ON HARMONIC NON-COMMUTATIVE $L^p$-OPERATORS ON LOCALLY COMPACT QUANTUM GROUPS

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Abstract. For a locally compact quantum group $G$ with tracial Haar weight $\varphi$ and a quantum measure $\mu$ on $G$, we study the space $\mathcal{H}_\mu^p(G)$ of $\mu$-harmonic operators in the non-commutative $L^p$-space $L^p(G)$ associated to the Haar weight $\varphi$. The main result states that if $\mu$ is non-degenerate, then $\mathcal{H}_\mu^p(G)$ is trivial for all $1 \leq p < \infty$.

1. Introduction and preliminaries

Non-commutative Poisson boundaries of (discrete) quantum groups $G$ was first introduced and studied by Izumi in [6]. Motivated by the classical setting, in fact, he defined the Poisson boundary of $G$ associated to a ‘quantum measure’ $\mu$ as the space of $\mu$-harmonic ‘functions’, i.e., the fixed point space of the Markov operator associated to $\mu$. For discrete quantum groups, this was further studied by several authors (cf. [7], [14], [15]). Poisson boundaries in the locally compact quantum group setting was studied by Neufang, Ruan and the author in [9]. Quantum versions of several important classical results regarding harmonic functions were proved there. In particular, triviality of special classes of harmonic functions, such as $C_0$-functions, was proved.

Another important fact regarding classical harmonic functions on locally compact groups is that for $1 \leq p < \infty$, any $L^p$-harmonic function associated to an adapted probability measure is trivial. The main result of this paper is a quantum version of this result. But, in order to talk about $\mu$-harmonic elements in the non-commutative $L^p$-spaces, we first need to define the convolution action by $\mu$ on such spaces.

In his PhD thesis [4], Cooney studied the non-commutative $L^p$-spaces associated to the Haar weight $\varphi$ of a locally compact quantum group $G$. He mainly considered Haagerup’s version and could prove that in the Kac algebra setting, the convolution action of an ‘absolutely continuous quantum measure’ can be extended to the Haagerup non-commutative $L^p$-spaces. So, we cannot consider harmonic operators in the general setting of all locally compact quantum groups. Moreover, in the case of non-tracial $\varphi$, there are different ways to define the non-commutative $L^p$-spaces. Although all these spaces are isometrically isomorphic as Banach spaces, the identifications are not necessarily compatible with the quantum group structure, so it is not clear whether the space of $\mu$-harmonic $L^p$-operators is the same, as a Banach space, for all different definitions of non-commutative $L^p$-spaces.
Therefore, in this paper, instead of restricting ourselves to the Kac algebra setting, we consider locally compact quantum groups $\mathbb{G}$ whose Haar weight $\varphi$ is a trace. In this case, the convolution action is extended to the non-commutative $L^p$-spaces, and the main result of the paper states that in the case of a non-degenerate quantum measure $\mu$, for $1 \leq p < \infty$, any $\mu$-harmonic element which lies in the non-commutative $L^p$-space of $\varphi$ is trivial.

First, let us introduce our terminology and recall some results on locally compact quantum groups which we will be using in this paper. For more details, we refer the reader to [11].

A locally compact quantum group $\mathbb{G}$ is a quadruple $(M, \Gamma, \varphi, \psi)$, where $M$ is a von Neumann algebra with a co-associative co-multiplication $\Gamma : M \to M \otimes M$, and $\varphi$ and $\psi$ are (normal faithful semi-finite) left and right Haar weights on $M$, respectively. We write $M_\varphi^+ = \{ x \in M^+ : \varphi(x) < \infty \}$ and $\mathfrak{N}_\varphi = \{ x \in M^+ : \varphi(x^* x) < \infty \}$, and we denote by $\Lambda_\varphi$ the inclusion of $\mathfrak{N}_\varphi$ into the GNS Hilbert space $H_\varphi$ of $\varphi$. For each locally compact quantum group $\mathbb{G}$, there exists a left fundamental unitary operator $W$ on $H_\varphi \otimes H_\varphi$ which satisfies the pentagonal relation and such that the co-multiplication $\Gamma$ on $M$ can be expressed as

$$\Gamma(x) = W^*(1 \otimes x)W \quad (x \in M).$$

There exists an anti-automorphism $R$ on $M$, called the unitary antipode, such that $R^2 = \iota$, and

$$\Gamma \circ R = \chi(R \otimes R) \circ \Gamma,$$

where $\chi(x \otimes y) = (y \otimes x)$ is the flip map. It can be easily seen that if $\varphi$ is a left Haar weight, then $\varphi R$ defines a right Haar weight on $M$.

