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A NOTE ON THE INVISCID LIMIT OF THE BENJAMIN-ONO-BURGERS EQUATION IN THE ENERGY SPACE

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Abstract. In this paper we study the inviscid limit of the Benjamin-Ono-Burgers equation in the energy space $H^{1/2}(\mathbb{R})$ or $H^{1/2}(\mathbb{T})$. We prove the strong convergence in the energy space of the solution to this equation toward the solution of the Benjamin-Ono equation as the dissipation coefficient converges to 0.

1. Introduction

The initial value problem (IVP) is associated to the Benjamin-Ono-Burgers (BOB) equation

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u - \varepsilon u_{xx} = u\partial_x u, \\ u(x,0) = \varphi(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}$ or $\mathbb{T}$, $t \in \mathbb{R}_+$, u is a real-valued function, ε is a positive real number and $\mathcal{H}$ is the Hilbert transform given for function on $\mathbb{R}$ by

$$\mathcal{H}f(x) = \text{p.v.}\, \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy. \tag{1.2}$$

This equation is a dissipative perturbation of the celebrated Benjamin-Ono (BO) equation and is proven to be physically relevant in [5]. Recall that the Benjamin-Ono equation was derived in [2] to modelize the unidirectional evolution of weakly nonlinear dispersive internal long waves at the interface of a two-layer system, one being infinitely deep.

It is easy to check that the BOB equation is globally well-posed in $H^s(\mathbb{R})$ and $H^s(\mathbb{T})$ for $s > -1/2$. One can use for instance a purely parabolic approach, as was used for the dissipative Burgers equation (cf. [1]). On the other hand, the IVP for the BO equation is more delicate to solve in Sobolev spaces with low indexes. Indeed, it was proven in [11] that this IVP cannot be solved by a Picard iteration scheme in any Sobolev space $H^s(\mathbb{R})$. However, in [12] Tao noticed that one can solve this IVP in $H^1(\mathbb{R})$ by introducing a suitable gauge transform. This approach has been pushed forward in [3], [7] and [10]. In these last two papers, the IVP is proved to be globally well-posed in $L^2(\mathbb{R})$ (see [9] or [10] for the global well-posedness in $L^2(\mathbb{T})$). Unfortunately this gauge transform does not behave well with respect to

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perturbations of the equation and in particular with respect to the BOB equation. It is thus not evident how to prove the uniform in ε wellposedness of the BOB equation in low regularity spaces and, as a consequence, how to study its inviscid limit behavior. However, in [6], Guo et al. used the variant of the Bourgain's spaces introduced in [8] to prove that the BOB equation is uniformly (in ε) well-posed in $H^s(\mathbb{R})$, $s \geq 1$, and to deduce the convergence of the solutions to this equation towards the one of the BO equation in $H^s(\mathbb{R})$, $s \geq 1$.

Our goal in this paper is to prove that this convergence result also holds in the energy space $H^{1/2}(\mathbb{R})$ by a very much simpler approach. This approach combines the conservation laws and the unconditional uniqueness in $H^{1/2}$ (cf. [3], [10]) of the Benjamin-Ono equation. Note that this approach also works in H^1 (and more generally in all $H^{n/2}$ for $n \geq 1$), where the unconditional uniqueness is a simple consequence of the L^2 Lipschitz bound established in [12]. Therefore, our approach also gives a great simplification of the inviscid limit result in $H^1(\mathbb{R})$ with only [12] in hand. Finally, it is worth noting that our method also works in the periodic setting.

2. Main result and proof

Our main result reads:

Theorem 2.1. *Let $K := \mathbb{R}$ or $\mathbb{T}$, $\varphi \in H^{1/2}(K)$ and $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers converging to 0. For any $T > 0$, the sequence $\{u_n\}_{n\in\mathbb{N}} \in C(\mathbb{R}_+; H^{1/2}(K))$ of the solution to $(1.1)_{\varepsilon_n}$ emanating from φ satisfies*

$$u_n \to u \text{ in } C([0,T]; H^{1/2}(K)), \tag{2.1}$$

where $u \in C(\mathbb{R}; H^{1/2}(K))$ is the unique solution to the BO equation emanating from φ.

Proof. We first give the complete proof in the real line case and then discuss the adaptation in the periodic case at the end of the paper. We will divide the proof into three steps.

