HEISENBERG UNIQUENESS PAIRS IN THE PLANE.
THREE PARALLEL LINES

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Abstract. A Heisenberg uniqueness pair is a pair \((\Gamma, \Lambda)\), where \(\Gamma\) is a curve in the plane and \(\Lambda\) is a set in the plane, with the following property: any bounded Borel measure \(\mu\) in the plane supported on \(\Gamma\), which is absolutely continuous with respect to arc length and whose Fourier transform \(\hat{\mu}\) vanishes on \(\Lambda\), must automatically be the zero measure. We characterize the Heisenberg uniqueness pairs for \(\Gamma\) as being three parallel lines \(\Gamma = \mathbb{R} \times \{\alpha, \beta, \gamma\}\) with \(\alpha < \beta < \gamma\), \((\gamma - \alpha)/(\beta - \alpha) \in \mathbb{N}\).

1. Introduction

The Heisenberg uncertainty principle states that both a function and its Fourier transform cannot be too localized at the same time (see \[2\] and \[3\]). M. Benedicks in \[1\] proved that given a nontrivial function \(f \in L^1(\mathbb{R}^n)\), the Lebesgue measure of the set of points where \(f \neq 0\) and the set of points where the Fourier transform \(\hat{f} \neq 0\) cannot be simultaneously finite. In this paper we consider a similar problem for measures supported on a subset of \(\mathbb{R}^2\).

Let \(\Gamma\) be a smooth curve in the plane \(\mathbb{R}^2\) and \(\Lambda\) a subset in \(\mathbb{R}^2\). In \[4\], Hedenmalm and Montes-Rodríguez posed the problem of deciding when it is true that
\[
\hat{\mu}|_{\Lambda} = 0 \implies \mu = 0
\]
for any Borel measure \(\mu\) supported on \(\Gamma\) and absolutely continuous with respect to the arc length measure on \(\Gamma\), where
\[
\hat{\mu}(\xi, \eta) = \int_{\mathbb{R}^2} e^{2\pi i ((x,y), (\xi,\eta))} d\mu(x,y).
\]
If this is the case, then \((\Gamma, \Lambda)\) is called a Heisenberg Uniqueness Pair (HUP).

When \(\Gamma\) is the circle, Lev \[7\] and Sjölin \[8\] independently characterized the HUP for some “small” sets \(\Lambda\).

In \[4\] Hedenmalm and Montes-Rodríguez characterized the HUP in the cases:
- \(\Gamma\) is the hyperbola \(xy = 1\) and \(\Lambda = (\alpha \mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta \mathbb{Z})\), for \(\alpha, \beta > 0\).
- \(\Gamma\) two parallel lines in \(\mathbb{R}^2\).
In this note we present a result generalizing the last case. We characterize the HUP for $\Gamma$ as being three parallel lines:

$$\Gamma = \mathbb{R} \times \{\alpha, \beta, \gamma\} \text{ with } \alpha < \beta < \gamma, \ (\gamma - \alpha)/(\beta - \alpha) \in \mathbb{N}.$$

### 2. THREE PARALLEL LINES

Given a set $E \subset \mathbb{R}$ and a point $\xi \in E$, let us define:

- $A^{E,\xi}_{loc} = \{\text{functions } \psi \text{ defined on } E \text{ such that there exist a small interval } I_\xi \text{ around } \xi \text{ and a function } \varphi \in L^1(\mathbb{R}) \text{ such that } \psi(\zeta) = \hat{\varphi}(\zeta), \text{ for } \zeta \in I_\xi \cap E\}.$
- $P^{1,p}[A^{E,\xi}_{loc}] = \{\text{functions } \psi \text{ defined on } E \text{ such that there exist an interval } I_\xi \text{ around } \xi \text{ and functions } \varphi_0, \varphi_1 \in L^1(\mathbb{R}) \text{ with } \psi^p(\zeta) + \hat{\varphi}_1(\zeta)\psi(\zeta) + \hat{\varphi}_0(\zeta) = 0, \text{ for } \zeta \in I_\xi \cap E\}.$

Wiener’s lemma [5 p. 57] states that if $\psi \in A^{E,\xi}_{loc}$ and $\psi(\xi) \neq 0$, then $1/\psi \in A^{E,\xi}_{loc}$. Observe also that if $\psi \in A^{E,\xi}_{loc}$, then $\psi \in P^{1,p}[A^{E,\xi}_{loc}]$. This is easy to see only if $p$ is natural.

