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STABLE SURFACES WITH CONSTANT ANISOTROPIC MEAN CURVATURE AND CIRCULAR BOUNDARY

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ABSTRACT. We show that for an axially symmetric anisotropic surface energy, only stable disc-type surfaces with constant anisotropic mean curvature bounded by a circle which lies in a plane orthogonal to the rotation axis of the Wulff shape are rescalings of parts of the Wulff shape and the flat disc.

1. INTRODUCTION

We begin with a question. If we are given a variational problem for surfaces with boundary and the variational problem and the boundary of a critical surface admit the same symmetry, must the critical surface be symmetric? In [1] it was shown that in the case where the functional is the area, any stable constant mean curvature immersion of a (topological) disc which is bounded by a round circle is necessarily axially symmetric and is hence a spherical cap or a flat disc. It is worth noting that earlier, the first author [6] obtained the same conclusion under the assumption that the surface is an absolute minimizer of the volume constrained boundary value problem. Also, Kapouleas [5] has produced examples of higher genus constant mean curvature surfaces bounded by a circle, although little is known about their stability. For more than one boundary component, Patnaik [11] has produced a remarkable example of a non-axially symmetric minimizer for the volume constrained Plateau problem where the boundary is prescribed to be two co-axial circles. In this paper, we obtain an extension of the result of [1] to the case of constant anisotropic mean curvature.

Let $\gamma : S^2 \to \mathbf{R}^+$ be a positive smooth function on the unit sphere $S^2 \subset \mathbf{R}^3$. We consider γ as an anisotropic surface density. This means that $\gamma(\nu)$ gives the unit energy per unit area of a surface element having normal ν. The (anisotropic surface) energy of a surface Σ is thus

$$\mathcal{F} = \int_\Sigma \gamma(\nu)\, d\Sigma\,.$$

There is a canonical closed convex surface associated with $\mathcal{F}$, known as the *Wulff shape*, which is defined by

$$W = \partial \cap_{n\in S^2} \{Y \in \mathbf{R}^3\ ;\ Y\cdot n \le \gamma(n)\}\,.$$

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The surface W is the absolute minimizer of $\mathcal{F}$ among all closed surfaces which enclose the same three dimensional volume as W. In this paper, we will assume a *convexity condition* that W is a smooth strictly convex surface. In particular, its curvature K_W is everywhere positive. Also, we will identify any translation of W with W.

Now let $X : \Sigma \to \mathbf{R}^3$ be a sufficiently smooth, oriented immersed surface. If $X_\epsilon := X + \epsilon \dot{X} + ...$ is a compactly supported variation of X, then the first variation formula

$$\delta \mathcal{F} := \partial_\epsilon \mathcal{F}(X_\epsilon)_{\epsilon=0} = -\int_\Sigma \Lambda \dot{X} \cdot \nu \, d\Sigma$$

defines the anisotropic mean curvature Λ (cf. [9]). The equation $\Lambda \equiv$ constant characterizes volume constrained equilibria of $\mathcal{F}$.

A surface with constant anisotropic mean curvature is said to be stable if the second variation of the anisotropic surface energy $\mathcal{F}$ is non-negative for all compactly supported variations of the surface which fix the enclosed oriented three-volume.

Theorem 1.1. *Let $\mathcal{F}$ be a convex anisotropic energy with axially symmetric Wulff shape W. Denote by D the unit disc in $\mathbf{R}^2$. Let S^1 be a round circle which lies in a plane orthogonal to the rotation axis of W. Let $X : (D, \partial D) \to (\mathbf{R}^3, S^1)$ be an immersion of a stable surface with constant anisotropic mean curvature. Then $X(D) \subset rW$ for some $r > 0$ or $X(D)$ is a flat disc.*

2. Preliminaries

We assume that $\gamma(\nu)$ is a convex anisotropic energy density. Let $\chi : S^2 \to \mathbf{R^3}$ be the embedding such that $\chi(S^2) = W$ and χ^{-1} coincides with the Gauss map of W. If $\Sigma \to \mathbf{R}^3$ is an immersed surface with Gauss map $\nu : \Sigma \to S^2$, then $\xi = \chi \circ \nu$ is the *Cahn-Hoffman field* [3], which may be thought of as an anisotropic Gauss map. Since $T_{\xi(p)}W = T_p\Sigma$, we can consider $d\xi_p$ as a linear map of $T_p\Sigma$ to itself. Unlike the isotropic case, this map is not necessarily self-adjoint.