Let $M_\ast$ be the predual of $M$. Then the pre-adjoint of $\Gamma$ induces on $M_\ast$ an associative completely contractive multiplication

$$\ast : M_\ast \otimes M_\ast \ni f_1 \otimes f_2 \mapsto f_1 \ast f_2 = (f_1 \otimes f_2) \circ \Gamma \in M_\ast.$$

The left regular representation $\lambda : M_\ast \to \mathcal{B}(H_\varphi)$ is defined by

$$\lambda : M_\ast \ni f \mapsto \lambda(f) = (f \otimes \iota)(W) \in \mathcal{B}(H_\varphi),$$

which is an injective and completely contractive algebra homomorphism from $M_\ast$ into $\mathcal{B}(H_\varphi)$. Then $\hat{M} = \{ \lambda(f) : f \in M_\ast \}''$ is the von Neumann algebra associated with the dual quantum group $\hat{\mathbb{G}}$. It follows that $W \in M \otimes \hat{M}$. We also define the completely contractive injection

$$\hat{\lambda} : \hat{M}_\ast \ni \hat{f} \mapsto \hat{\lambda}(\hat{f}) = (\iota \otimes \hat{f})(W) \in M.$$

The reduced quantum group $C^*$-algebra

$$C_0(\mathbb{G}) = \overline{\lambda(L_1(\hat{\mathbb{G}}))\| \cdot \|}$$

is a weak* dense $C^*$-subalgebra of $M$. Let $\mathcal{M}(\mathbb{G})$ denote the operator dual $C_0(\mathbb{G})^*$. There exists a completely contractive multiplication on $\mathcal{M}(\mathbb{G})$ given by the convolution

$$\ast : \mathcal{M}(\mathbb{G}) \otimes \mathcal{M}(\mathbb{G}) \ni \mu \otimes \nu \mapsto \mu \ast \nu = \mu(\iota \otimes \nu)\Gamma = \nu(\mu \otimes \iota)\Gamma \in \mathcal{M}(\mathbb{G})$$

such that $\mathcal{M}(\mathbb{G})$ contains $M_\ast$ as a norm closed two-sided ideal. Therefore, for each $\mu \in \mathcal{M}(\mathbb{G})$, we obtain a pair of completely bounded maps

$$f \mapsto \mu \ast f \quad \text{and} \quad f \mapsto f \ast \mu.$$
on $M_*$ through the left and right convolution products of $M(G)$. The adjoint maps give the convolution actions $x \mapsto \mu \ast x$ and $x \mapsto x \ast \mu$ that are normal completely bounded maps on $M$.

We denote by $\mathcal{P}(G)$ the set of all states on $C_0(G)$ (i.e., ‘the quantum probability measures’). For any such element the convolution action is a Markov operator, i.e., a unital normal completely positive map, on $M$.

Now assume that the left Haar weight $\varphi$ on $G$ is a trace, and let $\psi = \varphi R$ be the right Haar weight. We denote by $L^p(G)$ and $\tilde{L}^p(G)$ the non-commutative $L^p$-spaces associated to $\varphi$ and $\psi$, respectively. These spaces are obtained by taking the closure of $\mathfrak{M}_\varphi$ and $\mathfrak{M}_\psi$ under the norms $\|x\| = \varphi(|x|^p)^{\frac{1}{p}}$ and $\|x\| = \psi(|x|^p)^{\frac{1}{p}}$, respectively (see [12] for details). We denote by $L^\infty(G)$ the von Neumann algebra $M$. Similarly to the classical case, one can also construct the non-commutative $L^p$-spaces using the complex interpolation method (cf. [5], [10], [13]). The map
\begin{equation}
\mathfrak{M}_\varphi \ni x \mapsto \varphi \cdot x \in M_*
\end{equation}
extends to an isometric isomorphism between $L^1(G)$ and $M_*$, where $\langle \varphi \cdot x, y \rangle = \varphi(xy)$.

