SHEAVES ON $\mathbb{P}^1 \times \mathbb{P}^1$, BIGRADED RESOLUTIONS, AND COADJOINT ORBITS OF LOOP GROUPS

ROGER BIELAWSKI AND LORENZ SCHWACHHÖFER

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Abstract. We construct a canonical linear resolution of acyclic 1-dimensional sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ and discuss the resulting natural Poisson structure.

1. Introduction

The goal of this paper is to present a (yet another) variation on a theme developed by several authors, notably Moser, Adams, Harnad, Hurtubise, Previato [13], [1]–[5], and relating integrable systems, rank $r$ perturbations, spectral curves and their Jacobians, and coadjoint orbits of loop groups.

Let us briefly recall that, given matrices $A, Y, F, G$ of size, respectively, $N \times N$, $r \times r$, $N \times r$, and $r \times N$, one defines a $\mathfrak{gl}_r(\mathbb{C})$-valued rational map

\[ Y + G(A - \lambda)^{-1}F, \]

i.e. an element of the loop algebra $\widetilde{\mathfrak{gl}}(r)^{\mathbb{C}}$, consisting of loops extending holomorphically to the outside of some circle $S^1 \subset \mathbb{C}$. This determines a (shifted) reduced coadjoint orbit in $\widetilde{\mathfrak{gl}}(r)^{\mathbb{C}}$ (see Remark 4.5 for a definition). On the other hand, the polynomial (1.1) also determines (generically) a curve $S$ and a line bundle $L$ of degree $g + r - 1$: the curve is defined as the spectrum of (1.1), and $L$ is the dual of the eigenbundle of (1.1). This describes $S$ as an affine curve in $\mathbb{C}^2$, and the isospectral flows, corresponding to Hamiltonians on the space of rank $r$ perturbations, linearise on the Jacobian of the projective model of $S$.

In fact, as shown by Adams, Harnad, and Hurtubise [1,2], it is more convenient to compactify $S$ inside a Hirzebruch surface $F_d$, $d \geq 1$. This results in singularities, which may be partially resolved, but it gives a particularly nice description of $\text{Jac}^0(S)$, i.e. of the flow directions.

In this paper, we consider a different compactification of $S$, namely inside $\mathbb{P}^1 \times \mathbb{P}^1$ and defined as

\[ S = \left\{ (z, \lambda) \in \mathbb{P}^1 \times \mathbb{P}^1; \det \begin{pmatrix} Y - z & G \\ F & A - \lambda \end{pmatrix} = 0 \right\}. \]

This is a very natural thing to do, but we know of only one occurrence in the literature: the paper of Sanguinetti and Woodhouse [17] (we are grateful to Philip Boalch for this reference). In that paper, in addition to other results, the authors use the above compactification to give a nice picture of the duality phenomenon discussed in [3]. Our application is to another subtlety of the rank $r$ perturbation

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This gives us a linear polynomial matrix natural linear resolution of such a sheaf, very much in the spirit of Beauville [6].

Let \( L \in \text{Jac}^{g-r+1}(S) \) corresponds to \((A,Y,F,G)\) with rank \( F = \text{rank} \ G = r \) if and only if \( L \) satisfies:

\[
H^0(S,L(0,-1)) = H^1(S,L(0,-1)) = 0, \quad H^0(S,L(-1,0)) = 0, \quad H^1(S,L(1,-2)) = 0.
\]

We are interested in more than line bundles on smooth curves in \( \mathbb{P}^1 \times \mathbb{P}^1 \). The above approach generalises to acyclic (i.e. semistable) 1-dimensional sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \), with a fixed bigraded Hilbert polynomial. In §§2 and 3, we construct a natural linear resolution of such a sheaf, very much in the spirit of Beauville [6]. This gives us a linear polynomial matrix \( M(z,\lambda) \) (up to a certain group action). If the support of the sheaf is a smooth curve of bidegree \((r,N)\), then the matrix has size \( r \times N \). As long as the point \((\alpha,\infty)\) does not belong to the support of the sheaf, then the matrices \( M(z,\lambda) \) can be identified with the quadruples \( A,Y,F,G \).

The (generic) symplectic leaves are known, from [1,5], to be reduced coadjoint orbits of loop groups. Our aim is to describe these symplectic leaves directly in terms of sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \). We show that the (generic) symplectic leaves \( K^*_Q \simeq \mathcal{O}(2,2) \), one obtains a Poisson structure on \( M_Q(r,N) \) as a map

\[
T_{[\mathcal{F}]} M_Q(r,N) \simeq \text{Ext}^1_Q(\mathcal{F},\mathcal{F} \otimes K_Q) \xrightarrow{z^*} \text{Ext}^1_Q(\mathcal{F},\mathcal{F}) \simeq T_{[\mathcal{F}]} M_Q(r,N).
\]

2. Acyclic sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \) and their resolutions

**Definition 2.1.** Let \( X \) be a complex manifold and let \( \mathcal{F} \) be a coherent sheaf on \( X \). Then:

(i) The support of \( \mathcal{F} \) is the complex subspace \( \text{supp} \mathcal{F} \) of \( X \) defined as the zero-locus of the annihilator (in \( \mathcal{O}_X \)) of \( \mathcal{F} \). The dimension \( \dim \mathcal{F} \) of \( \mathcal{F} \) is the dimension of its support.