Let $\tilde{\gamma} : \mathbf{R}^3 - \{0\} \to \mathbf{R}^+$ denote the positive degree one homogeneous extension of γ, i.e. $\tilde{\gamma}(Y) = |Y|\gamma(Y/|Y|)$. The Cahn-Hoffman field ξ can be computed by [2]

$$\xi_p = (\nabla \tilde{\gamma}(\nu))_p \,,$$

and the anisotropic mean curvature is given by

$$\Lambda = -(\mathrm{div}\xi(\nu))_p \,.$$

We work locally on Σ and choose a complex coordinate so that the induced metric has the form $ds^2 = e^\mu |dz|^2$. We write

$$\xi_z = -\eta X_z - \beta e^{-\mu} X_{\bar{z}} \,, \quad \xi_{\bar{z}} = -\bar{\eta} X_{\bar{z}} - \bar{\beta} e^{-\mu} X_z \tag{1}$$

and

$$\Xi := -d\xi \cdot dX =: 2\Re\{\frac{\beta}{2}\, dz^2 + \frac{\eta\, e^\mu}{2}\, dz d\bar{z}\} \,. \tag{2}$$

The quantity η is called the *complex anisotropic mean curvature* [7], and it is given by

$$\eta = \frac{\Lambda}{2} + i\frac{\Gamma}{2} \,.$$

Here $\Gamma := \mathrm{trace}_\Sigma(d\xi \circ J)$, where J is the almost complex structure, i.e. $JX_z = iX_z$. The form Ξ is symmetric if and only if $\Gamma \equiv 0$. For example, $\Gamma \equiv 0$ always holds in the isotropic case, and it holds if both W and Σ are axially symmetric with the same rotation axis. However, if W is axially symmetric but not a sphere and X is an immersion of a helicoid, then $\Lambda \equiv 0$ holds but Γ is non-zero on Σ, [8].

The points on Σ where $d\xi_p = (-\Lambda/2)dX_p$ are called anisotropic umbilics (A-umbilics).

Lemma 2.1. *Let $X : \Sigma \to \mathbf{R}^3$ be an immersion with constant anisotropic mean curvature. Then the following are equivalent:*
(i) $p \in \Sigma$ *is an A-umbilic.*
(ii) $\beta(p) = 0$.
(iii) $(\Lambda^2 - 4K_\Sigma/K_W)(p) = 0$.

Proof. From (1), it is clear that a point $p \in \Sigma$ is A-umbilic if and only if $\beta(p) = \Gamma(p) = 0$. However, below we will show that $\beta(p) = 0$ implies $\Gamma(p) = 0$. Therefore, (i) and (ii) are equivalent.

Let $\{e_j\}_{j=1,2}$ be a positively oriented orthonormal basis for the tangent space at p which diagonalizes $d\chi_{\nu(p)}$, i.e. $d\chi_{\nu(p)}(e_j) = (1/\mu_j)e_i$. This is possible since $d\chi = D^2\gamma + \gamma I$, where $D^2\gamma$ denotes the Hessian of γ on S^2. Note that μ_j are positive, because the Wulff shape is strictly convex. Let $(-\sigma_{ij})$ be the matrix representation of $d\nu_p$ with respect to this basis. It is straightforward to check that

$$\Lambda = \frac{\sigma_{11}}{\mu_1} + \frac{\sigma_{22}}{\mu_2}\,, \quad \Gamma = \sigma_{12}\Big(\frac{-1}{\mu_1} + \frac{1}{\mu_2}\Big), \quad K_\Sigma/K_W = \frac{\sigma_{11}\sigma_{22} - \sigma_{12}^2}{\mu_1\mu_2}\,.$$

