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FINITE DIMENSIONAL MODULES FOR MULTILOOP SUPERALGEBRAS OF TYPES $A(m,n)$ AND $C(m)$

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Abstract. In this article we construct a new class of finite dimensional irreducible modules generalizing evaluation modules for multiloop superalgebras of types $A(m,n)$ and $C(m)$. We prove that these are the only such modules. In all other cases it is known that such modules are evaluation modules.

Introduction

In recent years there has been some interest in understanding finite dimensional modules for the Lie algebra $\mathfrak{g}\otimes A$ where $\mathfrak{g}$ is a simple finite dimensional Lie algebra and A is a commutative associative algebra with unit over complex numbers $\mathbb{C}$. See [CFK], [RK], [S], [C] and the references therein. Of particular interest is the classification of irreducible finite dimensional modules for $\mathfrak{g}\otimes A$. The classification of these modules can be found implictly in [CP] in the case $A=\mathbb{C}[t,t^{-1}]$. They turn out to be evaluation modules at finitely many points. A direct and simple proof is given in [E3]. A similar classification is given for the case $A=\mathbb{C}[t_1^{\pm1},\cdots,t_d^{\pm1}]$, the Laurent polynomial algebra in d commuting variables, in [E1] and [E2]. The most general case is done in [CFK] with the only assumption being that A is finitely generated.

In this article we consider $\mathfrak{g}$ to be a basic classical Lie superalgebra and $A=\mathbb{C}[t_1^{\pm1},\cdots,t_d^{\pm1}]$. The Lie superalgebra $\mathfrak{g}\otimes A$ is called a multiloop superalgebra. The main purpose of this article is to classify finite dimensional irreducible modules for multiloop superalgebras not covered in [EZ]. The classification of finite dimensional irreducible modules for multiloop superalgebra is obtained in [EZ] except for the types $A(m,n)$ and $C(m)$. They are all evaluation modules. Here we need to note that in the type $A(n,n)$, we have two Lie superalgebras. One is the basic classical Lie superalgebra of type $A(n,n)$ which is centerless, and the other is $sl(n,n)$ which is a one dimensional central extension of the earlier algebra. When we say a multiloop superalgebra, we include $sl(n,n)$.

As remarked in [EZ], the proof given there works for the basic classical Lie superalgebra of type $A(n,n)$. Thus we are left with two cases where $\mathfrak{g}=sl(m,n)$ or of type $C(m)$. We note that in these two cases the even part of $\mathfrak{g}$ is not semisimple but reductive with a one dimensional center. In all other cases the even part is semisimple. This is the reason why the proof given in [EZ] does not work for the two cases we consider in this article.

The article is organised as follows.

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In Section 1 we define basic classical Lie superalgebras and recall their classification due to Kac [K1]. We also define multiloop superalgebras. In Section 2 we construct evaluation modules for any multiloop superalgebra. It is proved in [EZ] that any finite dimensional irreducible module for a multiloop superalgebra is an evaluation module execept in the two special cases given above. In Section 3 we describe the Lie superalgebras $sl(m,n)$ and $C(m)$ in more detail and describe their root systems. In Section 4 we construct a new class of irreducible finite dimensional modules for the two cases above which are not evaluation modules.

This construcction, which is more general than evaluation modules, uses the special property that the even part has a one dimensional center.

In the last section we prove that any finite dimensional irreducible module for the multiloop superalgebra of types $sl(m,n)$ and $C(m)$ is one of the modules constructed in Section 4.

1. Multiloop algebras

All our vector spaces, algebras and tensor products are over the field of complex numbers $\mathbb{C}$. $\mathbb{Z}_2$ denotes the cyclic group of two elements $\{\overline{0},\overline{1}\}$.

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ equipped with $\mathbb{C}-$bilinear form $[,] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, called the Lie super bracket, satisfying the following conditions:

(1) $[\mathfrak{g}_{\overline{i}}, \mathfrak{g}_{\overline{j}}] \subseteq \mathfrak{g}_{\overline{i+j}}$
(2) $[X,Y] = -(-1)^{ij}\ [Y,X]$
(3) $\big[[X,Y],Z\big] = \big[X,[Y,Z]\big] - (-1)^{ij}\ \big[Y,[X,Z]\big]$

for all homogeneous elements $X \in \mathfrak{g}_{\overline{i}}, Y \in \mathfrak{g}_{\overline{j}}$ and $Z \in \mathfrak{g}_{\overline{k}}$.

