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A PROBABILISTIC VERSION OF ROSENTHAL'S INEQUALITY

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Abstract. We prove some probabilistic relations between sums of independent random variables and the corresponding disjoint sums, which strengthen the well-known Rosenthal inequality and its generalizations. As a consequence we extend the inequalities proved earlier by Montgomery-Smith and Junge for rearrangement invariant spaces to the quasi-normed case.

1. Introduction and preliminaries

Let $\{f_k\}_{k=1}^n$ $(n \in \mathbb{N})$ be a sequence of measurable functions on $[0,1]$. By $F(t)$ we denote its disjoint sum, i.e., the function

$$F(t) := \sum_{k=1}^n f_k(t-k+1)\chi_{(k-1,k]}(t) \tag{1}$$

defined on the semi-axis $(0,\infty)$. It is easy to see that

$$m\{t>0 : |F(t)|>\tau\} = \sum_{k=1}^n m\{t\in[0,1] : |f_k(t)|>\tau\} \ \ (\tau>0), \tag{2}$$

where m is the Lebesgue measure. Let $F^*(t)$ be the non-increasing left-continuous rearrangement of $F(t)$ and, as usual, χ_A be the characteristic function of a set A. In [1, Theorem 1], Johnson and Schechtman proved that for every quasi-normed rearrangement invariant (r.i.) space X on $[0,1]$ and for the arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ $(n\in\mathbb{N})$ of *non-negative* independent functions, the following inequality holds:

$$\|F^*\chi_{[0,1]}\|_X + \sum_{k=1}^n F^*(k) \le C_X \|\sum_{k=1}^n f_k\|_X. \tag{3}$$

In particular, in the classical case when $X = L_p$ $(1\le p<\infty)$, inequality (3) is a part of the remarkable Rosenthal inequality, which first appeared in [2]. Note that the other part of Rosenthal's inequality, opposite to (3), holds only for r.i. spaces that are situated sufficiently far from the extreme r.i. space L_∞ (see [1], [3] and [4]). Furthermore, [2] contains a similar inequality for sequences of *symmetrically distributed* independent functions. See [1] and [5] for generalizations and refinements of that version of Rosenthal's inequality.

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Let us return to inequality (3). In order to prove it we need to show that

$$\|F^*\chi_{[0,1]}\|_X \le A_X \|\sum_{k=1}^n f_k\|_X$$

and

$$\sum_{k=1}^n F^*(k) \le B_X \|\sum_{k=1}^n f_k\|_X. \tag{4}$$

The first of these inequalities is an immediate consequence of the definition of an r.i. space and the following distribution inequality (see, for instance, [6, Proposition 2.1]):

$$\begin{aligned}\frac{1}{2}m\{t\in[0,1]: F^*(t)\chi_{[0,1]}(t)>\tau\} &\le m\{t\in[0,1]: \max_{k=1,2,\dots,n} f_k(t)>\tau\}\\ &\le m\{t\in[0,1]: F^*(t)\chi_{[0,1]}(t)>\tau\} \quad (\tau>0).\end{aligned}$$

On the other hand, the proof of (4) in [1] is based on using some properties of quasi-normed r.i. spaces, in particular, on applying the Nikisin factorization theorem [7].

In [8], Montgomery-Smith proved a more general inequality than (4):

$$\|\sum_{k=1}^n F^*(k)e_k\|_E \le D_{X,E}\| \ \|\sum_{k=1}^n f_k e_k\|_E \ \|_X \tag{5}$$

(see also [9]). Here, X is a function r.i. space on $[0,1]$, E is a symmetric sequence space, $\{e_k\}_{k=1}^\infty$ is the standard basis in E and $\{f_k\}_{k=1}^n \subset X$ ($n\in\mathbb{N}$) is an arbitrary sequence of independent functions. A key point in the proof of inequality (5) in both [8] and [9] relies on the duality for E, and therefore the proof may not be extended to the quasi-normed case.

