A NON-UNITAL \(\ast\)-ALGEBRA HAS UC\(\ast\)NP
IF AND ONLY IF ITS UNITIZATION HAS UC\(\ast\)NP

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Abstract. The result stated in the title is proved, thereby disproving the result shown in a 1983 paper by B. A. Barnes (Theorem 4.1).

1. Introduction

Let \(A\) be a non-unital algebra such that \(a \in A\) and \(aA = \{0\}\) or \(Aa = \{0\}\) implies \(a = 0\). Let \(A_e = \{a + \lambda e : a \in A, \lambda \in \mathbb{C}\}\) be the unitization of \(A\) with the unit element denoted by \(e\). For an (algebra) norm \(\|\cdot\|\) on \(A\), define the algebra norms on \(A_e\) as

\[
\| a + \lambda e \|_{op} := \sup \{ \| ab + \lambda b \| : b \in A, \| b \| \leq 1 \} \quad \text{and} \quad \| a + \lambda e \|_1 := \| a \| + |\lambda| 
\]

for all \(a + \lambda e \in A_e\). We must note that throughout this paper, no norm on \(A\) is assumed to be complete.

A \(C^\ast\)-norm is a norm \(\|\cdot\|\) on an \(\ast\)-algebra \(A\) such that \(\| a^\ast a \| = \| a \|^2\) (\(a \in A\)). The \(\ast\)-algebra \(A\) has unique \(C^\ast\)-norm property (UC\(\ast\)NP) if \(A\) admits exactly one \(C^\ast\)-norm. The UC\(\ast\)NP and the twin property of \(\ast\)-regularity were discovered by Barnes [2]. They are of significance in harmonic analysis [6, Section 10.5] and have inspired the study of unique uniform norm property (UUNP) in Banach algebras [5, Section 4.6]. For a non-unital, commutative Banach \(\ast\)-algebra \(A\), Dabhi and Dedania have proved [4, Corollary 2.3(ii)] that \(A\) has UC\(\ast\)NP iff \(A_e\) has UC\(\ast\)NP. Here we prove the same for any non-unital \(\ast\)-algebra, not necessarily commutative. As a result, Theorem 4.1 in [2] is false.

2. Results

Suppose that \(\|\cdot\|\) is a unital norm on the algebra \(A_e\) (i.e., \(\| e \| = 1\)). Then \(\| a + \lambda e \|_{op} \leq \| a + \lambda e \| \leq \| a + \lambda e \|_1\) (\(a + \lambda e \in A_e\)). Theorem 2.1 below implies that the norm \(\|\cdot\|\) is equivalent to either \(\|\cdot\|_{op}\) or \(\|\cdot\|_1\). This theorem is inspired by [1].

**Theorem 2.1.** Let \(\|\cdot\|\) be a unital norm on \(A_e\).

1. If \(A\) is closed in \((A_e, \|\cdot\|)\), then \(\| a + \lambda e \|_1 \leq 3 \| a + \lambda e \| \) (\(a + \lambda e \in A_e\)).
2. If \(A\) is dense in \((A_e, \|\cdot\|)\), then \(\| a + \lambda e \|_{op} = \| a + \lambda e \| \) (\(a + \lambda e \in A_e\)).
Proof. (1) Let $a + \lambda e \in A_e$. We may assume that $\|a + \lambda e\|_1 = 1$. First suppose that $|\lambda| \leq 1/3$. Then $\|a\| = 1 - |\lambda| \geq 2/3$. So $\|a + \lambda e\| \geq \|a\| - |\lambda| \geq 2/3 - 1/3 = 1/3$. Hence $\|a + \lambda e\| \leq 3 \|a + \lambda e\|$. Secondly, suppose that $|\lambda| > 1/3$. Since $A$ is closed in $(A_e, \|\cdot\|)$, the multiplicative linear functional $\varphi_\infty(a + \lambda e) := \lambda (a + \lambda e) \in A_e$ is $\|\cdot\|$-continuous. Therefore $1 = |\varphi_\infty(e)| = |\varphi_\infty(b - e)| \leq b - e \| (b \in A)$. In particular, $1/3 \leq |\lambda| \leq |\lambda| - \frac{1}{3} = ||a + \lambda e||$. Thus $\|a + \lambda e\| \leq 3 \|a + \lambda e\|$. $\square$

Corollary 2.2. Let $A$ be a non-unital algebra.

