

A COHOMOLOGICAL INTERPRETATION OF BOGOMOLOV'S INSTABILITY

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ABSTRACT. We give a new proof of Bogomolov's instability theorem. Furthermore, we prove that it is equivalent to a statement which characterizes when the first cohomology group of a suitable divisor does not vanish.

1. INTRODUCTION

In the theory of stable vector bundles on surfaces, the following theorem, known as Bogomolov's instability theorem, plays a central role:

Theorem 1.1 (Bogomolov). *Let X be a smooth projective surface and V be a rank 2 vector bundle on X . If $c_1(V)^2 > 4c_2(V)$, then V is unstable.*

For the original proof we refer to [1]; see also [9]. This theorem was later proved by quite different techniques in [5] and [8]. Furthermore, Reider used Theorem 1.1 to study adjoint linear series on surfaces and to derive his famous theorem, [10].

The first cohomological proof of Reider's theorem was given by Sakai in [11]. His proof uses ideas of Serrano [12] and generalizes Reider's theorem to normal surfaces. The key point in Sakai's proof is the following theorem.

Theorem 1.2 (Sakai). *Let D be a big divisor with $D^2 > 0$ on a smooth projective surface X . If $H^1(X, \mathcal{O}_X(K_X + D)) \neq 0$, then there exists an effective divisor E such that*

- (1) $D - 2E$ is big;
- (2) $(D - E) \cdot E \leq 0$.

As shown in [11], Theorem 1.2 can be easily derived from Theorem 1.1. Moreover, Sakai gave an alternative proof based on Miyaoka's vanishing theorem for the Zariski decomposition of a divisor. Later Ein and Lazarsfeld showed how to apply the Kawamata-Viehweg vanishing theorem to prove a part of Reider's theorem in [2]. Based on these new techniques Fernández del Busto gave an elegant proof of Bogomolov's inequality which uses only the Kawamata-Viehweg theorem; see [3]. For a survey on these results we refer to [6].

On the other hand, Mumford showed that we can use Bogomolov's theorem for rank 2 vector bundles to give a short proof of a generalized Kodaira vanishing for surfaces; see [4]. This vanishing theorem is a little less general than the theorem of Kawamata-Viehweg. These results suggest that there should be a connection between Bogomolov's instability and some vanishing theorem.

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In this note we prove

Theorem 1.3. *Bogomolov’s instability theorem is equivalent to Theorem 1.2.*

Furthermore, using Sakai’s proof of Theorem 1.2, one gets a new proof of Bogomolov’s instability theorem which is entirely cohomological.

We now outline the proof of Theorem 1.3. After twisting the vector bundle V with a line bundle, we can assume that V has a global section. Using this section we have that the extension class of the vector bundle is nontrivial since V is locally free. The first step of our proof follows Fernández del Busto’s argument [3]. At this point we follow a different strategy. The numerical condition of Bogomolov’s inequality allows us to apply Theorem 1.2, and we show directly that the divisor E gives the destabilizing subsheaf.

2. PRELIMINARIES

For the convenience of the reader we sketch the proof of Theorem 1.2.

Proof. Let $D = P + N$ be the Zariski decomposition of D and write $N = \sum \alpha_j E_j$ with each α_j positive and rational. By Sakai’s lemma, Example 9.4.12 in [7], we know that $H^1(X, \mathcal{O}_X(K_X + D - \lfloor N \rfloor)) = 0$ so $\lfloor N \rfloor > 0$. Consider the following sequence of divisors:

$$D_0 = D - \lfloor N \rfloor, \dots, D_k = D_{k-1} + E_{j_k}, \dots, D_n = D.$$

If $D_k \cdot E_{j_k} > 0$ for any k , we get the vanishing of $H^1(X, \mathcal{O}_X(K_X + D))$. Thus we can collect all the E_{j_k} ’s with positive intersection to construct a sequence D_0, \dots, D_k such that $(D - D_k) \cdot E_j \leq 0$ for all irreducible components E_j of $D - D_k$. Now a computation shows that $E := D - D_k$ is the required divisor. \square

Corollary 2.1. *Let D and E be as above. Then*

$$H^1(X, \mathcal{O}_X(K_X + D - E)) = 0.$$

Proof. By the above construction,

$$H^1(X, K_X + D_0) = H^1(X, K_X + D_k).$$

Since $D_0 = D - \lfloor N \rfloor$ and $D_k = D - E$, the result follows from Sakai’s lemma. \square

In conclusion we recall two results which will be used in the proof of the main theorem.

Lemma 2.2. *Let $f : Y \rightarrow X$ be a birational morphism between smooth projective surfaces and \tilde{L} a divisor on Y . Set $L := f_*\tilde{L}$; if $\tilde{L}^2 > 0$ and L is big, then \tilde{L} is big.*

Proof. Lemma 3 in [11]. \square

Proposition 2.3. *Let $f : \tilde{X} \rightarrow X$ be a birational morphism between smooth projective surfaces. Let \tilde{D} be a divisor on \tilde{X} such that $\tilde{D}^2 > 0$. Suppose there is a divisor \tilde{E} which satisfies the conclusions of Theorem 1.2 and let $D := f_*\tilde{D}$, $E := f_*\tilde{E}$ and $\alpha := D^2 - \tilde{D}^2$. If D is nef and E is effective, we have*

$$0 \leq D \cdot E < \alpha/2.$$

Proof. See Proposition 2 in [11]. \square

3. MAIN THEOREM

We can now prove the main result of the paper.