The main goal of this paper is to show that inequalities similar to (4) and (5) are *only of probabilistic* nature and are consequences of some lower distribution estimates (Theorem 1), which may be also of independent interest. Using these estimates, we prove a probabilistic version of Rosenthal's inequality (Theorem 2), and, as a consequence, we obtain (5) for arbitrary quasi-normed spaces E and X (Corollary 2). Note that there are many probabilistic papers containing various lower distribution estimates. Nevertheless, it should be mentioned that the results obtained here are very special since they rely on comparing sums of non-negative random variables with corresponding disjoint sums.

Let $x(t)$ be a Lebesgue–measurable function on $(0,\infty)$. Then $|x(t)|$ *is equimeasurable* with its non-increasing left-continuous *rearrangement*

$$x^*(t) = \inf\{\tau\ge 0 \ : \ m\{s\in[0,1]: \ |x(s)|>\tau\}<t\} \ \ (t>0),$$

i.e., $m\{t>0: \ |x(t)|>\tau\} = m\{t>0: \ x^*(t)>\tau\}$, for every $\tau>0$. A Banach space X of real-valued Lebesgue–measurable functions on the interval $[0,\alpha)$, where $0<\alpha\le\infty$, is called a *rearrangement invariant (r.i.)* if: (1) from the conditions $|y(t)|\le|x(t)|$ a.e. and $x\in X$ it follows that $y\in X$ and $\|y\|_X\le\|x\|_X$; (2) if $|x(t)|$ and $|y(t)|$ are equimeasurable on $[0,\alpha)$ and $x\in X$, then $y\in X$ and $\|y\|_X=\|x\|_X$. If the norm in X satisfies only the generalized triangle inequality $\|x+y\|_X \le L(\|x\|_X+\|y\|_X)$, with some constant $L>1$, then X is said to be the *quasi-normed* r.i. space. We can define normed and quasi-normed symmetric sequence spaces similarly. For general properties of function r.i. spaces and symmetric sequence spaces, we refer the reader to the books [10] and [11].

Throughout this paper, by $[a]$ we will denote the integer part of a real number a.

2. Some lower distribution estimates

Here, we will consider arbitrarily distributed non-negative discrete random variables taking on the values from the same set.

For $n \in \mathbb{N}$ and any fixed vector $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $x_1 \geq x_2 \geq \dots \geq x_n \geq 0$, define the set $\mathcal{M}_n$ of all vectors $\xi = \{\xi_i\}_{i=1}^n$ consisting of independent random variables ξ_i, taking its values in the set $\{x_1, x_2, \dots, x_n\}$ and such that

$$\sum_{i=1}^n \mathbb{P}\{\xi_i = x_j\} = 1 \ (j = 1, 2, \dots, n). \tag{6}$$

Thereby, if $\xi \in \mathcal{M}_n$ and

$$a_{i,j} = \mathbb{P}\{\xi_i = x_j\} \ \ (i, j = 1, 2, \dots, n), \tag{7}$$

then $(a_{i,j})_{i,j=1}^n$ is a doubly stochastic matrix, i.e.,

$$0 \leq a_{i,j} \leq 1 \ , \ \sum_{i=1}^n a_{i,j} = 1 \ (j = 1, 2, \dots, n) \ , \ \sum_{j=1}^n a_{i,j} = 1 \ (i = 1, 2, \dots, n).$$

By $(\xi_i^\sharp(\omega))_{i=1}^n$ we denote the non-increasing rearrangement of the vector $(\xi_i(\omega))_{i=1}^n$ for every $\omega \in \Omega$.

Theorem 1. *There exist universal constants $h \in \mathbb{N}$ and $c_0 > 0$ such that for any $n \in \mathbb{N}$, $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, and $\xi \in \mathcal{M}_n$ we have*

$$\mathbb{P}\left\{\omega \in \Omega : \xi_i^\sharp(\omega) \geq x_{hi-h+1}, \, i = 1, 2, \dots, [(n+h-1)/h]\right\} \geq c_0.$$