Step 1. *Uniform bounds.* First we establish uniform in ε a priori estimates on solutions to $(1.1)_\varepsilon$. Taking the L^2-scalar product of the equation with u, it is straightforward to check that smooth solutions to $(1.1)_{\varepsilon_n}$ satisfy

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2_x}^2 + \varepsilon_n \|\partial_x u(t)\|_{L^2_x}^2 = 0 \, .$$

By the continuity of the flow-map of $(1.1)_{\varepsilon_n}$ in $H^{1/2}(\mathbb{R})$, it follows that, for any $\varphi \in H^{1/2}$, it holds that $u_n \in C_b(\mathbb{R}_+; L^2(\mathbb{R}))$ and $u_{n,x} \in L^2(\mathbb{R}_+; L^2(\mathbb{R}))$ with

$$\|u_n(t)\|_{L^2_x}^2 + 2\varepsilon_n \int_0^t \|\partial_x u_n(s)\|_{L^2_x}^2 \, ds \leq \|\varphi\|_{L^2_x}^2, \ \forall t \geq 0 \, . \tag{2.2}$$

Similarly, by denoting by D_x^s the operator with Fourier symbol $|\xi|^s$, taking the L^2-scalar product of the equation with $D_x u_n + u_n^2/2$ and setting

$$E(v) := \frac{1}{2}\int_{\mathbb{R}} |D_x^{1/2} v|^2 \, dx - \frac{1}{6}\int_{\mathbb{R}} v^3 \, dx \; ,$$

we obtain

$$\begin{aligned}\frac{d}{dt}E(u_n(t))+\varepsilon_n\|D_x^{3/2}u_n(t)\|_{L^2_x}^2 &= \frac{\varepsilon_n}{2}\int_{\mathbb{R}}u_n^2(t)u_{n,xx}(t)\\ &\lesssim \varepsilon_n\|u_n(t)\|_{L^\infty_x}\|u_{n,x}(t)\|_{L^2_x}^2\\ &\lesssim \varepsilon_n\|u_n(t)\|_{L^\infty_x}\|D_x^{1/2}u_n(t)\|_{L^2_x}\|D_x^{3/2}u_n(t)\|_{L^2_x}\ .\end{aligned}$$

Therefore by Young's inequality and then classical interpolation inequalities, we infer that

$$\begin{aligned}\frac{d}{dt}E(u_n(t))+\frac{\varepsilon_n}{2}\|D_x^{3/2}u_n(t)\|_{L^2_x}^2 &\lesssim \varepsilon_n\|u_n(t)\|_{L^\infty_x}^2\|D_x^{1/2}u_n(t)\|_{L^2_x}^2\\ &\lesssim \varepsilon_n\|u_n(t)\|_{L^2_x}^2\|u_{n,x}(t)\|_{L^2_x}^2\ .\end{aligned}\tag{2.3}$$

Gathering this last estimate with (2.2), we obtain

$$E(u_n(t))+\frac{\varepsilon_n}{2}\int_0^t\|D_x^{3/2}u_n(s)\|_{L^2_x}^2\,ds\lesssim\|\varphi\|_{L^2_x}^4+E(\varphi),\quad\forall t\geq 0\ .$$

By using the fact that $H^{1/6}(\mathbb{R})\hookrightarrow L^3(\mathbb{R})$, classical interpolation inequalities and again (2.2), we are eventually led to

$$\begin{aligned}&\|D_x^{1/2}u_n(t)\|_{L^2_x}^2+\varepsilon_n\int_0^t\|D_x^{3/2}u_n(s)\|_{L^2_x}^2\,ds\\ &\qquad\lesssim\|\varphi\|_{H^{1/2}_x}^4+\|\varphi\|_{L^2_x}^2\Big(\|D_x^{1/2}\varphi\|_{L^2_x}+\|D_x^{1/2}u_n(t)\|_{L^2_x}\Big),\quad\forall t\geq 0\ ,\end{aligned}$$

which ensures that

$$\|D_x^{1/2}u_n(t)\|_{L^2_x}^2+\varepsilon_n\int_0^t\|D_x^{3/2}u_n(s)\|_{L^2_x}^2\,ds\lesssim 1+\|\varphi\|_{H^{1/2}_x}^4,\quad\forall t\geq 0\ .\tag{2.4}$$