Due to invariance under translation and rescaling (see [4]) it will be sufficient to study the case when $\Gamma = \mathbb{R} \times \{0,1,p\}$ for $p \in \mathbb{N}$, $p > 1$.

Given a set $\Lambda \subset \mathbb{R}^2$, we say that $\mu$ is an admissible measure if $\mu$ is a Borel measure in the plane absolutely continuous with respect to arc length with $\text{supp } \mu \subset \Gamma$ and $\hat{\mu}|_{\Lambda} = 0$.

If $\mu$ is a measure absolutely continuous with respect to arc length on $\Gamma$, then there exist functions $f, g, h \in L^1(\mathbb{R})$ such that

$$\hat{\mu}(\xi, \eta) = \frac{f(\xi)}{\psi(\xi)} e^{\pi i \eta} g(\xi) + e^{\pi i \eta} h(\xi), \quad \text{for any } (\xi, \eta) \in \mathbb{R}^2.$$

In particular an admissible measure can be written in this form. Observe also that $\hat{\mu}$ is 2-periodic with respect to the second variable. So, for any set $\Lambda \subset \mathbb{R}^2$, we may consider the periodized set

$$\mathcal{P}(\Lambda) = \{ (\xi, \eta) \text{ such that } (\xi, \eta + 2k) \in \Lambda \text{ for some } k \in \mathbb{Z}\},$$

and it follows that $(\Gamma, \Lambda)$ is a HUP if and only if $(\Gamma, \overline{\mathcal{P}(\Lambda)})$ is a HUP, where $\overline{\mathcal{P}(\Lambda)}$ stands for the closure of $\mathcal{P}(\Lambda)$ in $\mathbb{R}^2$.

We may think without loss of generality that $\Lambda$ is a closed set in $\mathbb{R}^2$, 2-periodic with respect to the second coordinate.

We then have the following result.

**Theorem 1.** Let $\Gamma = \mathbb{R} \times \{0,1,p\}$, for some $p \in \mathbb{N}$, $p > 1$ and $\Lambda \subset \mathbb{R}^2$, closed and 2-periodic with respect to the second variable. Then $(\Gamma, \Lambda)$ is a Heisenberg uniqueness pair if and only if

$$\mathfrak{F} := \Pi^3(\Lambda) \cup (\Pi^2(\Lambda) \setminus \Pi^2(\Lambda)) \cup (\Pi^1(\Lambda) \setminus \Pi^1(\Lambda))$$

is dense in $\mathbb{R}$.

$\Pi(\Lambda)$ means the projection of $\Lambda$ on the axis $\mathbb{R} \times \{0\}$ and given a point $\xi \in \Pi(\Lambda)$, and $\text{Img}(\xi)$ corresponds to the set of points $\eta \in [0,2)$ with $(\xi, \eta) \in \Lambda$. The sets in $\mathfrak{F}$ are defined as follows:

- $\Pi^1(\Lambda) = \{ \xi \in \Pi(\Lambda) \text{ such that there is a unique } \eta_0 \in \text{Img}(\xi)\}$.
- $\Pi^2(\Lambda) = \{ \xi \in \Pi(\Lambda) \text{ such that there are two different points } \eta_0, \eta_1 \in \text{Img}(\xi) \text{, and if there is another point } \eta_2 \in \text{Img}(\xi) \text{, then } \frac{e^{\pi i \eta_2} - e^{\pi i \eta_0}}{e^{\pi i \eta_2} - e^{\pi i \eta_0}} = \frac{e^{\pi i \eta_1} - e^{\pi i \eta_0}}{e^{\pi i \eta_1} - e^{\pi i \eta_0}} \}.$


\begin{itemize}
\item $\Pi^3(\Lambda) = \{ \xi \in \Pi(\Lambda) \text{ such that there are at least three different points } \eta_0, \eta_1, \eta_2 \in \text{Img}(\xi) \text{ with } e^{\pi i \eta_1} - e^{\pi i \eta_0} \neq e^{\pi i \eta_2} - e^{\pi i \eta_0} \}.$
\end{itemize}