(ii) \( \mathcal{F} \) is pure if \( \dim \mathcal{E} = \dim \mathcal{F} \) for all nontrivial coherent subsheaves \( \mathcal{E} \subset \mathcal{F} \).

(iii) \( \mathcal{F} \) is acyclic if \( H^*(\mathcal{F}) = 0 \).

**Remark 2.2.** In the case of 1-dimensional sheaves on a smooth surface \( X \), purity of \( \mathcal{F} \) means that at every point \( x \in \text{supp} \mathcal{F} \), the skyscraper sheaf \( \mathcal{C}_x \) does not embed into \( \mathcal{F}_x \). In addition, a 1-dimensional sheaf \( \mathcal{F} \) on a smooth surface \( X \) is pure if and
only if it is reflexive; i.e., after performing the duality \( F \mapsto \text{Ext}^1_X(F, K_X) \) twice, we obtain \( F \) back (up to isomorphism) (see [9, \S 1.1]).

In the remainder of the paper, all sheaves are coherent.

We shall now consider sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \). For any \( p, q \in \mathbb{Z} \) we denote by \( \mathcal{O}(p, q) \) the line bundle \( \pi_1^* \mathcal{O}(p) \otimes \pi_2^* \mathcal{O}(q) \), where \( \pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \) are the two projections. We shall also denote by \( \zeta \) and \( \eta \) the two affine coordinates on \( \mathbb{P}^1 \times \mathbb{P}^1 \).

Let \( F \) be a sheaf on \( \mathbb{P}^1 \times \mathbb{P}^1 \). Associated to \( F \) is its bigraded Hilbert polynomial

\[
\sum_{x, y \in \mathbb{Z}} \chi(F(x, y)).
\]

(2.1) The sheaf \( F \) is 1-dimensional if and only if \( P_F \) is linear.

We begin by describing a canonical resolution of acyclic 1-dimensional sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \).

**Theorem 2.3.** Let \( F \) be a 1-dimensional acyclic sheaf on \( \mathbb{P}^1 \times \mathbb{P}^1 \). Then \( F \) has a linear resolution by locally free sheaves of the form

\[
0 \to \mathcal{O}(-2, -1)^{\oplus k} \oplus \mathcal{O}(-1, -2)^{\oplus l} \to \mathcal{O}(-1, -1)^{\oplus (k+l)} \to F \to 0,
\]

(2.2) for some \( k, l \geq 0 \).

Conversely, any \( F \) defined as a cokernel of a map \( M(\zeta, \eta) \) as above with \( \det M(\zeta, \eta) \neq 0 \) is acyclic and 1-dimensional.

**Remark 2.4.** Let \( F \) be a 1-dimensional acyclic sheaf on \( \mathbb{P}^1 \times \mathbb{P}^1 \) with \( P_F(x, y) = lx + ky \). Then \( F \) is semistable with respect to \( \mathcal{O}(1, 1) \).

**Remark 2.5.** This resolution is canonical, but not necessarily minimal, in the sense of being obtained from the minimal resolution of the bigraded module \( \bigoplus_{i, j \in \mathbb{Z}} H^0(F(i, j)) \).

**Proof.** Let \( h^0(F(0, 1)) = k \) and \( h^0(F(1, 0)) = l \) so that \( P_F = lx + ky \). Let \( E = F(1, 1) \), and let \( \Gamma_*(\mathcal{E}) = \bigoplus_{i, j \in \mathbb{Z}} H^0(F(i, j)) \) be the associated bigraded module over the bigraded ring \( S = \bigoplus_{i, j \in \mathbb{Z}} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(i, j)) \). Furthermore, let \( \Gamma_*(\mathcal{E})|_{i \geq 0} = \bigoplus_{i, j \geq 0} H^0(F(i, j)) \) be its truncation. Owing to [12, Lemma 6.8], the sheaf associated to \( \Gamma_*(\mathcal{E})|_{i \geq 0} \) is again \( \mathcal{E} \). Moreover, [12, Theorem 6.9] implies, as \( \mathcal{E}(-1, -1) \) is acyclic, that the natural map

\[
H^0(\mathcal{E}) \otimes H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(p, q)) \to H^0(\mathcal{E}(p, q))
\]

is surjective for any \( p, q \geq 0 \). Therefore, we have a surjective homomorphism

\[
S^{\oplus (k+l)} \to \Gamma_*(\mathcal{E})|_{i \geq 0} \to 0
\]

of bigraded \( S \)-modules. Since \( \mathcal{E} \) is of pure dimension 1, its projective dimension is 1, and, hence, the above homomorphism extends to a linear free resolution

\[
0 \to \bigoplus_{i=1}^{k+l} S(-p_i, -q_i) \to \bigoplus_{i=1}^{k+l} S \to \Gamma_*(\mathcal{E})|_{i \geq 0} \to 0,
\]

where \( p_i, q_i \geq 0 \) and \( p_i + q_i > 0 \) for each \( i \). The corresponding sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \) give us a locally free resolution of \( \mathcal{E} \):

\[
0 \to \bigoplus_{i=1}^{k+l} \mathcal{O}(-p_i, -q_i) \to \bigoplus_{i=1}^{k+l} \mathcal{O} \to \mathcal{E} \to 0.
\]

