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INVERTIBLE WEIGHTED SHIFT OPERATORS WHICH ARE m-ISOMETRIES

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Abstract. For a bounded linear operator T on a complex Hilbert space $\mathcal{H}$, let $\Delta_{T,m} = \sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*m-k}T^{m-k}$ for $m \in \mathbb{N}$. T is called an m-isometry if $\Delta_{T,m} = 0$. In this paper, for every even number m, we give an example of invertible $(m+1)$-isometry which is not an m-isometry. Next we show that if T is an m-isometry, then the operator $\Delta_{T,m-1}$ is not invertible.

1. Introduction

J. Agler and M. Stankus published excellent papers about m-isometric operators, [1], [2] and [3]. They showed that m-isometries have interesting spectral properties. For example, if T is an m-isometry, then the approximate point spectrum of T lies on the unit circle. In [10], Patel showed that Weyl's theorem holds for a 2-isometry. Applying Uchiyama and Tanahashi's result [11], in [8] we showed that if T is an m-isometry, then T has the single valued extension property. Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be a set of all bounded linear operators on $\mathcal{H}$. Let $\binom{m}{k}$ be the binomial coefficient. For an operator $T \in B(\mathcal{H})$, let

$$\Delta_{T,m} = \sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*m-k}T^{m-k}.$$

T is said to be an m-isometry if $\Delta_{T,m} = 0$. Agler and Stankus proved that if T is an m-isometry, then $\Delta_{T,m-1} \geq 0$ (Proposition 1.5, [1]). It is easy to see that $T^*\Delta_{T,m}T - \Delta_{T,m} = \Delta_{T,m+1}$. Hence it holds that if T is an m-isometry, then T is an $(m+1)$-isometry. In [7], T. Bermudez, A. Martinon and E. Negrin studied characterizations of weighted shift operators which are m-isometries. In [1], Agler and Stankus proved that if m is even and T is an invertible m-isometry, then T is an $(m-1)$-isometry. In [4] A. Athavale proved that if S is a unilateral weighted shift

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$Se_n = w_n e_{n+1}$ with weights $w_n = \sqrt{1 + \frac{m}{n}}$ for $n = 1, 2, 3, \cdots$, then S is an $(m+1)$-isometry which is not an m-isometry. The operator S is not invertible. We have not seen invertible $(m+1)$-isometries which are weighted shifts for m even. We give an example of an invertible $(m+1)$-isometry which is not an m-isometry for every even number m. Next we prove that power bounded m-isometries are isometries, and if T is an m-isometry for a natural number $m \geq 2$, then the operator $\Delta_{T,m-1}$ is not invertible.

2. Invertible weighted shift of an m-isometry

Agler and Stankus showed the following result.

Proposition A (Proposition 1.23, [1])**.** *If T is an invertible m-isometry and m is even, then T is an $(m-1)$-isometry.*

Their proof is fine. We give another proof.

Proof. Since T is an m-isometry, it holds that

$$\Delta_{T,m-1} = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} T^{*m-1-k} T^{m-1-k} \geq 0. \tag{2.1}$$

It is easy to see that T^{-1} also is an m-isometry. Since T^{-1} is an m-isometry, it holds that

$$\Delta_{T^{-1},m-1} = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} (T^{-1})^{*m-1-k} (T^{-1})^{m-1-k} \geq 0. \tag{2.2}$$

Since $m-1$ is an odd number, by (2.2) we have

$$\begin{aligned} -\Delta_{T,m-1} &= \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} T^{*k} T^k \\ &= \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} T^{*m-1} (T^{-1})^{*m-1-k} (T^{-1})^{m-1-k} T^{m-1} \\ &= T^{*m-1} (\Delta_{T^{-1},m-1}) T^{m-1} \geq 0. \end{aligned}$$

Hence $\Delta_{T,m-1} \leq 0$. By (2.1) we have $\Delta_{T,m-1} = 0$, so the proof is complete. □

For every even number m, we give an example of an invertible $(m+1)$-isometry T which is not an m-isometry.