Proof of Theorem 1.3. As mentioned before, Theorem 1.2 can be easily proved using Bogomolov's instability; see [11], p. 307.

We now want to show that Theorem 1.2 implies Bogomolov's theorem. Since the inequality in Theorem 1.1 is invariant under twisting with a line bundle, we can assume that V is globally generated, $\det(V)$ is ample and $c_2(V) > 0$. Taking a general section s of V , we get the following exact sequence:

$$0 \rightarrow \mathcal{O}_X \rightarrow V \rightarrow L \otimes I_Z \rightarrow 0,$$

where $L := \det(V)$ and Z is the zero locus of s . Then we have $c_2(V) = |Z|$, the length of Z .

Since V is locally free, the above extension is nontrivial and then

$$H^1(X, \mathcal{O}_X(K_X + L) \otimes I_Z) \neq 0.$$

Let $\pi : Y \rightarrow X$ be the blow up of X at all points in Z . Let E_j be the exceptional curve over $x_j \in Z$. Then

$$H^1(Y, \mathcal{O}_Y(K_Y + \pi^*L - 2 \sum_j E_j)) = H^1(X, \mathcal{O}_X(K_X + L) \otimes I_Z) \neq 0.$$

Define $\tilde{L} := \pi^*L - 2 \sum_j E_j$. Thus, we have

$$\tilde{L}^2 = (\pi^*L)^2 + 4 \sum_j E_j^2 = c_1^2(V) - 4c_2(V) > 0,$$

so \tilde{L} is big by Lemma 2.2.

By applying Theorem 1.2 we get an effective divisor \tilde{E}_s such that

- (1) $\tilde{L} - 2\tilde{E}_s$ is big;
- (2) $(\tilde{L} - \tilde{E}_s) \cdot \tilde{E}_s \leq 0$.

Note that \tilde{E}_s depends on the section s that we chose at the beginning. Let $E_s := \pi_*\tilde{E}_s$. We want to show that, for any s , E_s passes through at least one point of Z . Let $\tilde{E}_s := \pi^*E_s + \sum a_i E_i$, where E_i are the exceptional divisors. It suffices to show that there exists an index i such that $a_i < 0$. Write $\tilde{L} - \tilde{E}_s = \pi^*W_s - \sum (a_i + 2)E_i$, where $W_s := L - E_s$. Thus by (2) we have

$$E_s \cdot W_s + \sum_i a_i(a_i + 2) \leq 0.$$

Then if we show that $E_s \cdot W_s > 0$, we must have a negative a_i , and then $x_i \in \text{Supp}(E_s)$. By construction $L = E_s + W_s$, $L \cdot E_s > 0$ and

$$L \cdot W_s = (L - 2E_s) \cdot L + L \cdot E_s = (\tilde{L} - 2\tilde{E}_s) \cdot \pi^*L + L \cdot E_s > 0$$

by (1). From the Hodge index theorem we get $E_s \cdot W_s > 0$.

Now we need a result in [3], called the uniform multiplicity property. See also [6].

Lemma 3.1. *Choosing s and E_s generally we can assume that the multiplicity of E_s at every point of Z is the same.*

Since for any s there exists $x \in Z$ such that $x \in \text{Supp}(E_s)$, by the uniform multiplicity property, we can choose s and E_s generally such that $Z \subset \text{Supp}(E_s)$. For simplicity, we denote this divisor by E .

$Z \subset \text{Supp}(E)$ implies that the multiplication by E defines a map $\mathcal{O}_X(L - E) \rightarrow \mathcal{O}_X(L) \otimes I_Z$. Since the cohomology group in Corollary 2.1 vanishes, this map lifts to an injective map $\mathcal{O}_X(L - E) \rightarrow V$. Thus, $\mathcal{O}_X(L - E)$ is a subsheaf of V .

It remains to prove that V is unstable. This is equivalent to showing that

$$(L - 2E)^2 > 0, \quad (L - 2E) \cdot L > 0.$$

For the first inequality we consider the following exact sequence:

$$0 \rightarrow \mathcal{O}_X(L - E) \rightarrow V \rightarrow \mathcal{O}_X(E) \otimes I_{Z'} \rightarrow 0,$$

for some zero dimensional scheme Z' . Then $c_1(V) = L$ and $c_2(V) = (L - E) \cdot E + |Z'|$, and by hypothesis we get

$$(L - 2E)^2 > 4|Z'| > 0.$$

For the second one we note that

$$\alpha = c_1^2(V) - c_1^2(V) + 4c_2(V) = 4c_2(V),$$

and Proposition 2.3 gives the following:

$$L \cdot E < 2c_2(V).$$

Then

$$L^2 > 4c_2(V) > 2L \cdot E. \quad \square$$

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