(2.3)
Since \( H^*(\mathcal{E}(-1,-1)) = 0 \), either \( p_i = 0 \) or \( q_i = 0 \) for every \( i \). Since \( h^0(\mathcal{E}(-1,0)) = k \), we deduce, after tensoring \( E \) with \( \mathcal{O}(-1,0) \), that \( \sum p_i = k \). Similarly \( \sum q_i = l \). Since \( h^1(\mathcal{E}) = 0 \), none of the \( p_i \) or \( q_i \) can be greater than 1, and so all nonzero \( p_i \) and all nonzero \( q_i \) are equal to 1. This proves the existence of resolution \( 2.2 \).

Conversely, if \( \mathcal{F} \) admits a resolution of the form \( 2.2 \), then it is 1-dimensional.

The long exact cohomology sequence implies that \( \mathcal{F} \) is acyclic. \( \square \)

Let us write \( n = k + l \). The polynomial matrix \( M(\zeta, \eta) \) in \( 2.3 \) has size \( n \times n \) and is of the form

\[
2.4 \quad \begin{pmatrix} A_0 + A_1 \zeta & B_0 + B_1 \eta \end{pmatrix},
\]

with \( A_0, A_1 \in \text{Mat}_{n,k}(\mathbb{C}), B_0, B_1 \in \text{Mat}_{n,l}(\mathbb{C}) \). Let us denote by \( \mathcal{A}(k,l) \) the space of such matrices with nonzero determinant. The group \( GL_n(\mathbb{C}) \times GL_k(\mathbb{C}) \times GL_l(\mathbb{C}) \) acts on \( \mathcal{M}(k,l) \) via

\[
2.5 \quad (g, h_1, h_2) \cdot \begin{pmatrix} A(\zeta) & B(\eta) \end{pmatrix} = g \begin{pmatrix} A(\zeta) & B(\eta) \end{pmatrix} \begin{pmatrix} h_1^{-1} & 0 \\ 0 & h_2^{-1} \end{pmatrix},
\]

and we can restate Theorem \( 2.3 \) as follows:

**Corollary 2.6.** There exists a natural bijection between

(a) isomorphism classes of 1-dimensional acyclic sheaves \( \mathcal{F} \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) such that \( h^0(\mathcal{F}(0,1)) = k \), \( h^0(\mathcal{F}(1,0)) = l \)

and

(b) orbits of \( GL_{k+l}(\mathbb{C}) \times GL_k(\mathbb{C}) \times GL_l(\mathbb{C}) \) on \( \mathcal{A}(k,l) \).

\( \square \)

For a sheaf defined by \( 2.2 \), we can describe its support as follows. As a set, the support of \( \mathcal{F} \) is

\[
S = \{ (\zeta, \eta) \in \mathbb{P}^1 \times \mathbb{P}^1; \text{ det } M(\zeta, \eta) = 0 \}.
\]

Let us write \( \text{det } M(\zeta, \eta) = \prod_{i=1}^s q_i(\zeta, \eta)^{k_i} \), where \( q_i \) are irreducible polynomials.

We define the minimal polynomial \( p_M(\zeta, \eta) \) of \( M \) as \( \prod_{i=1}^s q_i(\zeta, \eta)^{r_i} \), where

\[
r_i = \max\{a_i b_i; \text{ at a generic point, } M(\zeta, \eta) \text{ has a Jordan block of size } a_i \text{ with eigenvalue } q_i(\zeta, \eta)^{b_i} \}.
\]

Then:

**Proposition 2.7.** The support of \( \mathcal{F} \) is the curve \( \{ (S, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}/(p_M)) \} \).

\( \square \)

Let us now fix the support \( S \). For simplicity, we shall assume that it is an integral curve in the linear system \( \mathcal{O}(k,l) \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \); i.e. \( S \) is given by an irreducible polynomial \( P(\zeta, \eta) \) of bidegree \( (k, l) \), \( k, l \geq 1 \). This immediately implies that the rank of \( \mathcal{F} \) is constant; i.e. \( \mathcal{F} \) is locally free. Theorem \( 2.3 \) and Corollary \( 2.6 \) imply

**Corollary 2.8.** Let \( P(\zeta, \eta) \) be an irreducible polynomial of bidegree \( (k, l) \) and \( S = \{ (\zeta, \eta); P(\zeta, \eta) = 0 \} \) be the corresponding integral curve of genus \( g = (k-1)(l-1) \). There exists a canonical biholomorphism

\[
\text{Jac}^{g-1}(S) - \Theta \simeq \{ M \in \mathcal{A}(k,l); \text{ det } M = P \} / GL_n(\mathbb{C}) \times GL_k(\mathbb{C}) \times GL_l(\mathbb{C}).
\]