The subspace $\mathfrak{g}_{\overline{0}}$ is called even and the subspace $\mathfrak{g}_{\overline{1}}$ is called odd. It is easy to see that $\mathfrak{g}_{\overline{0}}$ is the usual Lie algebra and $\mathfrak{g}_{\overline{1}}$ is a $\mathfrak{g}_{\overline{0}}$-module. Suppose X is a homogeneous element belonging to $\mathfrak{g}_{\overline{i}}$; then we denote $|X| = i$.

A Lie superalgebra is called basic classical if the algebra is simple, finite dimensional, and the even part is reductive and carries an even, non-degenerate supersymmetric invariant bilinear form. They have been classified by Kac [K1]. The following is the list of basic classical Lie superalgebras which are not Lie algebras and the decomposition of even part:

$A(m,n)$	$A_m + A_n + \mathbb{C}$	$m \geq 0, n \geq 0, m+n \geq 1, m \neq n$
$A(n,n)$	$A_n + A_n$	$n \geq 1$
$B(m,n)$	$B_m + C_n$	$m \geq 0, n \geq 1$
$C(n)$	$C_{n-1} + \mathbb{C}$	$n \geq 2$
$D(m,n)$	$D_m + C_n$	$m \geq 2, n \geq 1$
$D(2,1,a)$	$D_2 + A_1$	$a \neq 0, -1$
$F(4)$	$B_3 + A_1$	
$G(3)$	$G_2 + A_1$	

We will now define multiloop superalgebras. We fix a positive integer d and let $A = \mathbb{C}[t_1^{\pm 1}, \cdots, t_d^{\pm 1}]$ be a Laurent polynomial ring in d commuting variables. Let $\mathfrak{g}$ be a basic classical Lie superalgebra. In the case $\mathfrak{g}$ is of type $A(n,n), n \geq 1$, we will allow $\mathfrak{g}$ to be $sl(n+1,n+1), n \geq 1$. Note that $sl(n+1,n+1)$ has a one dimensional center and modulo the center it is basic classical. $\mathfrak{g} \otimes A$ has a natural Lie superalgebra structure.

(1.1) **Definition.** $\mathfrak{g} \otimes A$ is called a multiloop superalgebra.

Note that the universal central extensions of $\mathfrak{g} \otimes A$ for $d = 1$ are called affine superalgebras. They are widely studied. See [IK] and the references therein.

(1.2) The main purpose of this paper is to classify irreducible finite dimensional modules for multiloop superalgebras $\mathfrak{g} \otimes A_d$ in the following cases:

(1) $\mathfrak{g} = sl(m+1, n+1), m, n \geq 0, m+n \geq 1$
(2) $\mathfrak{g}$ is of type $C(m), m \geq 2$

We will describe these two algebras in more detail in Section 3.

In all other cases the classification has been obtained in [EZ] (Propositions 5.2 and 5.4). The case where $\mathfrak{g}$ is a basic classical Lie superalgebra of type $A(n, n)$ is similar, as noted in Remark 6.7(3) and Remark (7.5) of [EZ].

2. Evaluation modules

In this section we will define the all important evaluation modules for multiloop superalgebras. Before that we need to recall the notion of highest weight modules for $\mathfrak{g}$ as well as $\mathfrak{g} \otimes A$.

We fix a basic classical Lie superalgebra including the Lie superalgebra $sl(n+1, n+1)$. The following is well known. See [K2] for details.

Let $\mathfrak{g} = N^- \oplus \mathfrak{h} \oplus N^+$ be the standard decomposition where $\mathfrak{h}$ is a Cartan subalgebra which is actully a Lie algebra. For each $\lambda \in \mathfrak{h}^*$ there exists an irreducible $\mathfrak{g}$ module denoted by $V(\lambda)$ which is called a highest weight module with respect to the above decomposition.

Let $\mathfrak{g} \otimes A = N^- \otimes A \oplus \mathfrak{h} \otimes A \oplus N^+ \otimes A$ be the corresponding decomposition for $\mathfrak{g} \otimes A$.

(2.1) A module V of $\mathfrak{g} \otimes A$ is called a highest weight module if there exists a weight vector v (with respect to $\mathfrak{h}$) in V such that:

(1) $U(\mathfrak{g} \otimes A)v = V$,
(2) $N^+ \otimes Av = 0$ and
(3) $U(\mathfrak{h} \otimes A)v = \mathbb{C}v$.