We begin by proving an auxiliary proposition, where the following notation shall be used. For an arbitrary vector $\xi = \{\xi_i\}_{i=1}^n \in \mathcal{M}_n$ and any $k, j \in \mathbb{N}$, we set

$$p_{\xi,k,j} := \mathbb{P}\{\xi_k < x_j\} \ , \ P_{\xi,k,j} := \mathbb{P}\{\mathrm{card}\{i : \xi_i < x_j\} = k\},$$

and

$$S_{\xi,k,j} := \sum_{l=k}^n 2^{l-k} P_{\xi,l,j}.$$

It is easy to see that

$$P_{\xi,l,j} = \sum_{1 \leq i_1 < i_2 < \dots < i_l \leq n} \Big(\prod_{m=1}^l p_{\xi,i_m,j} \cdot \prod_{i \notin \{i_1,\dots,i_l\}} (1 - p_{\xi,i,j}) \Big).$$

Therefore,

$$S_{\xi,k,j} = \sum_{l=k}^n \sum_{1 \leq i_1 < i_2 < \dots < i_l \leq n} r_{\xi,k,j}(i_1, \dots, i_l), \tag{8}$$

where

$$r_{\xi,k,j}(i_1, \dots, i_l) := 2^{l-k} \prod_{m=1}^l p_{\xi,i_m,j} \cdot \prod_{i \notin \{i_1,\dots,i_l\}} (1 - p_{\xi,i,j}). \tag{9}$$

Proposition 1. *For arbitrary $k, j = 1, 2, \dots, n$ we have*

$$\max_{\xi \in \mathcal{M}_n} S_{\xi,k,j} = S_{\chi,k,j},$$

where the vector $\chi = \{\chi_i\}_{i=1}^n \in \mathcal{M}_n$ consists of independent random variables χ_i satisfying $\mathbb{P}\{\chi_i = x_j\} = 1/n$ $(i, j = 1, 2, \dots, n)$.

Proof. It is sufficient to prove that $S_{\xi,k,j}$ attains its maximum at a vector $\xi = \{\xi_i\}_{i=1}^n \in \mathcal{M}_n$ such that the corresponding probabilities $p_{\xi,i,j}$ do not depend on i. In fact, by (6), we deduce that $p_{\xi,i,j} = (n-j)/n$ $(i = 1, 2, \dots, n)$, i.e., $\xi = \chi$.

Suppose that for some k and j, the quantity $S_{\xi,k,j}$ is maximal at a vector $\xi = \{\xi_i\}_{i=1}^n \in \mathcal{M}_n$ and there are $u, v = 1, \dots, n$ such that $p_{\xi,u,j} \neq p_{\xi,v,j}$. Let $(a_{i,j})_{i,j=1}^n$ be a doubly stochastic matrix which corresponds to ξ according to (7). Set $b_{i,j} = a_{i,j}$ if $i \notin \{u, v\}$ and $b_{u,j} = b_{v,j} = (a_{u,j} + a_{v,j})/2$. Clearly, $(b_{i,j})_{i,j=1}^n$ is a doubly stochastic matrix as well. Denote by $\eta = \{\eta_i\}_{i=1}^n$ the corresponding vector from $\mathcal{M}_n$. Obviously,

$$p_{\eta,i,j} = p_{\xi,i,j} \ (i \neq \{u, v\}) \ \text{ and } \ p_{\eta,u,j} = p_{\eta,v,j} = (p_{\xi,u,j} + p_{\xi,v,j})/2. \tag{10}$$

We now prove

$$S_{\eta,k,j} \geq S_{\xi,k,j}. \tag{11}$$

First of all, using (8), we get

$$\begin{aligned} S_{\xi,k,j} \;=\; & \Big(\sum_{l=k}^{n-2} \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_l \leq n \\ i_m \notin \{u,v\}, m=1,\dots,l}} \big(r_{\xi,k,j}(i_1, \dots, i_l) + r_{\xi,k,j}(i_1, \dots, i_l, u) \\ & + \; r_{\xi,k,j}(i_1, \dots, i_l, v) + r_{\xi,k,j}(i_1, \dots, i_l, u, v) \big) \Big) \\ & + \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_{k-1} \leq n \\ i_m \notin \{u,v\}, m=1,\dots,k-1}} \big(r_{\xi,k,j}(i_1, \dots, i_{k-1}, u) + r_{\xi,k,j}(i_1, \dots, i_{k-1}, v) \\ & + \; r_{\xi,k,j}(i_1, \dots, i_{k-1}, u, v) \big) + \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_{k-2} \leq n \\ i_m \notin \{u,v\}, m=1,\dots,k-2}} r_{\xi,k,j}(i_1, \dots, i_{k-2}, u, v). \end{aligned}$$