Similarly, let \( \mathcal{U}_S(r,d) \) be the moduli space of semistable vector bundles (locally free sheaves) on \( S \). For \( d = r(g-1) \) define the generalised theta divisor \( \Theta \) as the set of bundles with nonzero section. Then we have:
Corollary 2.9. Let $P(\zeta, \eta)$ be an irreducible polynomial of bidegree $(k, l)$ and $S = \{(\zeta, \eta); P(\zeta, \eta) = 0\}$ be the corresponding integral curve of genus $g = (k - 1)(l - 1)$. There exists a canonical biholomorphism

$$U \simeq S \simeq \{ M \in \mathcal{A}(kr, lr); \ det M = P' \}/GL_{nr}(\mathbb{C}) \times GL_{kr}(\mathbb{C}) \times GL_{lr}(\mathbb{C}).$$

3. A geometric resolution

There is a much more geometric way of constructing resolution \[2.2\], which works under mild assumptions on the sheaf $F$ (cf. \[7\] for the case of $\sigma$-sheaves).

Definition 3.1. Let $F$ be a 1-dimensional sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. We say that $F$ is bipure if $F$ has no nontrivial coherent subsheaves supported on $\{(z) \times \mathbb{P}^1 \text{ or on } \mathbb{P}^1 \times \{z\}\}$ for any $z \in \mathbb{P}^1$.

Remark 3.2. Observe that bipure implies pure.

Now let $F$ be an acyclic and bipure sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert polynomial $lx + ky$. As in the proof of Theorem \[2.3\] we consider the sheaf $E = F(1, 1)$. Let $D_\zeta$ and $D_\eta$ denote the divisors $\{(\zeta) \times \mathbb{P}^1, \mathbb{P}^1 \times \{\eta\}\}$. We set

$$V_\zeta = \{s \in H^0(E); s|_{D_\zeta} = 0\}, \quad W_\eta = \{s \in H^0(E); s|_{D_\eta} = 0\}.$$

For any $\zeta$ and $\eta$, consider the maps

$$E(-1, 0) \to E, \quad E(0, -1) \to E$$

given by multiplication by global nonzero sections of $O(1, 0)$ and $O(0, 1)$, vanishing at $\zeta$ and $\eta$, respectively. Since $E$ is bipure, these maps are injective, and therefore $V_\zeta \simeq H^0(E(-1, 0))$, $W_\eta \simeq H^0(E(0, -1))$ for any $\zeta, \eta$. In particular, $\dim V_\zeta = k, \dim W_\eta = l$, for any $\zeta$ and $\eta$. Therefore, $\zeta \mapsto V_\zeta$ and $\eta \mapsto W_\eta$ are subbundles of $H^0(E) \otimes O$ on $\mathbb{P}^1$. They are isomorphic to $H^0(E(-1, 0)) \otimes O(-1)$ and to $H^0(E(0, -1)) \otimes O(-1)$. The isomorphism is realised explicitly via the map

$$H^0(E(-1, 0)) \otimes O(-1) \ni (s, (a, b)) \mapsto (b\zeta - a)s \in H^0(E)$$

(here $(a, b) \in l$, where $l$ is the fibre of $O(-1)$ over $l$), and similarly for the subbundle $W$. We now define a vector bundle $U$ on $\mathbb{P}^1 \times \mathbb{P}^1$, the fibre of which at $\zeta, \eta$ is $V_\zeta \oplus W_\eta$; i.e.

$$U \simeq (H^0(E(-1, 0)) \otimes O(-1)) \oplus (H^0(E(0, -1))) \otimes O(0, -1)).$$

We obtain an injective map of sheaves $U \to H^0(E) \otimes O$. Let $G$ be the cokernel, i.e.

$$0 \to U \to H^0(E) \otimes O \to G \to 0.$$

We claim that $G \simeq E$, and so $\[3.2\]$ is a natural resolution of $E$. To prove this, tensor the resolution $\[2.2\]$ by $O(1, 1)$ to obtain

$$0 \to O(-1, 0)^{\oplus l} \oplus O(0, -1)^{\oplus k} \to M(\zeta_0, \eta_0) \to G \to 0.$$

Clearly, the middle term is identified with $H^0(E) \otimes O$. For any $\zeta_0$, consider the image of $M(\zeta_0, \eta)$ restricted to $O(-1, 0)^{\oplus k}|_{\zeta_0} \oplus 0$. This image does not depend on $\eta$, and since $F$ is bipure, it is exactly $V_{\zeta_0}$, defined in $\[3.1\]$, i.e. sections vanishing on $\zeta_0 \times \mathbb{P}^1$. Similarly, for any $\eta_0$, the image of $M(\zeta, \eta_0)$ restricted to $0 \oplus O(0, -1)^{\oplus l}|_{\eta_0}$ is precisely $W_{\eta_0}$. Hence, there are canonical isomorphisms between both first and second terms in resolutions $\[3.2\]$ and $\[3.3\]$ which commute with the horizontal maps. Therefore $G \simeq E$. 