Here U denotes the universal enveloping algebra. By standard arguments we can show that for each $\psi \in (\mathfrak{h} \otimes A)^*$ there exists a unique irreducible highest weight module denoted by $W(\psi)$.

Now we recall the evaluation modules. For each $i, 1 \leq i \leq d$, let N_i be a positive integer. Let $\underline{a}_i = (a_{i1}, a_{i2}, \cdots, a_{iN_i})$ be non-zero distinct complex numbers.

Let $N = N_1 N_2 \cdots N_d$. Let $I = (i_1, \cdots, i_d)$ where $1 \leq i_j \leq N_j$.

There are N of them. Let $I_1, \cdots, I_N$ be some order.

Let $\underline{m} = (m_1, \cdots, m_d) \in \mathbb{Z}^d$ and define $a_I^{\underline{m}} = a_{1i_1}^{m_1} \cdots a_{di_d}^{m_d}$.

Let $t^{\underline{m}} = t_1^{m_1} \cdots t_d^{m_d} \in A$.

Let G be any Lie superalgebra and let φ be an algebra homomorphism defined by

$$\begin{aligned} \varphi : G \otimes A \to \oplus G &= G_N (N \text{ copies}) \\ \varphi(X \otimes t^{\underline{m}}) &= (a_{I_1}^{\underline{m}} X, \cdots, a_{I_N}^{\underline{m}} X) \end{aligned}$$

where $X \in G$ and $\underline{m} \in \mathbb{Z}^d$.

Let $P_j(t_j) = \prod_{k=1}^{N_j} (t_j - a_{jk})$ and note that P_j is a polynomial in t_j with non-zero distinct roots.

Let I be the co-finite ideal of A generated by $P_1(t_1), \cdots, P_d(t_d)$.

Lemma 2.2. (1) *φ is surjective.*
(2) *Ker $\varphi = G \otimes I$ and $G \otimes A/I \cong G_N$.*

Proof. The proof is similar to the Lie algebra case and can be found in Lemma 3.11 of [E2]. A more transparent proof can be found in Proposition 2.2 of [B].

Let $V(\lambda_1), \cdots, V(\lambda_N)$ be irreducible highest weight modules for $\mathfrak{g}$. Then $V(\underline{\lambda}, \underline{a}) = \bigotimes_{i=1}^{N} V(\lambda_i)$ is an irreducible highest weight module for the multiloop algebra $\mathfrak{g} \otimes A$ via the map φ taking $G = \mathfrak{g}$.

Let $\psi : \mathfrak{h} \otimes A \to \mathbb{C}$ be defined by

$$\psi(h \otimes t^{\underline{m}}) = \sum_{j=1}^{N} a_{I_j}^{\underline{m}} \lambda_j(h), h \in \mathfrak{h}.$$

Then it is easy to see that $W(\psi) \cong V(\underline{\lambda}, \underline{a})$.

(2.3) **Definition.** The modules $V(\underline{\lambda}, \underline{a})$ are called evaluation modules.

Note that $V(\underline{\lambda}, \underline{a})$ is finite dimensional if and only if each $V(\lambda_i)$ is finite dimensional. The condition for $V(\lambda_i)$ to be finite dimensional is given in [K2].

(2.4) **Proposition.** *Suppose $\mathfrak{g}$ is not of type $A(m,n)$ or $C(m)$. Then any finite dimensional irreducible module is an evaluation module.*

Proof. See Propositions 5.2 and 5.4 of [EZ].

In the case $\mathfrak{g}$ is of type $A(m,n)$ or $C(m)$, there exist irreducible finite dimensional modules which are not evaluation modules, which we will see in Section 4. Here we leave out the case where $\mathfrak{g}$ is a basic classical Lie superalgebra of type $A(n+1, n+1)$. In this case the classification is similar to the one given in Proposition 2.4.

3. Lie superalgebra $sl(m+1, n+1)$ and $C(m)$

In this section we assume $\mathfrak{g}$ is $sl(m+1, n+1), (m \geq 0, n \geq 0, m+n \geq 1)$, or $\mathfrak{g}$ is of type $C(m), m \geq 2$. We will now describe these two cases of Lie superalgebras and their root systems. We follow Kac [K2].