(1) Any norm $\|\cdot\|$ on $A_e$ is equivalent to either $\|\cdot\|_\text{op}$ or $\|\cdot\|_1$.

(2) [11 Corollary 2] Let $\|\cdot\|$ be a complete norm on $A$ such that $\|a\| = \|a\|_\text{op} = \|a\|$ $(a \in A)$. Then $\|a + \lambda e\|_1 \leq 3 \|a + \lambda e\|_\text{op} (a + \lambda e \in A_e)$.

Proof. (1) Without loss of generality, we may assume that $\|\cdot\|$ is unital. Now this is immediate from Theorem 2.1.

(2) Let $|\cdot| = \|\cdot\|_\text{op}$ on $A_e$. Then $|\cdot|$ is a unital norm on $A_e$. Since $\|\cdot\|$ is a complete norm on $A$, $|\cdot| = \|\cdot\|_\text{op}$ is complete on $A_e$, so $A$ is closed in $(A_e, |\cdot|)$. So, by Theorem 2.1(1), $|a + \lambda e|_1 \leq 3|a + \lambda e| (a + \lambda e \in A_e)$. Hence

$$\|a + \lambda e\|_1 = \|a\| + |\lambda| = \|a\|_\text{op} + |\lambda| \quad \text{(by hypothesis)}$$

$$= |a| + |\lambda| = |a + \lambda|_1 \leq 3|a + \lambda| = 3 \|a + \lambda\|_\text{op}.$$}

This proves (2). $\square$

Let $A$ be a non-unital $\ast$-algebra with UC$^*$NP. In Lemma 2.3 we show that $A_e$ cannot have more than two $C^*$-norms. Then in Theorem 2.6 we prove that, in fact, $A_e$ must have UC$^*$NP.

Lemma 2.3. Let $A$ be a non-unital $\ast$-algebra with UC$^*$NP. Then $A_e$ has at most two $C^*$-norms.

Proof. Assume that $A$ has UC$^*$NP. Let $\|\cdot\|$ be the unique $C^*$-norm on $A$. Then $\|\cdot\|_\text{op}$ is a $C^*$-norm on $A_e$ due to 3 Proposition 2.2(b)]. First we claim that $\|\cdot\|_\text{op}$ is the minimum $C^*$-norm on $A_e$. If $|\cdot|$ is a $C^*$-norm on $A_e$, then it is also a $C^*$-norm on $A$. Hence, $\|\cdot\|_\text{op} \leq |\cdot|$ on $A_e$. This proves our claim.

Now let $|||\cdot|||$ be a $C^*$-norm on $A_e$ other than $\|\cdot\|_\text{op}$. Because any two equivalent $C^*$-norms are identical, it is enough to show that $|||\cdot||| \cong \|\cdot\|_1$ on $A_e$. Since $A$ has UC$^*$NP, $|||\cdot||| = \|\cdot\|$ on $A$. Hence $|||\cdot|||_\text{op} = \|\cdot\|_\text{op}$ on $A_e$. Now $A$ must be closed in $(A_e, |||\cdot|||)$. Otherwise, by Theorem 2.1(2), $|||\cdot||| = \|\cdot\|_\text{op}$ on $A_e$, and so $|||\cdot||| = |||\cdot|||_\text{op} = \|\cdot\|_\text{op}$ on $A_e$, which is a contradiction. Then, by Theorem 2.1(1),

$$|||a + \lambda e||| \leq |||a||| + |\lambda| = \|a\| + |\lambda| = \|a + \lambda e\|_1 = \|a + \lambda e\|_1 \leq 3\|a + \lambda e\|.$$}