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4. Poisson structure and orbits of loop groups

According to Corollary 2.6 acyclic sheaves with the Hilbert polynomial $lx + ky$ correspond to orbits of $GL_{k+l}(\mathbb{C}) \times GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $\mathcal{A}(k, l)$, where $\mathcal{A}(k, l)$ is the set of polynomial matrices defined in (2.4) and the action is given in (2.5).

We now make the following assumption about the sheaf $\mathcal{F}$:

$$(\infty, \infty) \notin \text{supp} \mathcal{F}.$$ 

This can, of course, always be achieved via an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. In terms of the matrix $M(\zeta, \eta)$ corresponding to $\mathcal{F}$, (4.1) means that $\det(A_1, B_1) \neq 0$. We can, therefore, use the action of $GL_{k+l}(\mathbb{C})$ to make $(A_1, B_1)$ equal to minus the identity matrix so that $M(\zeta, \eta)$ becomes

$$
\begin{pmatrix}
X - \zeta & F \\
G & Y - \eta
\end{pmatrix}, \quad X \in \text{Mat}_{k,k}(\mathbb{C}), \quad Y \in \text{Mat}_{l,l}(\mathbb{C}), \quad G, F^T \in \text{Mat}_{l,k}(\mathbb{C}).
$$

The residual group action is that of conjugation by the block-diagonal $GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$. We denote this group by $K$.

**Remark 4.1.** We are, essentially, in the situation of [5]. The only difference is that we do not fix $X$ or $Y$.

We denote by $\mathcal{M}(k, l)$ the space of all matrices of the form (4.2), which we identify with quadruples $(X, Y, F, G)$ as above. The action of $K = GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $\mathcal{M}(k, l)$ is given by

$$(g, h)(X, Y, F, G) = (gXg^{-1}, hYh^{-1}, gFh^{-1}, hGg^{-1}).$$

Let us also write $\mathcal{S}(k, l)$ for the set of isomorphism classes of acyclic sheaves with Hilbert polynomial $lx + ky$ on $\mathbb{P}^1 \times \mathbb{P}^1$ which satisfy (4.1). The content of Corollary 2.6 is that there exists a natural bijection

$$\mathcal{M}(k, l)/K \simeq \mathcal{S}(k, l).$$

4.1. Poisson structure. The vector space $\text{Mat}_{k,l} \times \text{Mat}_{l,k}$ has a natural $K$-invariant symplectic structure: $\omega = \text{tr}(dF \wedge dG)$. On the other hand, $\text{Mat}_{k,k} \simeq \mathfrak{gl}_k(\mathbb{C})^*$ and $\text{Mat}_{l,l} \simeq \mathfrak{gl}_l(\mathbb{C})^*$ have canonical Poisson structures, and therefore $\mathcal{M}(k, l)$ has a natural $K$-invariant Poisson structure. If $\mathcal{M}(k, l)^0$ is the subset of $\mathcal{M}(k, l)$ on which the action of $K$ is free and proper, then $\mathcal{M}(k, l)^0/K$ is a Poisson manifold, and, consequently, we obtain a Poisson structure on the corresponding subset of acyclic sheaves with Hilbert polynomial $lx + ky$ and satisfying (4.1). We shall now describe symplectic leaves of $\mathcal{M}(k, l)^0/K$ in terms of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$.

First of all, let us describe sheaves corresponding to symplectic leaves in $\mathcal{M}(k, l)$. Such a leaf is determined by fixing conjugacy classes of $X$ and $Y$. On the other hand, conjugacy classes of $k \times k$ matrices correspond to isomorphism classes of torsion sheaves on $\mathbb{P}^1$, of length $k$. This correspondence is given by associating to a matrix $X \in \text{Mat}_{k,k}(\mathbb{C})$ the sheaf $\mathcal{G}$ via

$$0 \to \mathcal{O}(-1)^{\oplus k} \xrightarrow{X - \zeta} \mathcal{O}^{\oplus k} \to \mathcal{G} \to 0.$$ 

If, for example, $X$ is diagonalisable with distinct eigenvalues $\zeta_1, \ldots, \zeta_r$ of multiplicities $k_1, \ldots, k_r$, then $\mathcal{G} \simeq \bigoplus_{i=1}^r \mathbb{C}^{k_i}|_{\zeta_i}$; i.e. $\mathcal{G}|_{\zeta_i}$ is the skyscraper sheaf of rank $k_i$.

**Proposition 4.2.** Let $P$ be a conjugacy class of $k \times k$ matrices. The bijection (4.4) induces a bijection between
(i) orbits of $GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $\{(X,Y,F,G) \in \mathcal{M}(k,l); X \in P\}$ and
(ii) isomorphism classes of sheaves $\mathcal{F}$ in $\mathcal{S}(k,l)$ such that $\mathcal{F}|_{\eta=\infty}$ is isomorphic to $G$ defined by (4.5).