Case 1. $\mathfrak{g} = sl(m,n)$. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space and let $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$. The $\mathbb{Z}_2$-gradation on V naturally induces a $\mathbb{Z}_2$-gradation on $End(V) = (End(V))_{\bar{0}} \oplus (End(V))_{\bar{1}}$ by letting $End(V)_j = \{f \in End(V) : f(V_k) \subseteq V_{k+j}$ *for all* $k \in \mathbb{Z}_2\}$.

Then $End(V)$ becomes a Lie superalgebra by defining the Lie super bracket $[f, g] = f \circ g - (-1)^{ij} g \circ f$ for all $f \in (End(V))_i$ and $g \in (End(V))_j$.

We let $gl(m,n) = End(V)$. By fixing a basis for $V_{\bar{0}}$ and $V_{\bar{1}}$ we can write $f \in End(V)$ as

$$f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is an $m \times m$ matrix, B is an $n \times m$ matrix, C is an $m \times n$ matrix and D is an $n \times n$ matrix. It is easy to see that

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \big(End(V)\big)_{\bar{0}}$$

and

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in \big(End(V)\big)_{\bar{1}}.$$

We now define the supertrace of f denoted by strf = trace A-trace D. Define $sl(m,n) = \{X \in gl(m,n) | strX = 0\}$. It is known that $sl(m,n)$ is a simple Lie superalgebra if $m \neq n$. When $m = n, sl(n,n)$ has a one dimensional center if $n \geq 2$. Define

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix};\ A \text{ is an } m \times m \text{ matrix and } D \text{ is an } n \times n \text{ matrix} \right\}$$

$$\mathfrak{g}_{+1} = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix};\ B \text{ is an } n \times m \text{ matrix} \right\}$$

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix};\ C \text{ is an } m \times n \text{ matrix} \right\}$$

Then $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0, \mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{+1}$. It is easy to check that

$$[\mathfrak{g}_0, \mathfrak{g}_{-1}] \subseteq \mathfrak{g}_{-1}, [\mathfrak{g}_0, \mathfrak{g}_{+1}] \subseteq \mathfrak{g}_{+1}, [\mathfrak{g}_{-1}, \mathfrak{g}_{+1}] \subseteq \mathfrak{g}_0, [\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] = 0. \tag{3.1}$$

Thus $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ is a $\mathbb{Z}$-grading with $\mathfrak{g}_n = 0$ for $|n| > 1$. We can further check that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \nsubseteq [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$. This is the reason why the proof in [EZ] does not work for types $A(m,n)$ and $C(m)$. In all other cases (except in the cases 1.2(1) and 1.2(2)) $\mathfrak{g}_{\bar{0}}$ is semisimple and hence $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq \mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ which is contained in the even part $\mathfrak{g}_{\bar{0}}$. We can take $\mathfrak{h}$ to be the diagonal matrices in $\mathfrak{g}_{\bar{0}}$. Let ϵ_i be the ith projection from $\mathfrak{h}$ to $\mathbb{C}$. Let $\delta_i = \epsilon_{m+i}$ for $1 \leq i \leq n$. Then the root system of $sl(m,n)$ can be described in the following way:

$$\begin{aligned} \Delta_0 &= \{\epsilon_i - \epsilon_j, \delta_i - \delta_j, i \neq j\} \\ \Delta_1 &= \{\pm(\epsilon_i - \delta_j)\} \end{aligned}$$

Here Δ_0 denotes the even roots and Δ_1 denotes the odd roots.

Let $\Delta_1^+ = \{\epsilon_i - \delta_j\}$ and $\Delta_1^- = \{-(\epsilon_i - \delta_j)\}$. Then it is easy to see that the roots of the space $\mathfrak{g}_{+1}$ are Δ_1^+; similarly for $\mathfrak{g}_{-1}$ are Δ_1^-. From the above description of the root system one can easily see that (3.1) holds.

Case 2. $\mathfrak{g} = C(m), m \geq 2$. We will now describe the Lie superalgebra $C(m)$ which is known to be a subalgebra of $sl(2, 2m-2)$. We denote A^T for the transpose of the matrix A.