This completes the proof. $\square$
Lemma 2.4. Let $A$ be a non-unital $\ast$-algebra. Let $\| \cdot \|$ be a $C^\ast$-norm on $A$. Let $(C^\ast(A), \| \cdot \|)$ be the completion of $(A, \| \cdot \|)$. If $(C^\ast(A), \| \cdot \|)$ contains the identity, then $\| \cdot \|_{op} = \| \cdot \|_{ip}$ on $A_e$.

Proof. Clearly, $A_e \subset C^\ast(A)$. Let $a + \lambda e \in A_e \subset C^\ast(A)$ be non-zero. Then there exists a sequence $(a_n)$ in $A$ such that $a_n \longrightarrow a + \lambda e$ in $\| \cdot \|$. Let $b \in A$ be such that $\| b \| \leq 1$. Then $a_nb \longrightarrow ab + \lambda b$ in $\| \cdot \|$. So

$$\| ab + \lambda b \| = \lim_{n \rightarrow \infty} \| a_nb \| \leq \lim_{n \rightarrow \infty} \| a_n \| = \lim_{n \rightarrow \infty} \| a_n \| = \| a + \lambda e \|.$$ 

Hence $\| a + \lambda e \|_{op} \leq \| a + \lambda e \|_{ip}$. For the reverse inequality, consider a sequence $(c_n)$ in $A$ such that $\| c_n \| \leq 1$ and $c_n \longrightarrow (a + \lambda e)^*$, respectively. Then

$$\| a + \lambda e \|_{op} \geq \sup_n \| (a + \lambda e)c_n \| \geq \lim_{n \rightarrow \infty} \| (a + \lambda e)c_n \| = \lim_{n \rightarrow \infty} \| (a + \lambda e)c_n \| = \| \frac{(a + \lambda e)(a + \lambda e)^*}{\| a + \lambda e \|} \| = \| a + \lambda e \|_{ip}.$$ 

Thus $\| a + \lambda e \|_{op} \geq \| a + \lambda e \|_{ip}$.

Lemma 2.5. Let $A$ be a non-unital $\ast$-algebra. Let $\| \cdot \|$ be a $C^\ast$-norm on $A$. Let $(C^\ast(A), \| \cdot \|)$ be the completion of $(A, \| \cdot \|)$. If $(C^\ast(A), \| \cdot \|)$ does not contain the identity, then $\| \cdot \|_{op} = \| \cdot \|_{ip}$ on $A_e$.

Proof. Since $A \subset C^\ast(A)$, we have $A_e \subset C^\ast(A)_e$. Let $a + \lambda e \in A_e$. Then

$$\| a + \lambda e \|_{op} = \sup \{ \| (a + \lambda e)b \| : b \in A; \| b \| \leq 1 \} = \sup \{ \| (a + \lambda e)b \| : b \in A; \| b \| \leq 1 \} \leq \sup \{ \| (a + \lambda e)b \| : b \in C^\ast(A); \| b \| \leq 1 \} = \| a + \lambda e \|_{ip}.$$ 

For the reverse inequality, let $b \in C^\ast(A)$ be such that $\| b \| \leq 1$. Then there exists a sequence $(b_n)$ in $A$ such that $\| b_n \| \leq 1$ and $b_n \longrightarrow b$ in $\| \cdot \|$. So

$$\| (a + \lambda e)b \| = \lim_{n \rightarrow \infty} \| (a + \lambda e)b_n \| = \lim_{n \rightarrow \infty} \| (a + \lambda e)b_n \| \leq \sup_n \| (a + \lambda e)b_n \| \leq \| a + \lambda e \|_{op}.$$ 

Thus $\| a + \lambda e \|_{op} \leq \| a + \lambda e \|_{ip}$.