2. $\mu$-harmomic operators

We assume that $\mu \in \mathcal{P}(G)$ throughout this section. By invariance of the left Haar weight $\varphi$, we can easily see that $L^p(G) \cap L^\infty(G)$ is invariant under the left convolution action by $\mu$. Since $\varphi = \psi R$ is a trace, by [11, Proposition 5.20] we have

$$R((\mu \otimes \varphi)\Gamma(a)(1 \otimes b)) = (\mu \otimes \varphi)((1 \otimes a)\Gamma(b))$$

for all $a, b \in \mathfrak{M}_\varphi$. Therefore we obtain
\begin{align}
\langle \mu \ast (\varphi \cdot a), b \rangle &= \langle \mu, (\mu \otimes \varphi)((1 \otimes a)\Gamma(b)) \rangle = \langle \mu R, (\mu \otimes \varphi)((1 \otimes a)\Gamma(b)) \rangle \\
&= \langle (\mu R) \ast (b \cdot \varphi), a \rangle = \langle (b \cdot \varphi), (\mu R) \ast a \rangle = \langle \varphi \cdot ((\mu R) \ast a), b \rangle.
\end{align}

Since the map (1.1) is an isometry, this shows that the convolution action
\begin{equation}
\mathfrak{M}_\varphi \ni x \mapsto \varphi \cdot x \mapsto (\mu R) \ast (\varphi \cdot x) = \varphi \cdot (\mu \ast x) \mapsto \mu \ast x \in \mathfrak{M}_\varphi
\end{equation}
extends to an operator on $L^1(G)$ with the same norm as the convolution operator by $\mu$ on $L^\infty(G)$. Now, interpolating between $L^1(G)$ and $L^\infty(G)$, we can extend the convolution action

$$L^p(G) \cap L^\infty(G) \ni x \mapsto \mu \ast x \in L^p(G) \cap L^\infty(G)$$

to $L^p(G)$. An operator $x \in L^p(G)$ is called $\mu$-harmonic if $\mu \ast x = x$ and $\mathcal{H}_\mu^p(G) = \{x \in L^p(G) : \mu \ast x = x\}$ is the space of $\mu$-harmonic operators. It is easy to see that $\mathcal{H}_\mu^p(G)$ is a weak* closed subspace of $L^p(G)$ for all $1 < p \leq \infty$.

Similarly to the case $p = \infty$, we have a projection $E_\mu^p : L^p(G) \to \mathcal{H}_\mu^p(G)$ constructed as follows. Let $\mathcal{U}$ be a free ultra-filter on $\mathbb{N}$, and define $E_\mu^p : L^p(G) \to L^p(G)$ by the weak* limit

$$E_\mu^p(x) = \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^n \mu^k \ast x.$$

Then it is easy to see that $E_\mu^p \circ E_\mu^p = E_\mu^p$ and that $\mathcal{H}_\mu^p(G) = E_\mu^p(L^p(G))$. Moreover, by considering the convolution action on $L^p(G) \cap L^\infty(G)$ and passing to limits, we can see that $E_\mu^p$ is also positive.
Similarly, we can extend the right convolution action
\[ \tilde{L}^p(G) \cap L^\infty(G) \ni x \mapsto x \star \mu \in \tilde{L}^p(G) \cap L^\infty(G) \]
to \( \tilde{L}^p(G) \). Then \( \tilde{H}^p_\mu(G) = \{ x \in \tilde{L}^p(G) : x \star \mu = x \} \) is a weak* closed subspace of \( \tilde{L}^p(G) \) and there is a positive projection \( \tilde{E}^p_\mu \) on \( \tilde{L}^p(G) \) such that \( \tilde{H}^p_\mu(G) = \tilde{E}^p_\mu(\tilde{L}^p(G)) \).

**Proposition 2.1.** The unitary antipode \( R \) extends to an isometric isomorphism
\[ R : L^p(G) \to \tilde{L}^p(G) \]
such that \( R(H^p_\mu(G)) = \tilde{H}^p_\mu R \) for all \( 1 \leq p \leq \infty \).