The analogous equality holds also for $S_{\eta,k,j}$. Therefore, to prove (11) it is enough to check that for any admissible l and for arbitrary $1 \leq i_1 < i_2 < \dots < i_l \leq n$ such that $i_m \notin \{u, v\}$ $(1 \leq m \leq l)$, we have

(α)

$$r_{\xi,k,j}(i_1, \dots, i_l) \leq r_{\eta,k,j}(i_1, \dots, i_l) \ , \ r_{\xi,k,j}(i_1, \dots, i_l, u, v) \leq r_{\eta,k,j}(i_1, \dots, i_l, u, v)$$

and

$$\begin{aligned} (\beta) \qquad & r_{\xi,k,j}(i_1, \dots, i_l, u) + r_{\xi,k,j}(i_1, \dots, i_l, v) + r_{\xi,k,j}(i_1, \dots, i_l, u, v) \\ & \leq r_{\eta,k,j}(i_1, \dots, i_l, u) + r_{\eta,k,j}(i_1, \dots, i_l, v) + r_{\eta,k,j}(i_1, \dots, i_l, u, v). \end{aligned}$$

By (9) and (10), $r_{\xi,k,j}$ and $r_{\eta,k,j}$ differ from each other only in factors corresponding to the indices u and v. Therefore, inequality (α) is an immediate consequence of the arithmetic mean–geometric mean inequality. Further, taking into account that

$$\begin{aligned} & p_{\xi,u,j}(1 - p_{\xi,v,j}) + p_{\xi,v,j}(1 - p_{\xi,u,j}) + 2p_{\xi,u,j}p_{\xi,v,j} \\ & = p_{\eta,u,j}(1 - p_{\eta,v,j}) + p_{\eta,v,j}(1 - p_{\eta,u,j}) + 2p_{\eta,u,j}p_{\eta,v,j}, \end{aligned}$$

we obtain inequality (β). Thus, (11) is proved. Repeating this averaging procedure, we come to the desired result. $\square$

Proof of Theorem 1. Since for every $h \in \mathbb{N}$,

$$\mathbb{P}\Big\{\omega \in \Omega : \xi_i^\sharp(\omega) \geq x_{hi-h+1},\, i = 1, 2, \ldots, [(n+h-1)/h]\Big\}$$
$$\geq 1 - \sum_{i=1}^{[(n+h-1)/h]} \mathbb{P}\Big\{\omega \in \Omega : \xi_i^\sharp(\omega) < x_{hi-h+1}\Big\},$$

it is sufficient to prove that there exists a universal positive integer h such that the sum on the right hand side of the last inequality is less than $1/e + 1/2$.

First, from the definition of sums $S_{\xi,k,j}$ it follows that

$$\mathbb{P}\Big\{\omega \in \Omega : \xi_i^\sharp(\omega) < x_{hi-h+1}\Big\} \leq S_{\xi,n-i+1,hi-h+1}.$$

Applying Proposition 1, we get

$$S_{\xi,n-i+1,hi-h+1} \leq \sum_{l=n-i+1}^{n} 2^{l-(n-i+1)} C_n^l \Big(\frac{n-(hi-h+1)}{n}\Big)^l \Big(\frac{hi-h+1}{n}\Big)^{n-l}$$

(C_n^l are binomial coefficients). Therefore, by Stirling's formula,

$$\mathbb{P}\Big\{\omega \in \Omega : \xi_i^\sharp(\omega) < x_{hi-h+1}\Big\}$$
$$\leq K_1 \sum_{l=n-i+1}^{n} 2^{l-(n-i+1)} \Big(\frac{n-(hi-h+1)}{l}\Big)^l \Big(\frac{hi-h+1}{n-l}\Big)^{n-l}, \tag{12}$$

where K_1 is a universal constant.