The following technical lemma is easy to prove and shows that the functions $\tau$ and $\Phi$ are well defined for $\xi \in \Pi^3(\Lambda)$.

**Lemma 2.** Let $x, y, z \in \mathbb{C}$ be different with
\[
\tau = \frac{y^p - x^p}{y - x} = \frac{z^p - x^p}{z - x};
\]
then
\[
\frac{z^p - y^p}{z - y} = \tau \quad \text{and} \quad \Phi = x\tau - x^p = y\tau - y^p = z\tau - z^p.
\]

Let $\chi$ be a function defined on $\Pi^1(\Lambda)$ as $\chi(\xi) = e^{\pi i \eta}$, where $\eta \in \text{Img}(\xi)$. We define the set $\Pi^{1*}(\Lambda)$ as
\item $\Pi^{1*}(\Lambda) = \{ \xi \in \Pi^1(\Lambda) \text{ such that } \chi \in D^1_p[A^1_{\text{loc}}(\Lambda), \xi] \}$.

Let $\tau, \Phi$ be functions defined on $\Pi^2(\Lambda)$ as
\[
\tau(\xi) = \frac{e^{\pi i \eta_1} - e^{\pi i \eta_0}}{e^{\pi i \eta_1} - e^{\pi i \eta_0}} \quad \text{and} \quad \Phi(\xi) = e^{\pi i \eta_0} e^{\pi i \eta_1} - e^{\pi i \eta_0} e^{\pi i \eta_0},
\]
where $\eta_0, \eta_1 \in \text{Img}(\xi)$. We define the set $\Pi^{2*}(\Lambda)$ as
\item $\Pi^{2*}(\Lambda) = \{ \xi \in \Pi^2(\Lambda) \text{ such that } \tau, \Phi \in A^2_{\text{loc}}(\Lambda, \xi) \}$.

The next lemma will be needed for the proof of the necessity of condition (2.1) in Theorem 1.

**Lemma 3.** Let $I$ be an interval in $\mathbb{R}$ with $\Pi^{2*}(\Lambda)$ dense in $I$. Then there exists a subinterval $I' \subset I$ with $I' \subset \Pi^{2*}(\Lambda) \cup \Pi^3(\Lambda)$.

**Proof.** Pick an arbitrary point $\tilde{\xi} \in I \cap \Pi^{2*}(\Lambda)$. Since $\tau, \Phi \in A^2_{\text{loc}}(\Lambda, \tilde{\xi})$ and $\Pi^{2*}(\Lambda)$ is dense in $I$, we can extend the functions $\tau, \Phi$ continuously on a neighborhood of $\tilde{\xi}$. Let $\tilde{\eta} \neq \tilde{\phi} \in \text{Img}(\tilde{\xi})$. Then
\[
|\tau(\tilde{\xi})| = \left| \frac{e^{\pi i \tilde{\eta}} - e^{\pi i \tilde{\phi}}}{e^{\pi i \tilde{\eta}} - e^{\pi i \tilde{\phi}}} \right| < p,
\]
and since $\tau$ is continuous around $\tilde{\xi}$, there exists a small interval $I'$ around $\tilde{\xi}$ with $|\tau(\xi)| < p$ for $\xi \in I'$. We will see that $I' \subset \Pi^{2*}(\Lambda) \cup \Pi^3(\Lambda)$.