**Proof.** Since \( R \) is an anti-automorphism, we have
\[ \psi(|R(a)|^p) = \psi(R(|a|^p)) = \varphi(|a|^p) \]
for all \( a \in M_\varphi \). Therefore \( R \) extends to an isometry from \( L^p(G) \) onto \( \tilde{L}^p(G) \). Moreover, we have
\[
R(\mu \star a) = R((\mu \otimes \iota)\Gamma(a)) = R((\mu \otimes \iota)(R(\Gamma(a)))) \\
= R((\iota \otimes \mu)(R \otimes R)(\Gamma(a))) \\
= (\iota \otimes \mu R)\Gamma(R(a)) = R(a) \star \mu R,
\]
which implies that \( R(H^p_\mu(G)) = \tilde{H}^p_\mu R \).

Therefore, for \( 1 < p, q < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), we can identify each \( L^p(G) \) and \( \tilde{L}^q(G) \) with the dual space of the other via
\[ \langle a, b \rangle = \varphi(aR(b)) = \psi(R(a)b), \quad a \in L^p(G), \quad b \in \tilde{L}^q(G) \cdot \]

**Theorem 2.2.** Let \( 1 < p, q < \infty \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then we have linear isometric isomorphisms
\[ H^p_\mu(G)^* \cong H^q_\mu(G) \quad \text{and} \quad H^p_\mu(G) \cong H^q_\mu(G)^* \]

**Proof.** Denote
\[ J^p_\mu(G) := \{ x - \mu \star x : x \in \tilde{L}^p(G) \} \quad \text{and} \quad \tilde{J}^q_\mu(G) := \{ y - y \star \mu : y \in \tilde{L}^q(G) \} \cdot \]

Since
\[
\langle x, y \star \mu \rangle = \psi(R(x)(y \star \mu)) = \psi(R(x)(\iota \otimes \mu)\Gamma(y)) = \mu(\psi(\iota \otimes \mu)(R(x) \otimes 1)\Gamma(y)) \\
= \mu R(\psi(\iota \otimes \mu)\Gamma(R(x))(y \otimes 1)) = \psi(\iota \otimes \mu R)\Gamma(R(x)y) \\
= \psi(R((\mu \otimes \iota)\Gamma(x))y) = \psi(R(\mu \star x)y) = \langle \mu \star x, y \rangle
\]
for all \( x \in M_\varphi \) and \( y \in M_\psi \), it follows that \( H^p_\mu(G) = \tilde{J}^q_\mu(G)^{\perp} \), and therefore
\[ H^p_\mu(G)^* = \frac{\tilde{L}^q(G)}{H^p_\mu(G)^{\perp}} = \frac{\tilde{L}^q(G)}{J^p_\mu(G)} \cdot \]

In the following we show that the correspondence
\[ \frac{\tilde{L}^q(G)}{J^p_\mu(G)} \ni y + \tilde{J}^q_\mu(G) \mapsto \tilde{E}^q_\mu(y) \in \tilde{H}^q_\mu(G) \]
defines a linear isometric isomorphism. First we observe that
\[ \hat{E}_\mu^q(y \ast \mu - y) = \lim_{U} \left( (y \ast \mu - y) \ast \frac{\sum_{1}^{n} \mu^k}{n} \right) = 0 \]
for all \( y \in \tilde{L}^q(G) \), which implies that the above map is well-defined. It is obviously onto. To check the injectivity, first note that
\[ y - y \ast \mu^k = (y - y \ast \mu) + (y \ast \mu - y \ast \mu^2) + (y \ast \mu - y \ast \mu^2) \in \tilde{J}_\mu^q(G), \]
for all \( k \in \mathbb{N} \). Now suppose that \( \hat{E}_\mu^q(y) = 0 \). Then, by the above and by the weak* closeness of \( \tilde{J}_\mu^q(G) \), we have
\[ y = y - \hat{E}_\mu^q(y) = y - \left( \lim_{U} \frac{1}{n} \sum_{k=1}^{n} y \ast \mu^k \right) = \lim_{U} \frac{1}{n} \sum_{k=1}^{n} (y - y \ast \mu^k) \in \tilde{J}_\mu^q(G), \]
and therefore the injectivity of the map follows. Moreover, since \( \hat{E}_\mu^q \) is an idempotent, it follows that
\[ y + \tilde{J}_\mu^q(G) = \hat{E}_\mu^q(y) + \tilde{J}_\mu^q(G). \]
Therefore
\[ \|y + \tilde{J}_\mu^q(G)\| \leq \|\hat{E}_\mu^q(y)\|. \]
On the other hand, we have
\[ \|\hat{E}_\mu^q(y)\| = \sup \left\{ \left| \langle \hat{E}_\mu^q(y), x \rangle \right| : x \in L^p(G), \|x\| \leq 1 \right\} \]
\[ \overset{\text{continuity}}{=} \sup \left\{ \left| \langle y, E_\mu^p(x) \rangle \right| : x \in L^p(G), \|x\| \leq 1 \right\} \]
\[ \overset{\text{positivity of map}}{\leq} \|y + \tilde{J}_\mu^q(G)\|. \]
This shows that the map is isometric and so yields the first identification. The second identification is proved along similar lines. \( \square \)