Denote

$$A_l := 2^{l-(n-i+1)} \Big(\frac{n-(hi-h+1)}{l}\Big)^l \Big(\frac{hi-h+1}{n-l}\Big)^{n-l} \quad (l = n-i+1, \ldots, n).$$

At first, let $i > 1$. Then $hi - h + 1 \leq (h+1)(i-1)$, and hence

$$\begin{aligned} A_{n-i+1} &= \Big(\frac{n-(hi-h+1)}{n-i+1}\Big)^{n-i+1} \Big(\frac{hi-h+1}{i-1}\Big)^{i-1} \\ &\leq \Big(\frac{n-h(i-1)}{n-i+1}\Big)^{n-i+1} \cdot (h+1)^{i-1} \leq \big((h+1)e^{1-h}\big)^{i-1}. \end{aligned} \tag{13}$$

Next, we estimate the ratio A_{l+1}/A_l for every $l = n-i+1, \ldots, n$:

$$\begin{aligned} \frac{A_{l+1}}{A_l} &= 2 \cdot \frac{n-(hi-h+1)}{hi-h+1} \cdot \frac{l^l}{(l+1)^{l+1}} \cdot \frac{(n-l)^{n-l}}{(n-l-1)^{n-l-1}} \\ &\leq \frac{2n(n-l)}{(l+1)(hi-h+1)} \cdot \Big(\frac{l}{l+1}\Big)^l \cdot \Big(\frac{n-l}{n-l-1}\Big)^{n-l-1} \leq K_2 \frac{2n}{h(l+1)}, \end{aligned}$$

with a universal constant $K_2 > 0$ (in the case when $n = l+1$ we set $0^0 = 1$). Since $i \leq (n+h-1)/h$, we then have

$$l + 1 \geq n - i + 2 \geq n - \frac{n}{h} - 1 + \frac{1}{h} + 2 > \Big(1 - \frac{1}{h}\Big)n.$$

Combining this with the previous inequality, we infer that

$$\frac{A_{l+1}}{A_l} \leq \frac{2K_2}{h-1}. \tag{14}$$

Let h be an integer satisfying the conditions

$$h \geq 1 + 4K_2 \ \text{ and } \ (h+1)e^{1-h} \leq 1/(5K_1). \tag{15}$$

Then, by (13) and (14), we have

$$\sum_{l=n-i+1}^{n} A_l \le 2 \cdot \big((h+1)e^{1-h}\big)^{i-1},$$

and, therefore, from (12) it follows that

$$\begin{aligned}\sum_{i=2}^{[(n+h-1)/h]} \mathbb{P}\Big\{\omega \in \Omega : \xi_i^\sharp(\omega) < x_{hi-h+1}\Big\} &\le K_1 \sum_{i=2}^{[(n+h-1)/h]} \sum_{l=n-i+1}^{n} A_l \\ &\le 2K_1 \sum_{i=1}^{\infty} \big((h+1)e^{1-h}\big)^i \le \frac{1}{2}.\end{aligned}$$

Moreover, using the averaging procedure from the proof of Proposition 1, it is not hard to check that

$$\mathbb{P}\Big\{\omega \in \Omega : \xi_1^\sharp(\omega) < x_1\Big\} \le \mathbb{P}\Big\{\omega \in \Omega : \chi_1^\sharp(\omega) < x_1\Big\} \le \Big(\frac{n-1}{n}\Big)^n < \frac{1}{e}.$$

Thus, if h satisfies inequalities (15), then

$$\sum_{i=1}^{[(n+h-1)/h]} \mathbb{P}\Big\{\omega \in \Omega : \xi_i^\sharp(\omega) < x_{hi-h+1}\Big\} \le \frac{1}{2} + \frac{1}{e} < 1,$$

and the proof is completed. □

Remark 1. From the above proof, it follows that c_0 can be an arbitrary constant larger than $1 - 1/e$ provided $h \in \mathbb{N}$ is chosen sufficiently large.