If z is a complex coordinate near p with $z(p) = 0$, then there exists an angle θ such that at $z = 0$, $e^{-\mu/2}X_z = (1/2)e^{i\theta}(e_1 - ie_2)$. We compute at p:

$$\begin{aligned}
-\frac{\beta}{2}e^{-\mu} &= e^{-\mu}d\xi(X_z)\cdot X_z \\
&= \frac{e^{2i\theta}}{4}d\chi d\nu(e_1 - ie_2)\cdot(e_1 - ie_2) \\
&= \frac{-e^{2i\theta}}{4}[\frac{\sigma_{11}}{\mu_1}e_1 + \frac{\sigma_{12}}{\mu_2}e_2 - i(\frac{\sigma_{12}}{\mu_1}e_1 + \frac{\sigma_{22}}{\mu_2}e_2)]\cdot(e_1 - ie_2) \\
&= \frac{-e^{2i\theta}}{4}(\frac{\sigma_{11}}{\mu_1} - \frac{\sigma_{22}}{\mu_2} - i\sigma_{12}(\frac{1}{\mu_1} + \frac{1}{\mu_2}))\,,
\end{aligned}$$

$$\begin{aligned}
4|\beta|^2e^{-2\mu} &= (\frac{\sigma_{11}}{\mu_1} - \frac{\sigma_{22}}{\mu_2})^2 + \sigma_{12}^2(\frac{1}{\mu_1} + \frac{1}{\mu_2})^2 \\
&= (\frac{\sigma_{11}}{\mu_1} + \frac{\sigma_{22}}{\mu_2})^2 - \frac{4\sigma_{11}\sigma_{22}}{\mu_1\mu_2} + \frac{4\sigma_{12}^2}{\mu_1\mu_2} + \sigma_{12}^2(\frac{1}{\mu_1} - \frac{1}{\mu_2})^2 \\
&= (\Lambda^2 - 4\frac{K_\Sigma}{K_W} + \Gamma^2)\,. \qquad (3)
\end{aligned}$$

Since

$$\Lambda^2 - 4\frac{K_\Sigma}{K_W} = \Big(\frac{\sigma_{11}}{\mu_1} - \frac{\sigma_{22}}{\mu_2}\Big)^2 + \frac{4\sigma_{12}^2}{\mu_1\mu_2} \geq 0$$

holds, $\beta(p) = 0$ implies $\Gamma(p) = 0$. Moreover, one sees that (ii) and (iii) are equivalent. □

Proposition 2.1. *Let $X : \Sigma \to \mathbf{R}^3$ be an immersion with constant anisotropic mean curvature. If the immersion is not totally anisotropic umbilic, then the anisotropic umbilic points are isolated. If p is an anisotropic umbilic and C is a sufficiently small closed curve around p, then* $\mathrm{Var}_C(\arg\beta) :=$ *the total variation of*

$\arg\beta$ over C is equal to twice the negative of the winding number of the anisotropic principal direction fields around C. In particular, $\mathrm{Var}_C(\arg\beta) > 0$ holds.

Proof. Let $v := aX_z + \bar{a}X_{\bar{z}} \neq 0$ be a tangent vector. We obtain from (1) that

$$d\xi(v) = -2\Re\{[a\eta + \bar{a}\bar{\beta}e^{-\mu}]X_z\}\,, \quad Jv = 2\Re\{iaX_z\}\,.$$

The condition for v to be an anisotropic principal direction is that $d\xi v \cdot Jv = 0$ holds. This is the same as

$$\Re\{ia[\bar{a}\bar{\eta} + a\beta e^{-\mu}]e^{\mu}\} = 0\,,$$

which gives, using the definition of η,

$$|a|^2\frac{\Gamma}{2}e^{\mu} - \Im\{a^2\beta\} = 0\,.$$

(This agrees with the well known condition $\Im\{a^2\Phi\} = 0$ in the isotropic case, where Φ is the Hopf differential.)