**Proposition 2.3.** For \( 1 < p \leq \infty \) the space \( \mathcal{H}_\mu^p(G) \) is generated by its positive elements.

**Proof.** By considering the polar decomposition, we observe that \( L^p(G) \cap L^\infty(G) \) is self-adjoint. Let \( x \in \mathcal{H}_\mu^p(G) \), and \( L^p(G) \cap L^\infty(G) \ni x_n \to x \) in \( L^p(G) \). Using the continuity of the adjoint on \( L^p(G) \), we obtain
\[ \mu \ast x^* = \lim_n \mu \ast x_n^* = \lim_n (\mu \ast x_n)^* = (\lim_n \mu \ast x_n)^* = x^*, \]
where the limits are taken in \( L^p(G) \). Therefore, \( \mathcal{H}_\mu^p(G) \) is self-adjoint and so is generated by its self-adjoint elements. Now, let \( x \) be a self-adjoint element in \( L^p(G) \), and let \( x = x_+ - x_- \) where both \( x_+ \) and \( x_- \) are in \( L^p(G)^+ \). Then we have
\[ x = E_\mu^p(x) = E_\mu^p(x_+) - E_\mu^p(x_-), \]
which yields the result by positivity of the map \( E_\mu^p \). \( \square \)
Main Theorem: Case $1 < p < \infty$. A state $\mu \in \mathcal{P}(G)$ is called non-degenerate on $\mathcal{C}_0(G)$ if for every non-zero element $x \in \mathcal{C}_0(G)^+$ there exists $n \in \mathbb{N}$ such that $\langle x, \mu^n \rangle \neq 0$.

**Theorem 2.4.** Let $G$ be a non-compact, locally compact quantum group with a tracial (left) Haar weight $\varphi$, and let $\mu \in \mathcal{P}(G)$ be non-degenerate. Then for all $1 < p < \infty$ we have

\[ \mathcal{H}_\mu^p(G) = \{0\}. \]

**Proof.** First let $1 < p \leq 2$, and suppose $0 \leq x \in \mathcal{H}_\mu^p(G)$ with $\|x\|_p = 1$. Define

\[ \tilde{\mu} := \sum_{i=1}^{\infty} \frac{\mu^n}{2^n}. \]

Since $\mu$ is non-degenerate, $\tilde{\mu}$ is faithful, and $\tilde{\mu} \ast x = x$. Now, let $q \geq 2$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Using the duality between $\mathcal{L}^p(G)$ and $\mathcal{L}^q(G)$, we assign to each pair $a \in \mathcal{L}^p(G)$ and $b \in \mathcal{L}^q(G)$ an element $\Omega_{a,b} \in \mathcal{L}^\infty(G)$ defined by

\[ \langle f, \Omega_{a,b} \rangle = \langle f \ast a, b \rangle \quad (f \in M_\ast). \]