In order to state a combinatorial consequence of the last result we need more notation. Denote by $\mathcal{A}_n$ the set of all doubly stochastic matrices $A = (a_{i,j})_{i,j=1}^n$ of order n. For every $h \in \mathbb{N}$ define a function $\psi_h : \mathcal{A}_n \to \mathbb{R}$ as

$$\psi_h(A) := \sum_{E_1, E_2, \dots, E_n} \prod_{i \in E_1} a_{i,1} \prod_{i \in E_2} a_{i,2} \cdots \prod_{i \in E_n} a_{i,n},$$

where the summation is taken over all collections of subsets $E_1, E_2, \dots, E_n \subset \{1, 2, \dots, n\}$ such that $E_i \cap E_j = \emptyset$ $(i \ne j)$, $\bigcup_{i=1}^n E_i = \{1, 2, \dots, n\}$ and

$$\sum_{i=1}^{hk-h+1} \operatorname{card} E_i \ge k \ (k = 1, 2, \dots, [(n+h-1)/h])$$

(some of the sets E_i may be empty).

Corollary 1. *There exist universal constants $h \in \mathbb{N}$ and $c_0 > 0$ such that for all $n \in \mathbb{N}$,*

$$\min_{A \in \mathcal{A}_n} \psi_h(A) \ge c_0.$$

3. A strong version of the generalized Rosenthal inequality

In this section, we apply the above results to derive lower estimates for distributions of independent functions defined on $[0, 1]$ (equivalently, on a probabilistic space). These can be viewed as a strong probabilistic version of the generalized Rosenthal inequality (5). Let us recall that the disjoint sum $F(t)$ corresponding to a sequence $\{f_k\}_{k=1}^n$ of functions measurable on $[0, 1]$ is defined as in (1). As above,

$(f_j^\sharp(t))_{j=1}^n$ is the non-increasing rearrangement of the vector $(f_j^\sharp(t))_{j=1}^n$ for every fixed $t \in [0,1]$.

Theorem 2. *Let $n \in \mathbb{N}$ and $\{f_k\}_{k=1}^n$ be an arbitrary sequence of non-negative independent random functions defined on $[0,1]$. Then there exist universal constants $h \in \mathbb{N}$ and $c_0 > 0$ such that*

$$m\Big\{t \in [0,1] : f_j^\sharp(t) \geq F^*(hj-h+1), \ \ j = 1,2,\dots,[(n+h-1)/h]\Big\} \geq c_0. \tag{16}$$

Proof. Without loss of generality, we may assume that $m\{t \in [0,1] : f_i(t) = \tau\} = 0$ for any $i = 1,2,\dots,n$ and $\tau \in \mathbb{R}$. Therefore, setting $\tau_0 := \infty$, we can choose inductively reals $\tau_1 > \tau_2 > \cdots > \tau_n \geq 0$ as follows: if $k \in \{1,2,\dots,n\}$ and if τ_{k-1} is already defined, then

$$\tau_k := \max\left\{\tau \in [0,\tau_{k-1}) : \sum_{i=1}^n m\{t \in [0,1] : \tau \leq f_i(t) < \tau_{k-1}\} = 1\right\}. \tag{17}$$

If

$$a_{i,j} := m\{t \in [0,1] : \tau_j \leq f_i(t) < \tau_{j-1}\} \ \ (i,j = 1,2,\dots,n),$$

then $a_{i,j} \geq 0$ and from (17) it follows that $\sum_{i=1}^n a_{i,j} = 1$ $(j = 1,2,\dots,n)$. Moreover, since $f_i \geq 0$, we obtain

$$\sum_{j=1}^n a_{i,j} = \sum_{j=1}^n m\{t \in [0,1] : \tau_j \leq f_i(t) < \tau_{j-1}\} = 1 \ \ (i = 1,2,\dots,n).$$

Thus, $(a_{i,j})_{i,j=1}^n$ is a doubly stochastic matrix.