Proof. At $\eta = \infty$, the matrix (4.2) becomes \(\begin{pmatrix} X - \zeta & 0 \\ G & -1 \end{pmatrix}\). The statement follows from (4.3) and (2.2). \hfill \Box

Therefore symplectic leaves on $\mathcal{M}(k,l)$ correspond to fixing isomorphism classes of $\mathcal{F}|_{\eta=\infty}$ and of $\mathcal{F}|_{\zeta=\infty}$. Symplectic leaves on $\mathcal{M}(k,l)^0/K$ are of course smaller than $K$-orbits of symplectic leaves on $\mathcal{M}(k,l)^0$. They are obtained by fixing $X$ and $Y$ and taking the symplectic quotient of $\text{Mat}_{k,l} \times \text{Mat}_{l,k}$ by $\text{Stab}(X) \times \text{Stab}(Y)$. We shall describe sheaves corresponding to a particular symplectic leaf in the case when $X$ and $Y$ are diagonalisable.

4.2. Orbits of $GL_k(\mathbb{C})$ and matrix-valued rational maps. We now consider only the action of $GL_k(\mathbb{C}) \simeq GL_k(\mathbb{C}) \times \{1\} \subset K$ on $\mathcal{M}(k,l)$. We fix a semisimple conjugacy class of $X$; i.e. we suppose that $X$ is diagonalisable, with distinct eigenvalues $\zeta_1, \ldots, \zeta_r$ of multiplicities $k_1, \ldots, k_r$. The stabiliser of $X$ is then isomorphic to $\prod_{i=1}^r GL_{k_i}(\mathbb{C})$. If the action of $GL_k(\mathbb{C})$ is to be free, we must have $k_i \leq l$, $i = 1, \ldots, r$. Let us diagonalise $X$ so that $X$ has the block-diagonal form $(\zeta_1 \cdot 1_{k_1 \times k_1}, \ldots, \zeta_r \cdot 1_{k_r \times k_r})$, and let $F_i, G_i$ denote the $k_i \times l$ and $l \times k_i$ submatrices of $F,G$ such that rows of $F$ and the columns of $G$ have the same coordinates as the block $\zeta_i \cdot 1_{k_i \times k_i}$. The action of $GL_k(\mathbb{C})$ is free and proper at $(X,Y,F,G)$ if and only if rank $F_i = \text{rank} G_i = F_i$ for $i = 1, \ldots, r$.

As in [15], we can associate to each element of $\mathcal{M}(k,l)$ a $\text{Mat}_{l,l}(\mathbb{C})$-valued rational map:

\[
R(\zeta) = Y + G(\zeta - X)^{-1}F.
\]

The mapping $(X,Y,F,G) \mapsto R(\zeta)$ is clearly $GL_k(\mathbb{C})$-invariant. If $X$ is diagonalisable, as above, i.e. $X = (\zeta_1 \cdot 1_{k_1 \times k_1}, \ldots, \zeta_r \cdot 1_{k_r \times k_r})$, then

\[
R(\zeta) = Y + \sum_{i=1}^r G_i F_i / (\zeta - \zeta_i).
\]

We clearly have:

Lemma 4.3. Let $P$ be a semisimple conjugacy class of $k \times k$ matrices with eigenvalues $\zeta_1, \ldots, \zeta_r$ of multiplicities $k_1, \ldots, k_r$. The map $(X,Y,F,G) \mapsto R(\zeta)$ induces a bijection between

(i) $GL_k(\mathbb{C})$-orbits on $\{(X,Y,F,G) \in \mathcal{M}(k,l)^0; X \in P\}$ and
(ii) the set $\mathcal{R}_l(P)$ of all rational maps of the form

\[
R(\zeta) = Y + \sum_{i=1}^r \frac{R_i}{\zeta - \zeta_i},
\]

where $\text{rank} R_i = k_i$. \hfill \Box

4.3. Orbits of loop groups. A rational map of the form (4.6) may be viewed as an element of a loop Lie algebra $\overline{gl}(l)^+$, consisting of maps from a circle $S^1$ in $\mathbb{C}$, containing the points $\zeta_i$ in its interior, which extend holomorphically outside $S^1$ (including $\infty$). The group $GL(l)^+$, consisting of smooth maps $g : S^1 \to GL_l(\mathbb{C})$,
extending holomorphically to the interior of $S^1$, acts on $\mathfrak{gl}(l)^-$ by pointwise conjugation, followed by projection to $\mathfrak{gl}(l)^-$. In particular, if all eigenvalues of $X$ are distinct, then the action is

$$g(\zeta). \left( Y + \sum_{i=1}^{r} \frac{R_i}{\zeta - \zeta_i} \right) = Y + \sum_{i=1}^{r} \frac{g(\zeta_i)R_i g(\zeta_i)^{-1}}{\zeta - \zeta_i}. $$

Therefore, if we fix conjugacy classes of the $R_i$, we obtain an orbit of $\widetilde{GL}(l)^+ \times \mathfrak{gl}(l)^-$ in $\mathfrak{gl}(l)^-$. We shall now consider quotients of such orbits by $\text{Stab}(Y)$ and describe which sheaves correspond to elements of such an orbit. Let us give a name to such quotients:

**Definition 4.4.** The quotient of an orbit of $\widetilde{GL}(l)^+$ in $\mathfrak{gl}(l)^-$ by $GL_l(\mathbb{C})$ is called a semireduced orbit.