We clearly have $\|\Omega_{a,b}\| \leq \|a\|_p \|b\|_q$. Now, choose $y \in \mathcal{L}^q(G)$, $\|y\|_q = 1$, such that $\langle x, y \rangle = 1$. We claim that $\Omega_{x,y} \in \mathcal{C}_0(G)$ (in fact, $\Omega_{a,b} \in \mathcal{C}_0(G)$ for all $a \in \mathcal{L}^p(G)$ and $b \in \mathcal{L}^q(G)$). To see this, assume that

\[ x = \int_0^\infty \lambda \, d\lambda \]

is the spectral decomposition of $x$, and let

\[ x_n = \int_0^n \lambda \, d\lambda. \]

Then $x_n \in \mathcal{L}^p(G) \cap \mathcal{L}^\infty(G) \subseteq \mathcal{N}_\varphi$, $\|x_n\|_p \leq \|x\|_p$, and $\|x - x_n\|_p \to 0$. Also let $y_n \in \mathcal{N}_\varphi$ be such that $\|y_n - y\|_q \to 0$.

Denote by $\omega_{\eta,\zeta}$ the vector functional associated with $\eta, \zeta \in H_\varphi$. Then, for $f \in M_\ast$ we have

\[ \langle f, \Omega_{x_n,y_n} \rangle = \langle f \ast x_n, y_n \rangle = \langle \lambda(f) \Lambda_\varphi(x_n), \Lambda_\varphi(R(y_n)) \rangle \]

\[ = \langle \omega_{\Lambda_\varphi(x_n), \Lambda_\varphi(R(y_n))}, \lambda(f) \rangle = \langle f, \hat{\lambda}(\omega_{\Lambda_\varphi(x_n), \Lambda_\varphi(R(y_n))}) \rangle, \]

which implies that $\Omega_{x_n,y_n} = \hat{\lambda}(\omega_{\Lambda_\varphi(x_n), \Lambda_\varphi(R(y_n))}) \in \mathcal{C}_0(G)$. Moreover, it follows that

\[ \|\Omega_{x,y} - \Omega_{x_n,y_n}\|_\infty \leq \|\Omega_{x-x_n,y}\|_\infty + \|\Omega_{x_n,y-y_n}\|_\infty \]

\[ \leq \|x - x_n\|_p \|y\|_q + \|x_n\|_p \|y - y_n\|_q \to 0. \]

This shows that $\Omega_{x,y} \in \mathcal{C}_0(G)$, as claimed. But then we have $\|\Omega_{x,y}\| \leq \|x\|_p \|y\|_q = 1$, and

\[ \langle \tilde{\mu}, \Omega_{x,y} \rangle = \langle \tilde{\mu} \ast x, y \rangle = \langle x, y \rangle = 1. \]

Since $\tilde{\mu}$ is faithful, it follows that $\Omega_{x,y} = 1$, and therefore $1 \in \mathcal{C}_0(G)$, which contradicts our assumption of $G$ being non-compact, so $x = 0$. This shows, by Proposition 2.2, that $\mathcal{H}_\mu^p(G) = \{0\}$ for all $1 < p \leq 2$. Now, a similar argument yields $\mathcal{H}_\mu^q(G) = 0$ for all $1 < q \leq 2$, which implies by Theorem 2.2 that $\mathcal{H}_\mu^p(G) = \mathcal{H}_\mu^q(G)^* = \{0\}$ for all $2 \leq p < \infty$. \[\square\]
Main Theorem: Case $p=1$. Since $L^1(\mathbb{G})$ is not a dual Banach space, our proof for $1 < p < \infty$ does not work in this case, and so we have to treat this case separately. We do this by first proving a similar result for $M_*$ and then using the identification of the latter with $L^1(\mathbb{G})$. Note that for the following theorem we do not assume that the Haar weight is a trace.

**Theorem 2.5.** Let $G$ be a non-compact, locally compact quantum group, and let $\mu \in \mathcal{P}(G)$ be non-degenerate. If $\omega \in M(G)$ is such that $\mu \ast \omega = \omega$, then $\omega = 0$.