The next result disproves [2, Theorem 4.1]. The gap in that proof lies in the first line. It is claimed that $C^\ast(A)$ can be identified with a closed maximal ideal of $C^\ast(A_e)$ of codimension one. But this is not true. In [2, Example 4.4], we have $C^\ast(A_e) = C^\ast(A)$.

Theorem 2.6. A non-unital $\ast$-algebra $A$ has UCNP iff $A_e$ has UCNP.

Proof. Let $A$ have UCNP. Then, by Lemma 2.3, $A_e$ has at most two $C^\ast$-norms. Also, by the first paragraph in the proof of Lemma 2.3, $A_e$ has a minimum $C^\ast$-norm. Let $\| \cdot \|$ and $\| \cdot \|_{ip}$ be these two $C^\ast$-norms on $A_e$ with $\| \cdot \| \leq \| \cdot \|_{ip}$. Note that $\| \cdot \|_{op} = \| \cdot \|_{ip}$ on $A_e$. Since $A$ has UCNP, $\| \cdot \|_{ip} \leq \| \cdot \|_{ip}$ on $A$. Let $(C^\ast(A), \| \cdot \|)$ and $(C^\ast(A_e), \| \cdot \|_{ip})$ be the completions of $(A, \| \cdot \|)$ and $(A_e, \| \cdot \|_{ip})$, respectively.

We embed $C^\ast(A)$ into $C^\ast(A_e)$ as a $\ast$-subalgebra. Let $a \in C^\ast(A)$. Then there exists a sequence $(a_n)$ in $A$ converging to $a$ in $\| \cdot \|_{ip}$. Hence $(a_n)$ is a Cauchy
sequence in $\| \cdot \|$ is a Cauchy sequence in $\| \|_{\| \cdot \|}$ because we have $\| \cdot \|_{\| \cdot \|}$ on $A$. Thus it is a Cauchy sequence in $(A, \| \|_{\| \cdot \|})$ and so it converges to some $b$ in $C^*(A_e)$. Hence the map $j : C^*(A) \to C^*(A_e)$ defined by $j(a) = b$ is a well defined, one-to-one $*$-homomorphism. Thus we have $C^*(A) \hookrightarrow C^*(A_e)$.

Assume that $C^*(A)$ has identity. Then $A_e \hookrightarrow C^*(A) \hookrightarrow C^*(A_e)$. Since every $C^*$-algebra has UC*NP, we have $\| \cdot \|_{\| \cdot \|}$ on $C^*(A)$. So, by Lemma 2.4 we have $\| \cdot \|_{\| \cdot \|_{op}} = \| \cdot \|_{\| \cdot \|}$ on $C^*(A)$. Now, assume that $C^*(A)$ does not have the identity. Then $C^*(A) \hookrightarrow C^*(A)_e \hookrightarrow C^*(A_e)$. Since $C^*(A)_e$ is also a $C^*$-algebra with $\| \cdot \|_{op}$, we have $\| \cdot \|_{op} = \| \cdot \|_{\| \cdot \|}$ on $C^*(A)_e$ and hence on $A_e$ because $A_e \hookrightarrow C^*(A)_e$. By Lemma 2.5 $\| \cdot \|_{op} = \| \cdot \|_{op}$ on $A_e$. Therefore $\| \cdot \|_{\| \cdot \|_{op}} = \| \cdot \|_{\| \cdot \|}$ on $A_e$. Thus $A_e$ has UC*NP.

Conversely, let $A_e$ have UC*NP. Since $A$ is a $*$-ideal in $A_e$, $A$ has UC*NP due to [2, Theorem 2.2(1)].

A norm $\| \cdot \|$ on an algebra $A$ is **spectral** if $r_A(a) \leq \| a \|$ $(a \in A)$, where $r_A(\cdot)$ is the spectral radius on $A$. The algebra $A$ has **Spectral Extension Property** (SEP) if every norm on $A$ is spectral. This property arose in the investigation of incomplete algebra norms on Banach algebras [5, Section 4.5]. The next result is in the direction of a non-commutative analogue of [5 Corollary 3.2].