Given $\xi \in I'$, consider a sequence $\{\xi_k\} \subset \Pi^{2*}(\Lambda) \cap I'$ with $\xi_k \to \xi$, and for each $\xi_k$ let $\eta_k \neq \phi_k \in \text{Img}(\xi_k)$. There exist subsequences $\{\eta_k^\ast\}$ and $\{\phi_k^\ast\}$ such that $\eta_k^\ast \to \eta^\ast$ and $\phi_k^\ast \to \phi^\ast$ for some $\eta^\ast, \phi^\ast \in [0, 2]$. Since the set $\Lambda$ is closed and 2-periodic with respect to the second coordinate, we may assume WLOG that $\xi \in \Pi(\Lambda)$ with $\eta^\ast \neq \phi^\ast \in \text{Img}(\xi)$. Otherwise,
\[
|\tau(\xi_k)| = \left| \frac{e^{(p-1)\pi i \eta_k^\ast} + e^{(p-2)\pi i \eta_k^\ast} e^{\pi i \phi_k^\ast} + \cdots + e^{(p-1)\pi i \phi_k^\ast}}{e^{(p-1)\pi i \eta_k^\ast} + e^{(p-2)\pi i \eta_k^\ast} + \cdots + e^{(p-1)\pi i \eta_k^\ast}} \right| = p,
\]
which is a contradiction with the fact that $\xi \in I'$.

So $I' \subset \Pi^3(\Lambda) \cup \Pi^3(\Lambda)$, and since the extended functions $\tau, \Phi$ are continuous on $I'$, we also have that $\xi \in \Pi^{2*}(\Lambda)$ for any $\xi \in \Pi^2(\Lambda) \cap I'$. Also, we can conclude that $I' \subset \Pi^{2*}(\Lambda) \cup \Pi^3(\Lambda)$. 

\]
3. Proof of the main result

This section is devoted to the proof of Theorem \([\text{[2.1]}]\). The proof of the sufficiency of condition \((2.1)\) is rather easy. Let \(\mu\) be an admissible measure. Then there exist functions \(f, g, h \in L^1(\mathbb{R})\) such that

\[
\hat{\mu}(\xi, \eta) = \hat{f}(\xi) + e^{\pi i \eta} \hat{g}(\xi) + e^{\pi i \eta} \hat{h}(\xi),
\]

for any \((\xi, \eta) \in \mathbb{R}^2\).

Since \(\mathcal{F}\) is dense in \(\mathbb{R}\) we will be done if we show that \(\hat{\mu}(\xi, \eta) = \hat{f}(\xi) + e^{\pi i \eta} \hat{g}(\xi) + e^{\pi i \eta} \hat{h}(\xi) = 0\) for any \(\xi \in \mathcal{F} = \Pi^3(\Lambda) \cup \Pi^2(\Lambda) \cup \Pi^1(\Lambda) \cup \Pi^{1^*}(\Lambda)\).

If \(\xi \in \Pi^3(\Lambda)\), let \(\eta_0, \eta_1, \eta_2 \in \text{Im}(\xi)\) be different. Since \(\hat{\mu}|_{\Lambda} = 0\) and \(e^{\pi i \eta_0} \neq e^{\pi i \eta_1} - e^{\pi i \eta_0}\) or \(e^{\pi i \eta_2} - e^{\pi i \eta_0}\), it follows that \(\hat{f}(\xi) = \hat{g}(\xi) = \hat{h}(\xi) = 0\).

If \(\xi \in \Pi^2(\Lambda)\), let \(\eta_0 \neq \eta_1 \in \text{Im}(\xi)\). Since \(\hat{\mu}|_{\Lambda} = 0\), then \(\hat{g}(\xi) = -\tau(\xi) \hat{h}(\xi)\) and \(\hat{f}(\xi) = \Phi(\xi) \hat{h}(\xi)\). Suppose \(\hat{h}(\xi) \neq 0\). Then by Wiener’s lemma and Fubini’s theorem, \(\tau, \Phi \in A^{\Pi^2(\Lambda)}_{loc}\), which implies that \(\xi \in \Pi^2(\Lambda)\). So if \(\xi \in \Pi^2(\Lambda) \cap \Pi^{1^*}(\Lambda)\), then \(\hat{f}(\xi) = \hat{g}(\xi) = \hat{h}(\xi) = 0\).

