EMBEDDING UNIVERSAL COVERS OF GRAPH MANIFOLDS
IN PRODUCTS OF TREES

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Abstract. We prove that the universal cover of any graph manifold quasi-isometrically embeds into a product of three trees. In particular, we show that the Assouad-Nagata dimension of the universal cover of any closed graph manifold is 3, proving a conjecture of Smirnov.

A graph manifold is a compact connected 3-manifold (possibly with boundary) which admits a decomposition into Seifert fibred pieces when cut along a collection of embedded tori and/or Klein bottles. In particular, a graph manifold is a 3-manifold whose geometric decomposition admits no hyperbolic part. For this reason the class of graph manifold groups is rigid within the class of 3-manifold groups \cite{KL98}; moreover, such groups are classified up to quasi-isometry \cite{BN08}.

More details on graph manifolds and proofs of the above results can be found in \cite{BDM09}, \cite{Ger94} and \cite{KL98}.

We show the following:

Theorem 1. The universal cover of any graph manifold quasi-isometrically embeds in the product of three metric trees.

One may wish to compare this theorem with the result by Buyalo and Schroeder \cite{BS05} that $\mathbb{H}^3$ can be quasi-isometrically embedded in the product of three infinite valence simplicial trees. (This was refined to three infinite binary trees by \cite{BDS07}.)

As an application, we determine the Assouad-Nagata dimension ($\dim_{AN}$) - as defined by Assouad, \cite{Ass82} - of the universal cover of closed graph manifolds. We denote the asymptotic Assouad-Nagata dimension by $\asdim_{AN}$. Recall that the Assouad-Nagata dimension bounds from above the asymptotic dimension, first introduced by Gromov in \cite{Gro93}. However, asymptotic dimension and asymptotic Assouad-Nagata dimension can differ radically; see for instance the examples in \cite{BDL06}. The asymptotic Assouad-Nagata dimension of a group also bounds from above the dimension of its asymptotic cones \cite{DH08}, and if a group has finite Assouad-Nagata dimension, then it has compression exponent 1 \cite{Gal08}.

The asymptotic dimension of universal covers of closed graph-manifolds is known to be 3, as mentioned in \cite{Smi10}, in view of results in \cite{BD08}. Also, Smirnov \cite{Smi10} showed that their Assouad-Nagata dimension is finite (at most 7) and conjectured that it actually equals 3. Theorem 1 implies his conjecture:

Corollary 2. If $\tilde{M}$ is the universal cover of a closed graph-manifold, then

$$\dim_{AN}\tilde{M} = \asdim_{AN}\tilde{M} = 3.$$
Proof. Asymptotic dimension never exceeds either of the aforementioned dimensions, so this provides the lower bound of 3 in both cases. Also, as $M$ is a 3-manifold, $\dim_{\text{AN}} \widetilde{M} \leq \max\{\text{asdim}_{\text{AN}} \widetilde{M}, 3\}$. Results in [LS05] prove $\text{asdim}_{\text{AN}} X \leq n$ when $X$ is an $n$–fold product of trees and $\text{asdim}_{\text{AN}} A \leq \text{asdim}_{\text{AN}} B$ whenever $A$ admits a quasi-isometric embedding into $B$, so we get the upper bound using Theorem [LS05] □

A graph manifold is said to be non–geometric if its decomposition into Seifert fibred pieces is non-trivial. Notice that if the decomposition is trivial, then the universal cover is quasi-isometric to the product of a tree with $\mathbb{R}$.

Question 3. Does every non–geometric graph manifold have a fundamental group of asymptotic dimension 3?

Proof of Theorem [LS05]. We only have to consider non–geometric flip graph manifolds. In fact - at the level of universal covers - any graph manifold is quasi-isometric to a flip graph manifold [KL98]. We do not need the definition of such manifolds, as we will recall the essential properties required. Let $\widetilde{M}$ be a flip graph manifold and let $T$ be its Bass-Serre tree. The universal cover $\widetilde{M}$ of $M$ is constructed by suitably gluing certain metric spaces $X_v = F_v \times \mathbb{R}$, for $v$ a vertex in $T$. Each $F_v$ is the universal cover of a compact surface with non-empty boundary and so it admits a metric retraction $r_v : F_v \to T_v$, where $T_v \subseteq F_v$ is a tree, with the further properties that $r_v$ is injective when restricted to any boundary component of $F_v$ and there exists $\mu$ (not depending on $v$) such that for each $x \in F_v$ we have $d_{F_v}(x, r_v(x)) \leq \mu$. Finally, the gluings are performed as follows. Let $v, v'$ be adjacent vertices. Then there exist parametrisations $\gamma_v : \mathbb{R} \to F_v, \gamma_{v'} : \mathbb{R} \to F_{v'}$ of boundary components of $F_v, F_{v'}$ so that $(\gamma_v(t), u) \in F_v \times \mathbb{R}$ is identified with $(\gamma_{v'}(u), t) \in F_{v'} \times \mathbb{R}$ for each $t, u \in \mathbb{R}$. This is explained, for example, in [BN08].

Step 1. The trees. The first tree will just be the Bass-Serre tree $T_0 = T$. Let us define the other two trees, $T_1, T_2$, as follows.

