the subset of irreducible $W$ in $S^X_{(s)}(t)$ such that $\dim_k W = s + t$. Define, for $t \geq 0$,

$$z_s(X, t; \mathcal{A}) = \bigoplus_{W \in X_s(t)} K_0(k(W); \mathcal{A}).$$

See [11] Definition 5.6.1]. Kahn and Levine show that this actually becomes a complex, $z_s(X, *; \mathcal{A})$, and they define the higher Chow groups with coefficients in $\mathcal{A}$ as

$$\text{CH}_s(X, t; \mathcal{A}) = H_t(z_s(X, *; \mathcal{A})).$$

There are maps relating the complex $z_r(X, *; \mathcal{A})$ to $z_r(X, *)$, the untwisted complex that computes Bloch’s higher Chow groups. These are induced by the base-change map $B_\mathcal{E}$ and the forgetful map $F$ on $K$-theory:

$$B_\mathcal{E}^K : K_0(k(W)) \to K_0(k(W); \mathcal{A}),$$

$$F : K_0(k(W), \mathcal{A}) \to K_0(k(W)).$$

The map $B_\mathcal{E}$ takes a $k(W)$-vector space and tensors with $\mathcal{E}_{k(W)}$ to produce a left $\mathcal{A}_{k(W)}$-module. The norm map $F$ simply forgets the $\mathcal{A}_{k(W)}$-module structure on a vector space. The kernels of both of these maps are zero.

**Lemma 2.1.** The compositions $F \circ B_\mathcal{E}^z$ and $B_\mathcal{E}^z \circ F$ are multiplication by $na$ on $z_s(X, t)$ and $z_s(X, t; \mathcal{A})$.

**Proof.** Indeed, since the rank of $\mathcal{E}$ is $na$ as an $\mathcal{O}_X$-module, this follows immediately. □

**Corollary 2.2.** The cokernel of $F : z_s(X, t; \mathcal{A}) \to z_s(X, t)$ is a torsion group of exponent bounded above by $n^2$, and $B_\mathcal{E}^z : z_s(X, t) \to z_s(X, t; \mathcal{A})$ is a torsion group of exponent bounded above by $na$.

**Proof.** In the first case, one always has $\text{ind}(\mathcal{A}_{k(W)}) | n$, where $\text{ind}(\mathcal{A}_{k(W)})$ is the degree of the unique division algebra over $k(W)$ such that $\mathcal{A}_{k(W)} \cong M_m(D)$ for some $m$. Similarly,

$$\left(\frac{na}{\text{ind}(\mathcal{A}_{k(W)})^2}\right) | na,$$

so the second statement follows. □
Proposition 2.3. The kernels and cokernels of
\[ B^\text{CH}_E : \text{CH}_s(X, t) \to \text{CH}_s(X, t; A) \]
and of
\[ F : \text{CH}_s(X, t; A) \to \text{CH}_s(X, t) \]
are torsion groups of exponent at most na.

Proof. This follows immediately from Lemma 2.2. \square

Here is our main theorem. Theorem 1.1 follows from it by taking \( E = A \).

Theorem 2.4. Let \( X \) be a \( d \)-dimensional scheme of finite type over a field, and let \( A \) be an Azumaya algebra on \( X \). Then, the kernels and cokernels of
\[ B^E : \mathcal{G}_r(X) \to \mathcal{G}_r(X; A) \]
and of
\[ F : \mathcal{G}_r(X; A) \to \mathcal{G}_r(X) \]
are groups of exponent bounded above by \((na)^{d+1}\) for all \( r \geq 0 \).

Proof. Kahn and Levine [11] show that there is a convergent spectral sequence
\[ E^p_{2,q} = \text{CH}_q(X, -p - q; A) \Rightarrow \mathcal{G}_{-p-q}(X; A). \]
There is also the motivic spectral sequence
\[ E^p_{2,q} = \text{CH}_q(X, -p - q) \Rightarrow \mathcal{G}_{-p-q}(X). \]
The functors \( B^E : \mathcal{G}(X) \to \mathcal{G}(X; A) \) and \( F : \mathcal{G}(X; A) \to \mathcal{G}(X) \) are compatible with these spectral sequences and the functors \( B^\text{CH}_E \) and \( F \) on higher Chow groups. Note that \( E^p_{2,q} = E^p_{2,q}(A) = 0 \) whenever \( q < 0, -p < 0, \) or \( q > d \).

We will prove the theorem for the kernel of the functor \( B^E \). The other cases are entirely similar. On the \( E_\infty \)-page, the composition functor \( F \circ B^E \) is still multiplication by \( na \), so the kernels and cokernels of \( B^\text{CH}_E \) on \( E_\infty \) are still of exponent at most \( na \). The spectral sequences abut to filtrations \( F^s \mathcal{G}_r(X; A) \) and \( F^s \mathcal{G}_r(X) \), where
\[ F^{(s/s+1)} \mathcal{G}_r(X; A) = F^s \mathcal{G}_r(X; A) / F^{s+1} \mathcal{G}_r(X; A) \equiv E^{-r+s, -s}(A), \]
\[ F^{(s/s+1)} \mathcal{G}_r(X) = F^s \mathcal{G}_r(X) / F^{s+1} \mathcal{G}_r(X) \equiv E^{-r+s, -s}. \]
The filtration looks like
\[ 0 = F^0 \mathcal{G}_r(X) \subseteq F^{-1} \mathcal{G}_r(X) \subseteq \cdots \subseteq F^{-d} \mathcal{G}_r(X) = \mathcal{G}_r(X). \]
The filtration \( F^s \mathcal{G}_r(X) \) is of length \( d \) by the vanishing statements. Let \( z \in \mathcal{G}_r(X) \) be in the kernel of \( F \), and let \( \overline{z} \) be the image of \( z \) in \( E^{r-d,d}_\infty \). Then, by hypothesis, \( \overline{z} \) is in the kernel of \( F \), so that \( na \cdot \overline{z} = 0 \). Thus, \( na \cdot z \) is contained in \( F^{-d+1} \mathcal{G}_r(X) \). Continuing in this way, we see that \((na)^{d+1} \cdot z \) is contained in \( F^0 \mathcal{G}_r(X) = 0 \), so \((na)^{d+1} \cdot z = 0 \). \square

Corollary 2.5. The same result holds for \( K \)-theory when \( X \) is regular.

Corollary 2.6. The maps
\[ B^E_{(s/s+1)} : F^{(s/s+1)} \mathcal{G}_r(X) \to F^{(s/s+1)} \mathcal{G}_r(X; A), \]
\[ F : F^{(s/s+1)} \mathcal{G}_r(X; A) \to F^{(s/s+1)} \mathcal{G}_r(X) \]
have torsion kernels and cokernels of exponent at most \( na \).
Proof. This follows from the proof of the theorem. □

Corollary 2.7. For any commutative ring \( R \) in which \( na \) is invertible, the maps
\[
B_X^*: z_*(X, *; R) \rightarrow z_*(X, *; A; R),
\]
\[
B_E^*: G_r(X; R) \rightarrow G_r(X; A; R),
\]
\[
F^*: z_*(X, *; A; R) \rightarrow z_*(X, *; R),
\]
\[
F: G_r(X; A; R) \rightarrow G_r(X; R)
\]
are isomorphisms.

It is interesting that this method proves the isomorphisms by means of an isomorphism of cycle complexes, not just a quasi-isomorphism.

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REFERENCES


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