Finally, if \(\xi \in \Pi^1(\Lambda)\) and \(\eta_0 \in \text{Im}(\xi)\), since \(\hat{\mu}|_{\Lambda} = 0\), then \(\hat{f}(\xi) + \chi(\xi) \hat{g}(\xi) + \chi^p(\xi) \hat{h}(\xi) = 0\), where \(\chi(\xi) = e^{\pi i \eta_0}\). Suppose \(\hat{h}(\xi) \neq 0\); then \(\chi \in P^{1,p}[A^{\Pi^1(\Lambda)}_{loc}]\) and \(\xi \in \Pi^{1^*}(\Lambda)\). Otherwise, if \(\hat{g}(\xi) \neq 0\), by Wiener’s lemma and Fubini’s theorem, \(\chi \in A^{\Pi^1(\Lambda)}_{loc}\) and also \(\chi^p \in A^{\Pi^1(\Lambda)}_{loc}\), so \(\chi \in P^{1,p}[A^{\Pi^1(\Lambda)}_{loc}]\) and \(\xi \in \Pi^{1^*}(\Lambda)\). This means that if \(\xi \in \Pi^1(\Lambda) \cap \Pi^{1^*}(\Lambda)\), then \(\hat{f}(\xi) = \hat{g}(\xi) = \hat{h}(\xi) = 0\).

For the proof of the necessity of condition \((2.1)\), suppose that the set \(\mathcal{F}\) is not dense in \(\mathbb{R}\) and let us pick an open interval \(I\) that has empty intersection with \(\mathcal{F}\), i.e.,

\[
\Pi(\Lambda) \cap I = (\Pi^{1^*}(\Lambda) \cup \Pi^{2^*}(\Lambda)) \cap I.
\]

We consider three cases:

- **There exists a small interval \(I_\xi \subset I\) around \(\xi \in \Pi^{1^*}(\Lambda)\) such that all the points in \(I_\xi \cap \Pi(\Lambda)\) belong to \(\Pi^{1^*}(\Lambda)\).** Since \(\chi \in P^{1,p}[A^{\Pi^1(\Lambda)}_{loc}]\), there exist an interval \(I' \subset I_\xi\) around \(\xi\) and functions \(\varphi_0, \varphi_1 \in L^1(\mathbb{R})\) such that

\[
\chi^p(\xi^*) + \hat{\varphi}_1(\xi^*) \chi(\xi^*) + \hat{\varphi}_0(\xi^*) = 0
\]

for any \(\xi^* \in I' \cap \Pi(\Lambda)\). Let \(h \in L^1(\mathbb{R})\) with \(\hat{h}(\xi) \neq 0\) and \(\text{supp} \hat{h} \subset I'\), and define \(f, g \in L^1(\mathbb{R})\) via \(f = \hat{h} \hat{\varphi}_0\) and \(g = \hat{h} \hat{\varphi}_1\). Now,

\[
\hat{\mu}(\xi^*, \eta^*) = -\hat{f}(\xi^*) + \hat{g}(\xi^*) \chi(\xi^*) + \hat{h}(\xi^*) \chi^p(\xi^*) = 0
\]

for \(\xi^* \in I' \cap \Pi^{1^*}(\Lambda)\), \(\eta^* \in \text{Im}(\xi^*)\). Finally, since \(\text{supp} \hat{h} \subset I'\) and \(\Pi(\Lambda) = I' \cap \Pi^{1^*}(\Lambda)\), we can conclude that \(\hat{\mu}|_{\Lambda} \equiv 0\), and we have that \(\mu\) is a nontrivial admissible measure. So \((\Gamma, \Lambda)\) is not a Heisenberg uniqueness pair.