We can subdivide the vertices of $T$ into disjoint families $V_1, V_2$ such that if $v, v' \in V_i$, then $d_T(v, v')$ is even. Set $T_i = \bigcup_{v \in V_i} T_v$. We now wish to define an equivalence relation $\sim$ on $T_i$, and we will set $\sim_i = T_i/\sim$. Suppose that $v, v' \in V_i, v \neq v'$ and there exists $w$ such that $d_T(v, w) = d_T(v', w) = 1$. We will set $x \sim_d x'$, for $x \in T_v, x' \in T_{v'}$, if there exist $y, y'$ with $r_v(y) = x, r_{v'}(y') = x'$ such that the points $y, y'$ are identified with $(y, 0) \in F_v \times \mathbb{R}, (y', 0) \in F_{v'} \times \mathbb{R}$ have the same $\mathbb{R}$–coordinate. To ensure an equivalence relation, we set $\sim$ to be the transitive closure of $\sim_d$.

It is very easy to check that $T_i = T_i/\sim$ is a metric tree with only countably many branching points. In fact, it can be described as the increasing union of metric spaces $\{X_k\}_{k \in \mathbb{N}}$ such that $X_0$ is a tree and $X_{k+1}$ is obtained from $X_k$ by identifying a line in $X_k$ with a line in some tree.

Step 2. The components of the embedding. Define $f_0 : \widetilde{M} \to T_0$ to be any map such that for all $x \in M, x \in F_{f_0(x)} \times \mathbb{R}$ and define $f_i : \widetilde{M} \to T_i$ as follows. For each $v$, we let $\pi_v : F_v \times \mathbb{R} \to F_v$ be the projection on the first factor and as usual denote the equivalence classes of $\sim$ with square brackets.

If $x \in F_v \times \mathbb{R}$ for some $v \in V_i$, then set $f_i(x) = [r_v(\pi_v(x))].$ Otherwise we have $x \in F_w \times \mathbb{R}$ for $w \notin V_i$. Let $v \in V_i$ be any vertex such that $d_T(v, w) = 1$. Set $f_i(x) = [p]$, where $p \in T_v$ is such that $(p, 0)$ has, as a point in $F_w \times \mathbb{R}$, the same
$\mathbb{R}$–coordinate as $x$. This does not depend on the choice of $v$, by the equivalence relation.

**Step 3.** The product map is a quasi-isometric embedding. Define $f : \tilde{M} \to \prod T_i$ to be $\prod f_i$. We wish to show that $f$ is a quasi-isometric embedding. The easier inequality is $d(f(x), f(y)) \leq Kd(x, y) + C$: the maps $\pi_v$ and $r(v)$ are non-expanding, so $f_1$ and $f_2$ are readily checked to be $1$–Lipschitz, while $f_0$ satisfies $d_{T_0}(f_0(x), f_0(y)) \leq d_{\tilde{M}}(x, y)/\rho + 1$ where

$$0 < \rho = \inf\{d_{\tilde{M}}(x, x') : x \in X_v, x' \in X_{v'}, d_{T_0}(v, v') = 2\}.$$

Let us show the other inequality.

We will start with a geodesic $\delta$ in $\prod T_i$ connecting $f(x)$ to $f(y)$ and construct a path $\gamma$ in $\tilde{M}$ connecting $x$ to $y$ such that $l(\gamma) \leq Kl(\delta) + C$. Let $\delta_1, \delta_2$ be the projections of $\delta$ on the factors. One may wish to compare the paths we obtain in this way with the “special paths” described in [Sis11].

Suppose that $x \in X_\nu$, $y \in X_\mu$ and let $v_0, \ldots, v_n$ be the vertices of $T$ in the geodesic connecting $v_0$ to $v_n$. For $j = 0, \ldots, n$ let $i(j) \in \{1, 2\}$ be such that $v_j \in V_{i(j)}$ and choose $\alpha_j \subseteq \delta_{i(j)}$ so that $\alpha_j \subseteq [\nu_{v_j}(F_{v_j})]$. We will also require that the final point of $\alpha_j$ is the starting point of $\alpha_{j+2}$, that the starting point of $\alpha_0$ is $f_{i(0)}(x)$ and that the final point of $\alpha_n$ is $f_{i(n)}(y)$. This can be easily arranged using the fact that each $[\nu_v(F_v)]$ is convex in the corresponding $T_j$.

For $j = 0, \ldots, n − 1$, let $t_j$ be the $\mathbb{R}$–coordinate as a point in $F_{v_j} \times \mathbb{R}$ of $(p_j, 0) \in F_{v_{j+1}} \times \mathbb{R}$, where $p_j$ is the starting point of $\alpha_{j+1}$. Also, let $t_n$ be the $\mathbb{R}$–coordinate of $y \in F_{v_n} \times \mathbb{R}$.

For $j = 0, \ldots, n$ let $\gamma_j$ be the path $\alpha_j \times t_j$ in $X_{v_j}$. Notice that the distance between the final point of $\gamma_j$ and the starting point of $\gamma_{j+1}$ is at most $2\mu$. So, we can concatenate in a suitable order the $\gamma_j$’s and $n$ geodesics of length at most $2\mu$ to obtain a path $\gamma$ from $x$ to $y$. Clearly $l(\gamma_j) = l(\alpha_j)$, so

$$l(\gamma) = \sum l(\gamma_j) + 2n\mu = l(\delta_1) + l(\delta_2) + 2n\mu = d(f_1(x), f_1(y)) + d(f_2(x), f_2(y)) + 2\mu d(f_0(x), f_0(y)) + 4\mu,$$

as $d(f_0(x), f_0(y)) \geq n − 2$ we have

$$l(\gamma) \leq d(f_1(x), f_1(y)) + d(f_2(x), f_2(y)) + 2\mu d(f_0(x), f_0(y)) + 4\mu,$$

and we are done. \qed

**References**