**Remark 4.5.** In the literature (see, e.g., [1]–[5]) a reduced orbit is the symplectic quotient of an orbit by $H_Y = \text{Stab}(Y)$. The $GL_l(\mathbb{C})$-moment map on $\mathfrak{gl}(l)^-$ is identified with $Y + \sum_{i=1}^{r} R_i$ so that a reduced orbit is obtained by fixing the value of $a = \pi(\sum_{i=1}^{r} R_i)$, where $\pi$ is the projection $\mathfrak{gl}(l)(\mathbb{C}) \to \mathfrak{gl}(l)(\mathbb{C}) / h_Y$ (with $\perp$ taken with respect to tr), and dividing by $\text{Stab}(a) \subset \text{Stab}(Y)$. Therefore, if $\text{Stab}(Y)$ fixes $a$, then a reduced orbit can be identified with a subset of a semireduced orbit.

Let us, therefore, fix a semireduced orbit of $\widetilde{GL}(l)^+$. We choose $r$ distinct points $\zeta_1, \ldots, \zeta_r$ in $\mathbb{C}$. Furthermore, we choose $r + 1$ conjugacy classes $Q_0, Q_1, \ldots, Q_r$ of $l \times l$ matrices. This data determines a semireduced orbit $\Upsilon = \Upsilon(Q_0, \ldots, Q_r)$ of $\widetilde{GL}(l)^+$ defined as

$$\Upsilon = \left\{ R(\zeta) = Y + \sum_{i=1}^{r} \frac{R_i}{\zeta - \zeta_i}; \ Y \in Q_0, \ \forall i \geq 1 R_i \in Q_i \right\} / GL_l(\mathbb{C}).$$

Let

$$k_i = \text{rank } Q_i, \ i = 1, \ldots, r, \ k = \sum_{i=1}^{r} k_i.$$ 

In the notation of Lemma 4.3, $\Upsilon \subset R_l(P)$, where $P$ is the semisimple conjugacy class of $k \times k$ matrices with eigenvalues $\zeta_i$ of multiplicities $k_i$.

Thanks to Proposition 4.2, the conjugacy class $P$ determines $\mathcal{F}|_{\eta = \infty}$, which, in the case at hand, is $\bigoplus_{i=1}^{r} \mathbb{C}^{k_i}|_{(\zeta_i, \infty)}$. Similarly, $Q_0$ determines the isomorphism class of $\mathcal{F}|_{\zeta = \infty}$. We now discuss the significance of the other conjugacy classes $Q_1, \ldots, Q_l$.

We claim that these classes determine the isomorphism class of $\mathcal{F}|_{\eta^2 = \infty}$, i.e. of $\mathcal{F}$ restricted to the first order neighbourhood of $\eta = \infty$. This is only to be expected if one thinks in terms of the Mukai-Tyurin-Bottacin Poisson structure; cf. [11]. We again consider the canonical resolution (2.2) of $\mathcal{F}$ with $M(\zeta, \eta)$ given by (4.2). Let $\tilde{\eta} = 1/\eta$ be a local coordinate near $\eta = \infty$ so that

$$M(\zeta, \tilde{\eta}) = \begin{pmatrix} X - \zeta & \tilde{\eta}F \\ G & \tilde{\eta}Y - 1 \end{pmatrix}.$$
Using action $\text{(2.5)}$, we can multiply $M(\zeta, \tilde{\eta})$ on the right by $\begin{pmatrix} 1 & 0 \\ 0 & (1 - \tilde{\eta}^{-1}) \end{pmatrix}$. On the scheme $\tilde{\eta}^2 = 0$, we have $(1 - \tilde{\eta}^{-1}) = 1 + \tilde{\eta}$, and so $M(\zeta, \tilde{\eta})$ becomes

$$
\begin{pmatrix}
X - \zeta & \tilde{\eta}F \\
G & -1
\end{pmatrix}.
$$

To describe $\mathcal{F}|_{\tilde{\eta}^2 = 0}$, it is enough to describe it near each $\zeta_i$, i.e. to describe $\mathcal{G}_i = \mathcal{F}|_{U_i \times \{\tilde{\eta}^2 = 0\}}$, where $U_i$ is an open neighbourhood of $\zeta_i$ (not containing the other $\zeta_j$). The resolution $\text{(2.2)}$ of $\mathcal{F}$ restricted to $U_i \times \{\tilde{\eta}^2 = 0\}$ becomes

$$
0 \to \mathcal{O}(-2, -1)^{\oplus k_i} \oplus \mathcal{O}(-1, -2)^{\oplus l} \xrightarrow{M_i(\zeta, \tilde{\eta})} \mathcal{O}(-1, 0)^{\oplus (k_i + l)} \to \mathcal{G}_i \to 0,
$$

where

$$
M_i(\zeta, \tilde{\eta}) = \begin{pmatrix} \zeta_i - \zeta & \tilde{\eta}F_i \\
G_i & -1
\end{pmatrix}.
$$