- **There exists a small interval \(I_\xi \subset I\) around \(\xi \in \Pi^{2^*}(\Lambda)\) such that all the points in \(I_\xi \cap \Pi(\Lambda)\) belong to \(\Pi^{2^*}(\Lambda)\).** Now there exists a small interval \(I' \subset I_\xi\) around \(\xi\) and functions \(\Phi_1, \tau_1 \in L^1(\mathbb{R})\) such that \(\tau_1 = \tau\) and \(\Phi_1 = \Phi\) on \(I' \cap \Pi(\Lambda)\). Consider a function \(h \in L^1(\mathbb{R})\) with \(\text{supp} \hat{h} \subset I'\) and \(\hat{h}(\xi) \neq 0\), and define \(f, g \in L^1(\mathbb{R})\) as

\[
g = -h \ast \tau_1 \quad \text{and} \quad f = h \ast \Phi_1.
\]
Given a point \( \xi^* \in I' \cap \Pi^2(\Lambda) \), let \( \eta^* \neq g^* \in \text{Img}(\xi^*) \). Since \( \tau(\xi^*) = e^{\pi i \eta^*} - e^{\pi i g^*} \) and \( \Phi(\xi) = e^{\pi i \eta^*} e^{\pi i \eta^*} - e^{\pi i g^*} \), we have that
\[
\tilde{\mu}(\xi^*, \eta^*) = \tilde{f}(\xi^*) + \tilde{g}(\xi^*) e^{\pi i \eta^*} + \tilde{h}(\xi^*) e^{\pi i g^*} = 0
\]
and also that \( \tilde{\mu}(\xi^*, g^*) = 0 \). So, the corresponding measure \( \mu \) is a nontrivial admissible measure and \((\Gamma, \Lambda)\) is not a Heisenberg uniqueness pair.

- All the intervals \( I_2 \subset I \) contain points in \( \Pi^1(\Lambda) \) and points in \( \Pi^2(\Lambda) \). That is, the sets \( \Pi^1(\Lambda) \) and \( \Pi^2(\Lambda) \) are dense in \( I \cap (\Pi^1(\Lambda) \cup \Pi^2(\Lambda)) = I \cap \Pi^1(\Lambda) \). But this is not possible. In fact, if \( \Pi^2(\Lambda) \) is dense in \( I \), by Lemma 3 there exists a subinterval \( I' \subset I \) such that \( I' \subset \Pi^2(\Lambda) \cup \Pi^2(\Lambda) \).

This finishes the proof of the theorem.

### 4. Examples and Further Results

Given a point \( \xi \in \Pi(\Lambda) \) such that \( \sharp \{ \eta \in \text{Img}(\xi) \} \geq 3 \), we will state a criteria to decide whether the point \( \xi \) belongs to \( \Pi^3(\Lambda) \) or to \( \Pi^2(\Lambda) \). But before this we prove the following lemma.

**Lemma 4.** Given \( C \in \mathbb{C} \), there exist at most \( p \) different points \( \rho^{(k)} \in [0, 2) \) such that for any \( j \neq k \),
\[
\frac{x^p - y^p}{x - y} = C, \quad \text{where} \quad x = e^{\pi i \rho^{(k)}}, \ y = e^{\pi i \rho^{(j)}}.
\]

**Proof.** Observe that for fixed \( C \), there exists a constant \( C^* \in \mathbb{C} \) such that
\[
x C - x^p = C^*
\]
for any \( x = e^{\pi i \rho^{(k)}} \) solution of \( (4.1) \). Now it is obvious that there are at most \( p \) different solutions \( \rho^{(k)} \in [0, 2) \) of the equation \( (4.2) \).

**Corollary 5.** Given a point \( \xi \in \Pi(\Lambda) \), if \( \sharp \{ \eta \in \text{Img}(\xi) \} > p \), then \( \xi \in \Pi^3(\Lambda) \).