**Proof.** Assume that $\mu \ast \omega = \omega$, and let $\tilde{\mu}$ be as in the proof of Theorem 2.4. So, $\tilde{\mu}$ is faithful, and $\tilde{\mu} \ast \omega = \omega$. Therefore we have

$$\lambda(\tilde{\mu})\lambda(\omega)\xi = \lambda(\tilde{\mu} \ast \omega)\xi = \lambda(\omega)\xi$$

for all $\xi \in H_\omega$. Now if $\omega \neq 0$, there exists $\xi \in H_\omega$ such that $\|\lambda(\omega)\xi\| = 1$. Denote by $\lambda$ the restriction of $\omega_{\lambda(\omega)}$ to $\tilde{M}$. Then $\|\lambda\| = 1$, and

$$\langle \tilde{\mu} , \lambda(\omega) \rangle = \langle \lambda(\tilde{\mu}) , \lambda(\omega) \rangle = \langle \lambda(\tilde{\mu})\lambda(\omega)\xi , \lambda(\omega)\xi \rangle = \langle \lambda(\omega)\xi , \lambda(\omega)\xi \rangle = 1.$$ 

Since $\|\lambda(\omega)\| \leq 1$ and $\tilde{\mu}$ is faithful, it follows that $\lambda(\omega) = 1$. But this implies that $1 \in C_0(G)$, which contradicts our assumption of $G$ being non-compact. Hence, $\omega = 0$. \hfill $\square$

**Theorem 2.6.** Let $G$ be a non-compact, locally compact quantum group with a tracial (left) Haar weight $\varphi$, and let $\mu \in \mathcal{P}(G)$ be non-degenerate. Then

$$\mathcal{H}_\mu^1(\mathbb{G}) = \{0\}.$$ 

**Proof.** Let $x \in \mathcal{H}_\mu^1(G)$. We have that $\mu R \in \mathcal{P}(G)$ is non-degenerate, and from equations (2.1) and (2.2) we get

$$\mu R \ast (\varphi \cdot x) = \varphi \cdot (\mu \ast x) = \varphi \cdot x.$$ 

Hence, $\varphi \cdot x = 0$ by Theorem 2.5 and therefore $x = 0$. \hfill $\square$

**Remark 2.7.** The statements of Theorems 2.4 and 2.6 are not true in general for the case $p = \infty$. Any non-degenerate probability measure on a non-amenable discrete group is a counterexample [8].

**Compact Case.** We conclude by proving the triviality of $\mu$-harmonic operators in the compact quantum group setting.

**Theorem 2.8.** Let $G$ be a compact quantum group with tracial Haar state, and let $\mu \in \mathcal{P}(\mathbb{G})$ be non-degenerate. Then $\mathcal{H}_\mu^p(\mathbb{G}) = C_1$ for all $1 \leq p \leq \infty$.

**Proof.** The case $p = \infty$ was proved in the general case in [9]. Let $1 \leq p < \infty$, and assume that $x \in \mathcal{H}_\mu^p(\mathbb{G})$, $x \notin C_1$ and $\|x\|_p = 1$. Then there exists $y \in \mathcal{L}^q(\mathbb{G})$ with $\|y\|_q = 1$ such that $\langle x, y \rangle = 1$ (we let $q = \infty$ for $p = 1$) and $\langle 1, y \rangle = 0$. Then from the proof of Theorem 2.4 (which we can also apply to the case of $p = 1$ and $q = \infty$, since $\mathcal{L}_\infty(\mathbb{G}) \subseteq \mathcal{L}^2(\mathbb{G})$ for a compact quantum group) we have $\Omega_{x,y} = 1$ and

$$\langle \varphi \ast x , y \rangle = \langle \varphi , 1 \rangle = 1,$$

where $\varphi$ is the Haar state on $\mathbb{G}$. Now, let $x_n \in \mathcal{L}_\infty(\mathbb{G})$ be such that $\|x_n - x\|_p \to 0$. Then

$$\langle \varphi \ast x , y \rangle = \lim_n \langle \varphi \ast x_n , y \rangle = \lim_n \langle \varphi , x_n \rangle \langle 1 , y \rangle = 0.$$ 

But this contradicts (2.3), and therefore $x = 0$. Hence, $\mathcal{H}_\mu^p(\mathbb{G}) = C_1$. \hfill $\square$
Remark 2.9. All of our results in this paper can be proved, by slight modifications of the arguments, for a state $\mu$ on the universal $C^*$-algebra $C_u(\mathbb{G})$.

References


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