Step 2. *Convergence in the weak topology.*

Proposition 2.1. *Let* $\varphi\in H^{1/2}(\mathbb{R})$ *and* $\{\varepsilon_n\}_{n\in\mathbb{N}}$ *be a decreasing sequence of real numbers converging to* 0. *The sequence* $\{u_n\}_{n\in\mathbb{N}}$ *of the solution to* $(1.1)_{\varepsilon_n}$ *emanating from* φ *satisfies*

$$u_n\rightharpoonup u\ \textit{weak star in}\ L^\infty(\mathbb{R};H^{1/2}(\mathbb{R}))\tag{2.5}$$

and

$$(u_n,\phi)_{H^{1/2}}\to(u,\phi)_{H^{1/2}}\ \textit{in}\ C([-T,T]),\forall\phi\in H^{1/2}(\mathbb{R}),\ \forall T>0\ ,\tag{2.6}$$

where $u\in C(\mathbb{R};H^{1/2}(\mathbb{R}))$ *is the unique solution to the BO equation emanating from* φ.

Proof. According to (2.2) and (2.4), the sequence $\{u_n\}$ is bounded in $L^\infty(\mathbb{R};H^{1/2}(\mathbb{R}))$. Moreover, in view of equation (1.1), the sequence $\{\partial_t u_n\}$ is bounded in $L^\infty(\mathbb{R};H^{-2}(\mathbb{R}))$. By the Aubin-Lions compactness theorem, we infer that for any $T>0$ and $R>0$, $\{u_n\}$ is relatively compact in $L^2(]-T,T[\times]-R,R[)$. Therefore, using a diagonal extraction argument, we obtain the existence of an increasing sequence $\{n_k\}\subset\mathbb{N}$ and $u\in L^\infty(\mathbb{R};H^{1/2}(\mathbb{R}))$ such that

$$u_{n_k}\rightharpoonup u\ \text{weak star in}\ L^\infty(\mathbb{R};H^{1/2}(\mathbb{R}))\ ,\tag{2.7}$$

$$u_{n_k}\to u\ \text{in}\ L^2_{loc}(\mathbb{R}^2)\ ,\tag{2.8}$$

$$u_{n_k}\to u\ \text{a.e. in}\ \mathbb{R}^2\ ,\tag{2.9}$$

$$u_{n_k}^2\rightharpoonup u^2\ \text{weak star in}\ L^\infty(\mathbb{R};L^2(\mathbb{R}))\ .\tag{2.10}$$

These convergence results enable us to pass to the limit on the equation and to obtain that the limit function u satisfies the BO equation in the distributional sense. Now, the crucial argument is that, according to [4] and [10], the BO equation is unconditionally well-posed in $H^{1/2}(\mathbb{R})$ (and even in $H^s(\mathbb{R})$ for $s > 1/4$) in the sense that the solution constructed in [10] is the only function belonging to $L^\infty(-T,T;H^{1/2}(\mathbb{R}))$ that satisfies the BO equation in the distributional sense and is equal[1] to φ at $t = 0$. By the uniqueness of the possible limit, this ensures that, actually, $\{u_n\}$ converges to u in the sense of (2.7)-(2.10).

Now, using the equation and the bound on $\{u_n\}$, it is clear that for any $\phi \in C_0^\infty(\mathbb{R})$ and any $T > 0$, the sequence $\{t \mapsto (u_n,\phi)_{H^{1/2}}\}$ is uniformly equi-continuous on $[-T,T]$. By Ascoli's theorem it follows that $\{(u_n,\phi)_{H^{1/2}}\} \to (u,\phi)$ in $C([-T,T])$. Since $\{u_n\}$ is bounded in $L^\infty(\mathbb{R};H^{1/2}(\mathbb{R}))$, this yields (2.6).

Step 3. *Making use of the conservation laws of the BO equation.* We start by proving a strong convergence result in L^2. A first idea would be to derive a Lipschitz bound in L^2. Note that, following [12], such a L^2-Lipschitz bound would be available at the H^1-regularity. However, we do not know how to get it at the $H^{1/2}$-regularity. Instead, we will rely on the nonincreasing property of the L^2-norm of solutions to $(1.1)_{\varepsilon_n}$ established in (2.2). According to the L^2-conservation law of the BO equation, this proves that for any $n \in \mathbb{N}$ and $t > 0$,

$$\|u_n(t)\|_{L^2_x} \le \|\varphi\|_{L^2_x} = \|u(t)\|_{L^2_x} \, . \tag{2.11}$$