Theorem 1. *For any even number m, there exists an invertible $(m+1)$-isometry T which is not an m-isometry.*

Proof. Let m be any even number and put $\psi(x) = x(x+1)\cdots(x+m-1)$. Then $\psi(x)$ is a polynomial of even degree m. Hence $\psi(n) \geq 0$ for $n = 0, \pm 1, \pm 2, \cdots$. So, for any $k > 0$, we have $\psi(n) + k > 0$ for $n = 0, \pm 1, \pm 2, \cdots$. Let

$$w_n = \sqrt{\frac{\psi(n+1) + k}{\psi(n) + k}} > 0 \quad \text{for } n = 0, \pm 1, \pm 2, \cdots.$$

Let $\{e_n\}_{n=-\infty}^{\infty}$ be an orthonormal basis of $\mathcal{H} = \ell^2$ and T be a bilateral weighted shift such that $Te_n = w_n e_{n+1}$. We define

$$\begin{aligned} I_{m,n} &= w_n^2 w_{n+1}^2 \cdots w_{n+m-1}^2 - \binom{m}{1} w_n^2 w_{n+1}^2 \cdots w_{n+m-2}^2 \\ &+ \binom{m}{2} w_n^2 w_{n+1}^2 \cdots w_{n+m-3}^2 + \cdots + (-1)^{m-1} \binom{m}{m-1} w_n^2 + (-1)^m. \end{aligned}$$

Note that T is an m-isometry if and only if $I_{m,n} = 0$ for $n = 0, \pm 1, \pm 2, \cdots$. To show T is an $(m+1)$-isometry, we will show that $I_{m+1,n} = 0$ for $n = 0, \pm 1, \pm 2, \cdots$. We define a function $f(x)$ as

$$\begin{aligned} f(x) &= x^{n+m-1}(1-x)^{m+1} \\ &= x^{n+m-1} - \binom{m+1}{1} x^{n+m} + \binom{m+1}{2} x^{n+m+1} \\ &\quad + \cdots + (-1)^m \binom{m+1}{m} x^{n+2m-1} + (-1)^{m+1} x^{n+2m}. \end{aligned}$$

Note that $f^{(m)}(1) = 0$. By differentiating m times, we have

$$\begin{aligned} 0 &= f^{(m)}(1) \\ &= (n+m-1)(n+m-2)\cdots n \\ &\quad - \binom{m+1}{1} (n+m)(n+m-1)\cdots(n+1) \\ &\quad + \cdots + (-1)^m \binom{m+1}{m} (n+2m-1)(n+2m-2)\cdots(n+m) \\ &\quad + (-1)^{m+1}(n+2m)(n+2m-1)\cdots(n+m+1). \end{aligned}$$

Hence

$$\begin{aligned} &(-1)^{m+1} I_{m+1,n}\left(\psi(n)+k\right) \\ &= (n+m+1)(n+m+2)\cdots(n+2m) + k \\ &\quad - \binom{m+1}{1} \{(n+m)(n+m+1)\cdots(n+2m-1)+k\} \\ &\quad + \cdots + (-1)^m \binom{m+1}{m} \{(n+1)(n+2)\cdots(n+m)+k\} \\ &\quad + (-1)^{m+1} \{n(n+1)(n+2)\cdots(n+m-1)+k\} \\ &= f^{(m)}(1) + k(1-1)^{m+1} = 0. \end{aligned}$$

This implies that $I_{m+1,n} = 0$. Hence T is an $(m+1)$-isometry.

To show that T is not an m-isometry, we will show that $I_{m,n} \neq 0$ for $n = 0, \pm 1, \pm 2, \cdots$. We define a function $g(x)$ as

$$\begin{aligned} g(x) &= x^{n+m-1}(1-x)^m \\ &= x^{n+m-1} - \binom{m}{1} x^{n+m} + \binom{m}{2} x^{n+m+1} \\ &\quad + \cdots + (-1)^{m-1} \binom{m}{m-1} x^{n+2m-2} + (-1)^m x^{n+2m-1}. \end{aligned}$$

Note that $g^{(m)}(1) = m!(-1)^m$. By differentiating m times, we have

$$\begin{aligned} m!(-1)^m &= g^{(m)}(1) \\ &= (n+m-1)(n+m-2)\cdots n \\ &\quad - \binom{m}{1}(n+m)(n+m-1)\cdots(n+1) \\ &\quad + \cdots + (-1)^{m-1}\binom{m}{m-1}(n+2m-2)(n+2m-3)\cdots(n+m-1) \\ &\quad + (-1)^m(n+2m-1)(n+2m-2)\cdots(n+m). \end{aligned}$$

Hence

$$\begin{aligned} &(-1)^m I_{m,n}\left(\psi(n)+k\right) \\ &= (n+m)(n+m+1)\cdots(n+2m-1)+k \\ &\quad - \binom{m}{1}\{(n+m-1)(n+m)\cdots(n+2m-2)+k\} \\ &\quad + \cdots + (-1)^{m-1}\binom{m}{m-1}\{(n+1)(n+2)\cdots(n+m)+k\} \\ &\quad + (-1)^m\{n(n+1)(n+2)\cdots(n+m-1)+k\} \\ &= g^{(m)}(1) + k\,(1-1)^m = m!(-1)^m. \end{aligned}$$

This implies that $I_{m,n} = m!\left(\psi(n)+k\right)^{-1} \neq 0$. Hence T is not an m-isometry.