Let

$$\mathfrak{g}_0 = \left\{ \left(\begin{array}{cc|cc} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ \hline 0 & 0 & A & B \\ 0 & 0 & C & -A^T \end{array} \right) : \begin{array}{l} \alpha \in \mathbb{C}, A, B, C \quad \text{are} \quad (m-1) \times (m-1) \\ \text{matrices}, B^T = B, C^T = C \end{array} \right\},$$

$$\mathfrak{g}_{+1} = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & A & B \\ 0 & 0 & 0 & 0 \\ \hline 0 & -B^T & 0 & 0 \\ 0 & \mathrm{A} & 0 & 0 \end{array} \right); A \text{ and } B \text{ are } (m-1) \times 1 \text{ matrices} \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & A & B \\ \hline -B^T & 0 & 0 & 0 \\ A & 0 & 0 & 0 \end{array} \right) ; A, B \text{ are } (m-1) \times 1 \text{ matrices} \right\}.$$

Then it is easy to check that

$$[\mathfrak{g}_0, \mathfrak{g}_{+1}] \subseteq \mathfrak{g}_{+1}, [\mathfrak{g}_0, \mathfrak{g}_{-1}] \subseteq \mathfrak{g}_{-1}, [\mathfrak{g}_{-1}, \mathfrak{g}_{+1}] \subseteq \mathfrak{g}_0, [\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] = 0. \tag{3.2}$$

From this we can see that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}$-gradation with $\mathfrak{g}_n = 0$ for $|n| > 1$. Further we have $\mathfrak{g}_0 = \mathfrak{g}_{\overline{0}}, \mathfrak{g}_{\overline{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{+1}$. We can also check that

$$[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] \nsubseteq [\mathfrak{g}_{\overline{0}}, \mathfrak{g}_{\overline{0}}].$$

We will now describe the root system of $C(m)$. As in the earlier case we can take the Cartan subalgebra $\mathfrak{h}$ to be diagonal matrices in $\mathfrak{g}_{\overline{0}}$. Let ϵ_i be the ith projection from $\mathfrak{h}$ to $\mathbb{C}$.

Let $\delta_1 = \epsilon_3, \cdots, \delta_{m-1} = \epsilon_{m+1}$. Let $\Delta_0, \Delta_1, \Delta_1^+, \Delta_1^-$ be even, odd, positive odd and negative odd roots respectively. Then

$$\begin{array}{lcl} \Delta_0 & = & \{\pm 2\delta_i, \pm\delta_i \pm \delta_j, i \neq j\} \\ \Delta_1 & = & \{\pm\epsilon_1 \pm \delta_i\} \\ \Delta_1^+ & = & \{\epsilon_1 \pm \delta_i\} \\ \Delta_1^- & = & \{-\epsilon_1 \pm \delta_i\} \end{array}$$

It is easy to verify that Δ_1^+ is the set of roots for $\mathfrak{g}_{+1}$ and Δ_1^- is the set of roots for $\mathfrak{g}_{-1}$. One can also verify (3.2) using the above description.

4. A new class of finite dimensional modules for multiloop superalgebras of types $A(m,n)$ and $C(m)$

Let $\mathfrak{g} = sl(m+1, n+1), m \geq 0, n \geq 0, m+n \geq 1$, or $\mathfrak{g} = C(m), m \geq 2$. Recall that $\mathfrak{g}_{\overline{0}}$ is reductive but not semisimple. Let $\mathfrak{g}_{ss}$ be the semisimple part of $\mathfrak{g}_{\overline{0}}$ and $\mathbb{C}z$ be the one dimensional center of $\mathfrak{g}_{\overline{0}}$, so that we have

$$\mathfrak{g}_{\overline{0}} = \mathfrak{g}_{ss} \oplus \mathbb{C}z.$$

Let V be a finite dimensional evaluation module for $\mathfrak{g}_{ss} \otimes A$. Thus there exists a co-finite ideal $I^{'}$ of A generated by polynomial $P_j^{'}(t_j) = \prod_{k=1}^{N_j}(t_j - a_{jk})$, where a_{jk} and N_j are given in Section 3 such that $\mathfrak{g}_{ss} \otimes I^{'}.V = 0$

Further, $V \cong W(\psi)$, where $\psi \in (\mathfrak{h} \otimes A)^*$ and $\psi(h \otimes t^{\underline{m}}) = \sum a_{I_j}^m \lambda_j(h)$. (See Section 3 for details.) We will allow ψ to be zero and note that $W(0)$ is a trivial one dimensional module.