In particular, if \( \Gamma \) consists of three parallel equidistant lines in the plane \( (p = 2) \), we have
\[
\Pi^3(\Lambda) = \{ \xi \in \Lambda \text{ such that } \sharp \{ \eta \in \text{Img}(\xi) \} \geq 3 \},
\]
\[
\Pi^2(\Lambda) = \{ \xi \in \Lambda \text{ such that } \sharp \{ \eta \in \text{Img}(\xi) \} = 2 \}.
\]

**Example 6.** The following example shows that Corollary 5 is sharp:

- Let \( \Lambda = \mathbb{R} \times \{2k/p\}_{k=0, \ldots, p-1} \). Then for any \( \xi \in \mathbb{R} \),
\[
\sharp \{ \eta \in \text{Img}(\xi) \} = p
\]
and \( \xi \in \Pi^2(\Lambda) \). Observe that in this case, \((\Gamma, \Lambda)\) is not an HUP.

This lemma will be useful for another example.

**Lemma 7.** For any \( z \in \mathbb{C} \) with \( |z| < 1 \), there exist \( w_1, w_2 \in \mathbb{C} \) unimodular with \( z = w_1 + w_2 \).

**Proof.** Let \( z = re^{i\sigma} \) and let \( v \in [0, \pi/2] \) with \( \cos v = r/2 \). Let’s take
\[
w_1 = e^{i(v+\sigma)}, \ w_2 = e^{i(-v+\sigma)}.
\]
Then,
\[
w_1 + w_2 = e^{i(v+\sigma)} + e^{i(-v+\sigma)} = e^{i\sigma} 2 \cos(v) = re^{i\sigma} = z,
\]
and this finishes the proof. \( \square \)
**Example 8.** Let $p = 2$. Let $g$ be a bounded, continuous function with $|g| < 1$ that is nowhere locally the Fourier transform of an $L^1$ function. There exists a set $\Lambda \subset \mathbb{R} \times [0, 2)$ such that $\Pi(\Lambda) = \Pi^2(\Lambda)$ is dense in $\mathbb{R}$ and the function $\Phi \equiv g$ on $\Pi^2(\Lambda)$. So, $(\Gamma, \Lambda)$ is not a HUP.

Let’s first prove the existence of the function $g$. Let $E$ be a dense set of measure zero on the circle $\mathbb{T}$. By [6] there exists a continuous function $f$ such that the Fourier series of $f$ fails to converge on any point of $E$. Now let $g : \mathbb{R} \to \mathbb{C}$ be the $2$-periodic function defined as $g(t) = f(e^{\pi it})$. It is easy to see that this function $g$ is continuous but it is not a Fourier transform of an $L^1$ function locally at any point. By a standard argument we can think that $g$ is bounded with $|g| < 1$.

Now we will define the set $\Lambda$. By Lemma [7] for any $\xi \in \mathbb{R}$ there exist $w_1(\xi) = e^{\pi i \eta_0}, w_2(\xi) = e^{\pi i \eta_1}$ with $w_1(\xi) + w_2(\xi) = g(\xi)$. Observe also that there is a dense set $\Psi$ of $\mathbb{R}$ such that $\eta_0 \neq \eta_1$ for any $\xi \in \Psi$. Otherwise the function $g$ is constant on an interval, and we get a contradiction with the fact that $g$ is not locally the Fourier transform of an $L^1$ function.

We define $\Lambda = \{(\xi, \eta_0) \cup (\xi, \eta_1)\}_{\xi \in \Psi}$. Now $\Pi(\Lambda) = \Pi^2(\Lambda)$ and

$$\Phi(\xi) = e^{\pi i \eta_0} + e^{\pi i \eta_1} = g(\xi),$$

for any $\xi \in \Psi$.

Since $\Phi \notin \mathcal{A}_{loc}^{\Pi^2(\Lambda), \xi}$ for any $\xi \in \Pi^2(\Lambda)$, we have that $\Pi(\Lambda) = \Pi^2(\Lambda)$, and so $(\Gamma, \Lambda)$ is not a HUP.

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**References**


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