Finally, since $w_n \to 1$ $(n \to \pm\infty)$, we have $\sigma(T) = \{\lambda : |\lambda| = 1\}$ by Proposition 2.6.8 (b) of [9]. Therefore T is invertible. The proof is complete. □

Remark 1. This result provides an example of an $(m+1)$-isometry T which is not an m-isometry if we take $Te_n = w_n e_{n+1}$ for $n = 1, 2, \cdots$ where

$$w_n = \sqrt{\frac{(n+1)(n+2)\cdots(n+m)+k}{n(n+1)\cdots(n+m-1)+k}}$$

with $0 \leq k$. If $k = 0$, then

$$w_n = \sqrt{\frac{(n+1)(n+2)\cdots(n+m)+0}{n(n+1)\cdots(n+m-1)+0}} = \sqrt{\frac{n+m}{n}}.$$

This is the result of A. Athavale, Proposition 8 of [4].

3. Some properties of m-isometries

An operator $T \in B(\mathcal{H})$ is said to be power bounded if there exists a positive number M such that $\|T^n\| \leq M$ for every $n \in \mathbb{N}$.

Theorem 2. *A power bounded m-isometry is an isometry.*

To prove this result, we will use Berberian's method (cf. [6], [12]).

Proposition (Lemma 2.7, [12])**.** *Let $\mathcal{H}$ be a complex Hilbert space. Then there exist a Hilbert space $\mathcal{H}^\circ \supset \mathcal{H}$ and a unital linear map $\circ : B(\mathcal{H}) \to B(\mathcal{H}^\circ)$ such that*

$$\begin{aligned}
&\text{(i) } (ST)^\circ = S^\circ T^\circ, (T^*)^\circ = (T^\circ)^*, \|T\| = \|T^\circ\|,\\
&\text{(ii) } S \le T \Longrightarrow S^\circ \le T^\circ,\\
&\text{(iii) } \sigma(T) = \sigma(T^\circ), \sigma_a(T) = \sigma_a(T^\circ) = \sigma_p(T^\circ).
\end{aligned}$$

We prepare notation and a lemma. For a unit vector $x \in \mathcal{H}$, assume that $(T^*T - I)x = a_1 x$. Let $a_n = \langle \Delta_{T,n} x, x\rangle$ and $b_n = \|T^n x\|^2$ for $n = 1, 2, 3, \cdots$. Then we have the following lemma.

Lemma. *For an operator $T \in B(\mathcal{H})$ with the above notation, it holds that*

$$b_n = a_n + \binom{n}{n-1} a_{n-1} + \binom{n}{n-2} a_{n-2} + \cdots + n a_1 + 1. \tag{3.1}$$

Proof. Note that

$$b_1 = \|Tx\|^2 = \langle T^*Tx, x\rangle = \langle a_1 x + x, x\rangle = a_1 + 1.$$

Since

$$\begin{aligned}
T^{*2}T^2 &= (T^{*2}T^2 - 2T^*T + I) + 2(T^*T - I) + I\\
&= \Delta_{T,2} + 2\Delta_{T,1} + I,
\end{aligned}$$

we have

$$\begin{aligned}
b_2 &= \langle T^{*2}T^2 x, x\rangle\\
&= \langle \Delta_{T,2} x, x\rangle + 2\langle \Delta_{T,1} x, x\rangle + \langle x, x\rangle\\
&= a_2 + 2a_1 + 1.
\end{aligned}$$

Hence (3.1) holds for $n = 1, 2$. Note that

$$\begin{aligned}
t^n &= (t - 1 + 1)^n\\
&= (t-1)^n + \binom{n}{1}(t-1)^{n-1} + \binom{n}{2}(t-1)^{n-2} + \cdots + n(t-1) + 1.
\end{aligned}$$