Let b_{jk} be some positive integers for $1 \leq j \leq d$ and $1 \leq k \leq N_j$. Let $P_j(t_j) = \prod_{k=1}^{N_j}(t_j - a_{jk})^{b_{jk}}$. Let I be the ideal generated by $P_j(t_j), j = 1, 2, \cdots, d$. Clearly $I \subseteq I^{'}$. Let $\lambda \in (\mathbb{C}z \otimes A)^*$ be such that $\lambda(z \otimes I) = 0$. Now consider $W(\psi)$ as a $D_0 := \mathfrak{g}_{\overline{0}} \otimes A/I = \mathfrak{g}_{ss} \otimes A/I \oplus \mathbb{C}z \otimes A/I$-module. Let $D_+ = \mathfrak{g}_{+1} \otimes A/I (D_- = \mathfrak{g}_{-1} \otimes A/I)$ act on $W(\psi)$ trivially. Since $[D_0, D_+] \subseteq D_+$ and $[D_+, D_+] = 0$, we see

that $W(\psi)$ is a well defined module for $D_0 \oplus D_+$. Now consider the induced module for $\mathfrak{g} \otimes A/I$,

$$M(\psi, \lambda) = U(\mathfrak{g} \otimes A/I) \bigotimes_{D_0 \oplus\ D_+} W(\psi),$$

and as a vector space $\cong U(D_-) \otimes W(\psi)$. Since D_- is finite dimensional and odd, by the PBW theorem for Lie superalgebras, we conclude that $M(\psi, \lambda)$ is finite dimensional. By standard argument we see that $M(\psi, \lambda)$ has a unique irreducible quotient, say $V(\psi, \lambda)$. By the surjective map $\mathfrak{g} \otimes A \to \mathfrak{g} \otimes A/I$ we see that $V(\psi, \lambda)$ is an irreducible finite dimensional module for $\mathfrak{g} \otimes A$. It is not hard to see that $V(\psi, \lambda)$ is not an evaluation module if $\lambda(z \otimes I^{'}) \neq 0$. In the case when $\lambda(z \otimes I^{'}) = 0$, $V(\psi, \lambda)$ is an evaluation module. This follows from Lemma 5.1.

5. Classification theorem

In this section we recall some generalities on highest weight modules for multiloop superalgebras. We will also state and prove our main result on the classification.

Let $\mathfrak{g}$ be any finite dimensional Lie superalgebra as mentioned in Section 1. Consider the standard decomposition

$$\mathfrak{g} \otimes A = N^- \otimes A \oplus \mathfrak{h} \otimes A \oplus N^+ \otimes A.$$

Note that $\mathfrak{h} \otimes A$ is a Lie algebra (abelian). Let $\psi \in (\mathfrak{h} \otimes A)^*$ and $\mathbb{C}_\psi$ be a one dimensional representation of $\mathfrak{h} \otimes A$ via ψ. Let $N^+ \otimes A$ act trivially on $\mathbb{C}_\psi$. Consider the Verma module

$$M(\psi) = U(\mathfrak{g} \otimes A) \bigotimes_{N^+ \otimes A + \mathfrak{h} \otimes A} \mathbb{C}_\psi.$$

By standard arguments we see that $M(\psi)$ has a unique irreducible quotient, say $V(\psi)$, which need not have finite dimensional weight spaces (with respect to $\mathfrak{h}$).

Lemma 5.1. *$V(\psi)$ has finite dimensional weight spaces if and only if ψ factors through $\mathfrak{h} \otimes A/I$ for some co-finite ideal I of A. In this case $\mathfrak{g} \otimes I.V(\psi) = 0$.*

Proof. Follows from Lemma 4.7 and Remark 4.8 of [EZ]. See Lemma 3.7 of [E2] for the original proof. Note that the $\mathfrak{g}'$ that occurs in [E2] is $[\mathfrak{g}, \mathfrak{g}]$ and equals $\mathfrak{g}$ in our case.

Remark 5.2. From the proof of Lemma 3.7 of [E2] it follows that the co-finite I can be chosen to to be generated by polynomials $P_1, \cdots, P_d$ in variables $t_1, \cdots, t_d$ respectively. We can also assume that they have non-zero roots.

From now on we assume $\mathfrak{g} = sl(m+1, n+1)$ or $C(m)$.