**Theorem 2.7.** Let $A$ be a semisimple, non-unital algebra.

1. If $A_e$ has SEP, then $A$ has SEP and $A$ is closed in every norm on $A_e$.
2. If $A$ has SEP and if $A$ is closed in every norm on $A_e$, then $A_e$ has SEP.

**Proof.** (1) Let $\| \cdot \|$ be any norm on $A$. Then, by the hypothesis, $\| \cdot \|_{1}$ is a spectral norm on $A_e$ so that $r_{A_e}(a + \lambda e) \leq \| a + \lambda e \|_{1}$ $(a + \lambda e \in A_e)$, and hence $r_A(a) = r_{A_e}(a) \leq \| a \|$ $(a \in A)$. Thus $A$ has SEP. Now let $\| \cdot \|$ be any norm on $A_e$. Since $A_e$ has SEP, $\| \cdot \|$ is a spectral norm on $A_e$. Define $\varphi_\infty(a + \lambda e) := \lambda (a + \lambda e \in A_e)$. Then $\varphi_\infty$ is a multiplicative linear functional on $A_e$ and $\ker \varphi_\infty = A$. Since $\| \cdot \|$ is a spectral norm on $A_e$, $\varphi_\infty$ is $\| \|_{\| \cdot \|}$-continuous. Hence $A = \ker \varphi_\infty$ is closed in $(A_e, \| \cdot \|)$.

(2) Let $A$ have SEP. Let $| \cdot |$ be any norm on $A_e$. It is enough to show that $r_{A_e}(a + \lambda e) \leq |a + \lambda e|_{op}$ $(a + \lambda e \in A_e)$.

Set $\| \cdot \| = | \cdot |_{op}$ on $A_e$. Then $\| \cdot \|$ is a unital norm on $A_e$. By the hypothesis, $A$ is closed in $(A_e, \| \cdot \|)$. Hence, by Theorem 2.1(1), we have

$$\| (a + \lambda e) \|_{1} \leq 3 \| a + \lambda e \|$ $(a + \lambda e \in A_e)$.

Therefore,

$$r_{A_e}(a + \lambda e) \leq r_{A_e}(a) + |\lambda| = r_A(a) + |\lambda| \leq \| a + |\lambda| = \| a + \lambda e \|_{1} \leq 3 \| a + \lambda e \|.$$

Since $r_{A_e}((a + \lambda e)^n) = r_{A_e}(a + \lambda e)^n$ and $(a + \lambda e)^n \| (a + \lambda e)^n \|$ for all $n \in \mathbb{N}$, we get $r_{A_e}(a + \lambda e) \leq \| a + \lambda e \|_{op}$ $(a + \lambda e \in A_e)$. □

**Remark 2.8.** (1) Even if $A$ is a non-unital $*$-algebra with UC*NP, the $C^*$-algebra $C^*(A)$ may have the identity. For example, let $D := \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\}$, let $R := \{(x, 0, t) \in \mathbb{R}^3 : -1 \leq x \leq 1, 0 \leq t \leq 1\}$, and let $\Omega := D \cup R$. Let $A$ be the algebra of all continuous complex-valued functions $f$ on $\Omega$ such that $f$ is analytic on the interior of $D$ and $f(0, 1, 0) = f(0, -1, 0) = 0$. For $f \in A$, define $f^*(x, y, t) = \overline{f(x, -y, t)}$ $(x, y, t) \in \mathbb{R}^3)$. Then $A$ is a non-unital, commutative Banach $*$-algebra with UC*NP. As in [2, Example 4.4], the $C^*(A)$ has identity.
(2) We do not know whether the assumption that “A is closed in every norm on $A_e$” in Theorem 2.7(2) can be omitted.

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