Hence

$$T^{*n}T^n = \Delta_{T,n} + \binom{n}{1}\Delta_{T,n-1} + \binom{n}{2}\Delta_{T,n-2} + \cdots + n\Delta_{T,n-1} + I$$

and

$$\begin{aligned}
b_n &= \langle T^{*n}T^n x, x\rangle\\
&= a_n + \binom{n}{1} a_{n-1} + \binom{n}{2} a_{n-2} + \cdots + n a_1 + 1\\
&= a_n + \binom{n}{n-1} a_{n-1} + \binom{n}{n-2} a_{n-2} + \cdots + n a_1 + 1.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 2. Let T be a power bounded m-isometry. Since $T^*T - I$ is self-adjoint, it suffices to show that

$$\sigma(T^*T - I) = \sigma_a(T^*T - I) = \{0\}.$$

Assume that there exists a non-zero real number $a \in \sigma(T^*T - I)$. Since a belongs to the approximate point spectrum of $T^*T - I$, by Berberian's method we consider

an extension $\mathcal{H}^\circ$ of $\mathcal{H}$ and the mapping $S \to S^\circ$ of $B(\mathcal{H})$ into $B(\mathcal{H}^\circ)$. Then a is an eigen-value of $T^{*\circ}T^\circ - I^\circ$ and T° is also a power bounded m-isometry. For simplification, we denote T° by T. Since a is an eigen-value of $T^*T - I$, there exists a unit vector x such that $(T^*T - I)x = ax$. Hence

$$\langle (T^*T - I)x, x\rangle = \langle ax, x\rangle = a.$$

Let $a_1 = a$ and $a_n = \langle \Delta_{T,n}x, x\rangle$. Then since T is an m-isometry, we have $a_m = a_{m+1} = \cdots = a_n = 0$ for $n > m$. By the Lemma, we have

$$\|T^n x\|^2 = \binom{n}{m-1} a_{m-1} + \binom{n}{m-2} a_{m-2} + \cdots + na_1 + 1.$$

Hence

$$\frac{\|T^n x\|^2}{\binom{n}{m-1}} = a_{m-1} + \frac{1}{\binom{n}{m-1}} \left\{ \binom{n}{m-2} a_{m-2} + \cdots + na_1 + 1 \right\}.$$

Since T is power bounded, we have $a_{m-1} = 0$ by $n \to \infty$. Repeating this, we have

$$a_{m-1} = a_{m-2} = \cdots = a_1 = a = 0.$$

This is a contradiction, so the proof is complete. □

Patel (Corollary 2.8, [10]) proved that if T is a 2-isometry, then $1 \in \sigma(T^*T)$. That is, if T is a 2-isometry, then $0 \in \sigma(\Delta_{T,1})$. We now generalize this result as follows.

Theorem 3. *If T is an m-isometry for $m \geq 2$, then $0 \in \sigma(\Delta_{T,m-1})$.*

Proof. For simplification, we denote $\Delta_{T,m-1}$ by Δ. Assume that $0 \notin \sigma(\Delta)$. Since $\Delta_{T,m} = T^*\Delta T - \Delta = 0$, it holds that $T^*\Delta T = \Delta$. Let $S = \Delta^{1/2}T\Delta^{-1/2}$. Then

$$S^*S = (\Delta^{-1/2}T^*\Delta^{1/2})(\Delta^{1/2}T\Delta^{-1/2}) = I.$$

Hence S is an isometry and $T = \Delta^{-1/2}S\Delta^{1/2}$. So T is similar to the isometry S. Since S is power bounded, T is an isometry by Theorem 3. Hence $\Delta_{T,1} = 0$, and so $\Delta_{T,m-1} = \Delta = 0$. This contradicts our assumption that $0 \notin \sigma(\Delta)$, so the proof is complete. □

Remark 2. Since Weyl's theorem holds for an isometry, Weyl's theorem holds for a power bounded m-isometry by Theorem 3 (cf. [5], [10]).

Finally we show the following result. For a 2-isometry, Patel proved it (Theorem 2.1, [10]).

Theorem 4. *A power of an m-isometry is again an m-isometry.*

Proof. Let T be an m-isometry, i.e.,

$$\Delta_{T,m} = T^{*m}T^m - mT^{*m-1}T^{m-1} + \cdots + (-1)^m I = 0.$$

We prove T^k is also an m-isometry. Write

$$\begin{aligned}(t^k - 1)^m &= (t-1)^m(t^{k-1} + t^{k-2} + \cdots + t + 1)^m \\ &= (t^m - mt^{m-1} + \cdots + (-1)^m)(a_{m(k-1)}t^{m(k-1)} + \cdots + a_1 t + a_0).\end{aligned}$$

Hence we have

$$\Delta_{T^k,m} = \sum_{j=0}^{m(k-1)} a_j T^{*j} \left(\Delta_{T^k,m}\right) T^j = 0.$$

So the proof is complete. □

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