Theorem 5.3. *Any finite dimensional irreducible module for $\mathfrak{g} \otimes A$ is isomorphic to $V(\psi, \lambda)$ as defined in Section 4.*

Proof. Let V be a finite dimensional irreducible module for $\mathfrak{g} \otimes A$. Since V is finite dimensional, $V \cong V(\phi)$ for some $\phi \in (\mathfrak{h} \otimes A)^*$.

Let v be the highest weight vector. Recall that $\mathfrak{g}_{ss}$ is the semisimple part of $\mathfrak{g}_{\overline{0}}$. Let $M = U(\mathfrak{g}_{ss} \otimes A)v$, which is a finite dimensional $\mathfrak{g}_{ss} \otimes A$ module. In fact it is a $\mathfrak{g}_{\overline{0}} \otimes A$ module, as the additional center $z \otimes A$ act as scalars.

Claim. M is irreducible as a $\mathfrak{g}_{ss} \otimes A$ module.

To see the claim, recall that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{+1}$ and $\mathfrak{g}_{+1}$ is the sum of positive odd root spaces. Let $\mathfrak{g}_{ss} = \mathfrak{g}_{ss}^{-} \oplus \mathfrak{h}_{ss} \oplus \mathfrak{g}_{ss}^{+}$ be the standard triangular decomposition. Since v is a highest weight vector we have

$$M = U(\mathfrak{g}_{ss}^{-} \otimes A)v.$$

Let $w \in M$ be a weight vector of weight μ_1 and let $\phi|\mathfrak{h} = \mu$. Then $w = Xv$ for $X \in U(\mathfrak{g}_{ss}^{-} \otimes A)$.

Without loss of generality, by assuming X is a monomial and of weight $-\beta$ where β is a non-negative linear combination of simple roots coming from $\mathfrak{g}_{ss}$, we have $\mu_1 = \mu - \beta$.

Since $V(\phi)$ is $\mathfrak{g} \otimes A$-irreducible there exists $Y \in U\big((\mathfrak{g}_{+1} \oplus \mathfrak{g}_{ss}^{+}) \otimes A\big)$ such that $Yw = v$. We can assume Y is monomial and of weight β. By looking at the root systems, this is not possible unless $Y \in U(\mathfrak{g}_{ss}^{+} \otimes A)$. This proves that M is $\mathfrak{g}_{ss} \otimes A$ irreducible and hence the claim.

Now by Lemma 5.1 and Remark 5.2 there exists a co-finite ideal I of A such that

$$\mathfrak{g} \otimes I.V(\phi) = 0.$$

We further can choose polynomials $P_1, \cdots, P_d$ in variables $t_1, \cdots, t_d$ such that I is generated by $P_1, \cdots, P_d$.

Let $P_j(t_j) = \prod_{j=1}^{N_j} (t_j - a_{jk})^{b_{jk}}$, where for each $j, a_{j1}, \cdots, a_{jN_j}$ are distinct non-zero complex numbers, and for b_{jk}, N_j are some positive integers.

Let $P_j^{'}(t_j) = \prod_{j=1}^{N_j} (t_j - a_{jk})$ and let $I^{'}$ be the ideal generated by $P_1^{'}, \cdots, P_d^{'}$; clearly $I \subseteq I^{'}$. Now from the classification of finite dimensional irreducible modules for $\mathfrak{g}_{ss} \otimes A$ (see Remark 5.5 of [E2]; there the proof is given for simple Lie algebras, but the same proof works for any finite dimensional semisimple Lie algebra) it follows that

$$(\mathfrak{g}_{ss} \otimes I^{'})M = 0.$$

The space $\mathbb{C}z \otimes A$, which is central in $\mathfrak{g}_{\overline{0}} \otimes A$, acts on M as scalars via the map ϕ and factors through $\mathbb{C}z \otimes A/I$. Further, $\mathfrak{g}_{+1} \otimes A$ acts trivially on M as $[\mathfrak{g}_{+1} \otimes A, \mathfrak{g}_{ss} \otimes A] \subseteq \mathfrak{g}_{+1} \otimes A$.

Thus M satisfies all the properties of $W(\psi)$ in Section 4. Thus by the universal property of induced modules, $V(\phi)$ is the quotient of $U(\mathfrak{g} \otimes A) \bigotimes_{D_\circ \oplus D_1} M$. Hence $V(\phi) \cong V(\psi, \lambda)$ for some ψ and λ as in Section 4.

This completes the proof of the theorem.

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