Next, by (2) and (17), for every $\tau \in (\tau_k, \tau_{k-1})$ we have that

$$\begin{aligned} m\{t > 0 : |F(t)| > \tau\} &= \sum_{j=1}^{k-1}\sum_{i=1}^n m\{t \in [0,1] : \tau_j \leq f_i(t) < \tau_{j-1}\} \\ &+ \sum_{i=1}^n m\{t \in [0,1] : \tau < f_i(t) < \tau_{k-1}\} < k, \end{aligned}$$

whence from the definition of the non-increasing rearrangement of a measurable function it follows that

$$F^*(k) \leq \tau_k \ \ (k = 1,2,\dots,n). \tag{18}$$

Now, define the following sequence of step functions:

$$g_i(t) := \sum_{j=1}^n \tau_j \chi_{\{\tau_j \leq f_i(t) < \tau_{j-1}\}}(t) \ \ (i = 1,2,\dots,n).$$

These functions are independent, $m\{t \in [0,1] : g_i(t) = \tau_j\} = a_{i,j}$ $(i,j = 1,2,\dots,n)$, and for all $t \in [0,1]$,

$$g_i(t) \leq f_i(t) \ \ (i = 1,2,\dots,n). \tag{19}$$

Finally, applying Theorem 1 to the sequence $\{g_i\}_{i=1}^n$ and using (18) and (19), we obtain that

$$\begin{aligned} &m\Big\{t \in [0,1] : f_j^\sharp(t) \geq F^*(hj-h+1), \ \ j = 1,2,\dots,[(n+h-1)/h]\Big\} \\ &\geq m\Big\{t \in [0,1] : g_j^\sharp(t) \geq \tau_{hj-h+1}, \ \ j = 1,2,\dots,[(n+h-1)/h]\Big\} \geq c_0. \end{aligned}$$

□

From the last theorem, we are able to deduce the following extension of the lower estimates obtained earlier by Montgomery-Smith [8, Theorem 1] and Junge [9, Theorem 0.2] to the quasi-normed case.

Corollary 2. *Suppose X is a quasi-normed function r.i. space on $[0,1]$, E is a quasi-normed symmetric sequence space, and $\{e_k\}_{k=1}^{\infty}$ is the standard basis in E. Then there exists a constant $D_{X,E} > 0$ such that inequality* (5) *holds for any $n \in \mathbb{N}$ and any arbitrary sequence of non-negative independent functions $\{f_k\}_{k=1}^{n} \subset X$.*

Proof. Let $h \in \mathbb{N}$ and $c_0 > 0$ be universal constants from Theorem 2. Denote

$$A := \left\{ t \in [0,1] : \|\sum_{k=1}^{n} f_k e_k\|_E \geq \Big\| \sum_{k=1}^{[(n+h-1)/h]} F^*(hk-h+1)e_k \Big\|_E \right\}.$$

Since E is a symmetric space, then from Theorem 2 it follows that $m(A) \geq c_0 > 0$. Moreover, taking into account that the function F^* decreases, we obtain

$$\begin{aligned} \|\sum_{k=1}^{n} F^*(k)e_k\|_E &\leq \Big\| \sum_{k=1}^{h} \sum_{j=1}^{[(n+h-1)/h]} F^*(hj-h+k)e_{hj-h+k} \Big\|_E \\ &\leq hL^{\log_2 h+1} \Big\| \sum_{j=1}^{[(n+h-1)/h]} F^*(hj-h+1)e_j \Big\|_E, \end{aligned}$$

where L is the constant from the triangle inequality for the quasi-norm of the space E. Therefore,

$$\| \|\sum_{k=1}^{n} f_k e_k\|_E\|_X \geq \| \|\sum_{k=1}^{n} f_k e_k\|_E \cdot \chi_A\|_X \geq h^{-1}L^{-\log_2 h-1} \|\sum_{k=1}^{n} F^*(k)e_k\|_E \cdot \|\chi_A\|_X,$$

which gives inequality (5) with the constant $D_{X,E} := hL^{\log_2 h+1}\|\chi_A\|_X^{-1}$. □

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