Let p be an A-umbilic and let C be a positively oriented curve around p which does not contain or pass through any other A-umbilic. This is possible since it was shown in [10] that the A-umbilic points are isolated. We assume that v now represents a vector in a continuous anisotropic direction field along C. Write $a = |a|e^{i\theta}$, $\beta = |\beta|e^{i\vartheta}$. We can write the previous equation as

$$\sin(\vartheta + 2\theta) = \frac{\Gamma e^{\mu}}{2|\beta|} = \frac{\Gamma}{\sqrt{\Lambda^2 - 4(K_\Sigma/K_W) + \Gamma^2}} < 1\,, \tag{4}$$

by the lemma and the assumption that C contains no A-umbilics. Note that the first equality in (4) is the same as

$$\vartheta + 2\theta - \arcsin\Big(\frac{\Gamma e^{\mu}}{2|\beta|}\Big) = 0\,.$$

However the last term on the left is a well defined continuous function along C, so its variation over C vanishes and we get

$$\mathrm{Var}_C\vartheta = -2\mathrm{Var}_C\theta\,.$$

Since it was shown in [10] that the winding number of the direction fields around an A-umbilic is negative, this completes the proof. $\square$

Corollary 2.1. *Let $\tilde{\Xi}$ denote the symmetrization of Ξ, i.e. $\tilde{\Xi}(u,v) = (1/2)(\Xi(u,v) + \Xi(v,u)$, and let T be an eigendirection of $\tilde{\Xi}$. Then the singularities of T are exactly the A-umbilic points, and the winding number of T around any A-umbilic is equal to $-(1/2)\mathrm{Var}_C\arg\beta$, where C is as above.*

Proof. From (2) and (3), it is seen that the singularities of T are exactly the A-umbilic points. The last statement is proved in the same way as the proposition except that now the Γ term is missing. $\square$

3. Proof of Theorem 1.1

Let $X : (D, \partial D) \to (\mathbf{R}^3, S^1)$ be an immersion with constant anisotropic mean curvature Λ. If we consider a smooth variation field $\dot{X} = u\nu + T$ where T is tangent to the immersion, then the pointwise variation of Λ is given by [9]

$$\dot{\Lambda} = \frac{1}{2}J[u]\,, \tag{5}$$

where J is the Jacobi operator of the immersion. This operator is given by

$$J[u] = \operatorname{div}[(D^2\gamma + \gamma I)\nabla u] + \langle (D^2\gamma + \gamma I)\, d\nu, d\nu\rangle u\,.$$

The endomorphism $(D^2\gamma + \gamma I)$ is positive definite at each point; this is just the convexity condition for the Wulff shape W. It follows that J is elliptic and self-adjoint.

The second variation of $\mathcal{F}$ for a volume-preserving variation which fixes the boundary with $\dot{X}$ as a variation vector field is $I[u] := -\int_D uJ[u]\, d\Sigma$, where $d\Sigma$ is the area element of X. Denote by λ_j the jth eigenvalue of the Dirichlet eigenvalue problem for J. If $\lambda_2 < 0$, then a suitable linear combination f of eigenfunctions of λ_1, λ_2 satisfies

$$f|_{\partial D} = 0, \quad I[f] < 0, \quad \int_D f\, d\Sigma = 0.$$

One obtains a volume-preserving variation of X which fixes the boundary with variation vector field $f\nu$. Hence X is unstable.

Proof of Theorem 1.1. The proof closely follows the proof of the result in [1]. As in [1] we consider the variation of X given by

$$X_\epsilon := \sigma_\epsilon X = X + \epsilon E_3 \times X + \mathcal{O}(\epsilon^2), \quad E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \sigma_\epsilon = \begin{pmatrix} \cos\epsilon & -\sin\epsilon & 0 \\ \sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is just a one parameter family of rotations with vertical axis applied to X. Therefore, if $\psi := E_3 \times X \cdot \nu$, then $J[\psi] = 0$ by (5) and $\psi|_{\partial D} \equiv 0$, since the boundary is set-wise fixed by the variation.

We will show that ψ is identically zero. We first assume that ψ is not identically zero, and we will show that $D \setminus \{\psi = 0\}$ has at least three components. (In fact, if there are three components, then there must be four, since ψ changes sign from one component to the next.) If this is true, then ψ is an eigenfunction belonging to some $\lambda_j = 0$ with $j \geq 3$ by Courant's Nodal Domain Theorem. Hence, X is unstable by the above remark.