We claim that (2.11), together with (2.6), ensures that for any $T > 0$,

$$u_n \to u \text{ in } C([0,T];L^2(\mathbb{R})). \tag{2.12}$$

To see this, we proceed by contradiction assuming that $u_n \not\to u$ in $L^\infty(0,T;L^2(\mathbb{R}))$. Then, there exist $t_0 \in [0,T]$, $\{t_n\} \subset [0,T]$ and $\alpha > 0$ such that $t_n \to t_0$ as $n \to \infty$ and

$$\|u_n(t_n) - u(t_n)\|_{L^2_x} \ge \alpha, \quad \forall n \in \mathbb{N} \, . \tag{2.13}$$

Now, from (2.11) we infer that

$$\begin{aligned}\|u_n(t_n)-u(t_n)\|^2_{L^2_x} &= 2\Big(u(t_n) - u_n(t_n), u(t_n)\Big)_{L^2_x} + \|u_n(t_n)\|^2_{L^2_x} - \|u(t_n)\|^2_{L^2_x} \\ &\le 2\Big(u(t_n) - u_n(t_n), u(t_0)\Big)_{L^2_x} + 2\Big(u(t_n) - u_n(t_n), u(t_n) - u(t_0)\Big)_{L^2_x} \, .\end{aligned}$$

The first term of the right-hand side of the above inequality tends to 0 as $n \to \infty$ thanks to (2.6), whereas the absolute value of the second term is dominated by $4\|\varphi\|_{L^2_x}\|u(t_n) - u(t_0)\|_{L^2_x} \to 0$ as $n \to \infty$. This contradicts (2.13) and thus proves (2.12).

Combining (2.12) with Proposition 2.1 we infer that for any $0 < s < 1/2$ and any $T > 0$,

$$u_n \to u \text{ in } L^\infty(0,T;H^s(\mathbb{R})) \, . \tag{2.14}$$

[1] Note that according to the equation, the time derivative of such a function must belong to $L^\infty(-T,T;H^{-2})$ and thus such a function has to belong to $C(-T,T;H^{-2})$.

Now, from interpolation inequalities, (2.2) and (2.4), we infer that for any fixed $t>0$,

$$\begin{aligned}\varepsilon_n \int_0^t \|u_n(s)\|_{L^2_x}^2 \|\partial_x u_n(s)\|_{L^2_x}^2\, ds &\lesssim \varepsilon_n \int_0^t \|u_n(s)\|_{L^2_x}^{8/3} \|D_x^{3/2} u_n(s)\|_{L^2_x}^{4/3}\, ds \\ &\lesssim \|\varphi\|_{L^2_x}^{8/3} \varepsilon_n^{1/3} t^{1/3} \Big[\varepsilon_n \int_0^t \|D_x^{3/2} u_n(s)\|_{L^2_x}^2\, ds\Big]^{2/3} \\ &\lesssim O(\varepsilon_n^{1/3})\ .\end{aligned}$$

We thus deduce from (2.3) that for any $t \geq 0$,

$$E(u_n(t)) \leq E(\varphi) + O(\varepsilon_n^{1/3}) \longrightarrow E(\varphi) = E(u(t))\ .$$

Moreover, according to (2.14),

$$\sup_{t\in[-T,T]} \Big| \int_{\mathbb{R}} u_n^3(t,x)\, dx - \int_{\mathbb{R}} u^3(t,x)\, dx \Big| \to 0 \text{ as } n \to \infty\ .$$

Gathering the two convergence results above, we deduce that for all $t \in \mathbb{R}$,

$$\limsup_{n\to\infty} \int_{\mathbb{R}} |D_x^{1/2} u_n(t)|^2\, dx \leq \int_{\mathbb{R}} |D_x^{1/2} u(t)|^2\, dx\ .$$

In the same way as for (2.12), this last inequality, combined with (2.6) and (2.12), ensures that for any $T>0$,

$$u_n \to u \text{ in } C([0,T]; H^{1/2}(\mathbb{R}))\ .$$

This concludes the proof of the theorem when $K=\mathbb{R}$. $\square$

Finally, it is easy to check that the case $K=\mathbb{T}$ works exactly as well, since according to [10], the Benjamin-Ono equation is unconditionnaly well-posed in $H^{1/2}(\mathbb{T})$. Actually, this case is even simpler since the weak convergence in $H^{1/2}(\mathbb{T})$ directly implies the strong convergence in $H^s(\mathbb{T})$ for $s<1/2$.

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