We compute

$$(6) \qquad (\frac{1}{\mu_1} + \frac{1}{\mu_2})\partial_n \psi = 2e^{-\mu}\Im\{z^2\beta\}\,,$$

where n is the outward pointing unit normal along ∂D. The equality (6) will be proved in the Appendix. It is enough to show that $\partial_n\psi$ has at least three zeros on ∂D, since at each of the zeros, a branch of the nodal set must enter into D by the Hopf maximum principle [4]. To do this, we consider $\mathrm{Var}_{S^1}(\arg(z^2\beta)) = 4\pi + \mathrm{Var}_{S^1}(\arg\beta)$. It is enough to show that $\mathrm{Var}_{S^1}(\arg\beta) \geq 0$ holds, since then $\Im\{z^2\beta\}$ must have at least three zeros.

First assume that there are no A-umbilics on ∂D. By the corollary and general facts about indices of direction fields, $\mathrm{Var}_{S^1}(\arg\beta)$ is equal to -2 times the sum of the indices of a direction field of $\tilde{\Xi}$ in D. This is clearly non-negative.

The case where there are A-umbilics on ∂D is handled by the usual type of indentation argument. For each A-umbilic point of the boundary, $\mathrm{Var}_{S^1}(\arg\beta)$ is incremented by minus the winding number of a direction field of $\tilde{\Xi}$ around such a point, so again $\mathrm{Var}_{S^1}(\arg\beta)$ is non-negative.

Since the assumption that ψ is not identically zero implies that the surface is unstable, we can conclude that $\psi \equiv 0$ holds. From this it follows that the immersion X is axially symmetric. However, axially symmetric surfaces with constant anisotropic mean curvature are classified ([9]), and the only ones of disc type are either subsets of rW in the case $\Lambda = -2/r$ or they are flat discs in the case $\Lambda = 0$. The former are known to be stable, in fact minimizing, by a result known as Winterbottom's Theorem [12], and it is easy to show that planar surfaces are also stable. $\square$

Appendix

We will prove (6). Let $z = x + iy$ be the usual coordinate in the disc and let $\zeta = \log z = \log r + i\theta$ in $D \setminus \{0\}$. Then using the fact that $r \equiv 1$ on ∂D, we have

$$\frac{1}{2}(X_r - iX_\theta) = X_\zeta = zX_z\,, \qquad \frac{1}{2}(X_r + iX_\theta) = X_{\bar\zeta} = \bar z X_{\bar z}\,.$$

If n, t denote the unit conormal and tangent to ∂D, we have

$$n = e^{-\mu/2}X_r = e^{-\mu/2}(zX_z + \bar z X_{\bar z})\,, \qquad t = e^{-\mu/2}X_\theta = e^{-\mu/2}(izX_z - i\bar z X_{\bar z})\,.$$

We then obtain from (1) that

$$d\xi(t)\cdot n + d\xi(n)\cdot t = 2e^{-\mu}\Im\{z^2\beta\}\,. \tag{7}$$

Next note that since the surface is bounded by a circle, both n and t are principal directions of W. Specifically we have

$$d\chi(n) = \frac{1}{\mu_1}n\,, \qquad d\chi(t) = \frac{1}{\mu_2}t\,.$$

Let $(-\sigma_{ij})$ denote the matrix representing $d\nu$ with respect to the basis $\{n, t\}$. We easily obtain, using $d\xi = d\chi \circ d\nu$,

$$d\xi(t)\cdot n + d\xi(n)\cdot t = -(\frac{1}{\mu_1} + \frac{1}{\mu_2})\,\sigma_{12}\,. \tag{8}$$

Finally, we compute

$$\partial_n\psi = \partial_n(E_3 \times X \cdot \nu) = (E_3 \times n)\cdot\nu + (E_3 \times X)\cdot d\nu(n) = t\cdot d\nu(n) = -\sigma_{12}\,.$$

This with (7), (8) implies (6). $\square$

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