This implies that we have an exact sequence

$$
\text{(4.10)} \quad 0 \to \mathcal{O}(-2, -1)^{\oplus k_i} \oplus \mathcal{O}(-1, -2)^{\oplus l} \xrightarrow{M_i(\zeta, \tilde{\eta}) + \tilde{\eta}F_i G_i} \mathcal{O}(-1, 0)^{\oplus (k_i + l)} \to \mathcal{G}_i \to 0
$$
on $U_i \times \{\tilde{\eta}^2 = 0\}$. Therefore $\mathcal{G}_i$ is determined by the $GL_{k_i}(\mathbb{C})$-conjugacy class of $F_i G_i$, which is the same as the $GL_{l_i}(\mathbb{C})$-conjugacy class of $G_i F_i$. Lemma $\text{[4.3]}$ and formula $\text{(4.7)}$ imply that the conjugacy class of $G_i F_i$ is $Q_i$. Thus, the conjugacy classes $Q_1, \ldots, Q_r$, which determine the orbit $\text{(4.8)}$, correspond to the isomorphism class of $\mathcal{F}|_{\tilde{\eta}^2 = 0}$. Observe that the support of $\mathcal{G}_i$ is given by $\det((\zeta_i - \zeta) + \tilde{\eta}F_i G_i) = 0$. In other words, the eigenvalues of $F_i G_i$ give $\frac{\zeta_i - \zeta}{\tilde{\eta}}$ at $(\zeta, \tilde{\eta}) = (\zeta_i, 0)$, i.e. the first order neighbourhood of $\text{supp} \mathcal{F}$ at $(\zeta_i, \infty)$.

Summing up, we have:

**Theorem 4.6.** There exists a natural bijection between elements of the semireduced rational orbit $\text{(4.8)}$ of $GL(l)^+ \subset \mathfrak{g}(l)^{-}$ and isomorphism classes of 1-dimensional acyclic sheaves $\mathcal{F}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ such that

1. The Hilbert polynomial of $\mathcal{F}$ is $P_{\mathcal{F}}(x, y) = lx + ky$.
2. $(\infty, \infty) \notin \text{supp} S$, and $\mathcal{F}|_{\eta = \infty} \simeq \bigoplus_{i = 1}^r \mathbb{C}^{k_i} \big|_{(\zeta_i, \infty)}$.
3. The isomorphism class of $\mathcal{F}|_{x = \infty}$ corresponds to $Q_0$, as in Proposition $\text{4.2}$.
4. The isomorphism class of $\mathcal{F}|_{\eta^2 = \infty}$ corresponds to conjugacy classes $Q_1, \ldots, Q_r$, as described above.

**Remark 4.7.** A variation of this result is probably well known to the integrable systems community (at least when $\mathcal{F}$ is a line bundle supported on a smooth curve $S$). We think it useful, however, to state it in this language and in full generality.

### 4.4. Symplectic leaves of $\mathcal{M}(k, l)^0/K$. We can finally describe symplectic leaves of $\mathcal{S}(k, l)$, i.e. sheaves corresponding to a particular symplectic leaf $L$ in $\mathcal{M}(k, l)/K$, at least in the case when $L \subset \mathcal{M}(k, l)^0/K$ and $X$ and $Y$ are semisimple. As we have already mentioned in $\text{[4.1]}$ a symplectic leaf in $\mathcal{M}(k, l)^0/K$ is obtained by fixing $X$ and $Y$, as well as a coadjoint orbit $\Lambda \subset \mathfrak{h}^*$ of $H = \text{Stab}(X) \times \text{Stab}(Y)$. If $\mu : \text{Mat}_{k, l} \times \text{Mat}_{l, k} \to \mathfrak{h}^*$ is the moment map for $H$, then the symplectic leaf determined by these data is

$$
\text{(4.11)} \quad L = \{(X, Y, F, G) \in \mathcal{M}(k, l)^0; X \text{ and } Y \text{ are given}, \mu(F, G) \in \Lambda\}/H.
$$

Let $X$ be diagonal, written as in $\text{[4.2]}$ i.e. $X = (\zeta_1 \cdot 1_{k_1 \times k_1}, \ldots, \zeta_r \cdot 1_{k_r \times k_r})$, and let $F_i, G_i, i = 1, \ldots, r$, be the corresponding submatrices of $F$ and $G$. Then $\text{Stab}(X) \simeq \prod_{i = 1}^r GL_{k_i}(\mathbb{C})$, and the moment map is the projection of the $GL_{k_i}(\mathbb{C})$-moment map,
(4.12) \[ \mu_X(F, G) = (F_1G_1, \ldots, F_rG_r). \]

Similarly, if \( Y \) is diagonal with \( s \) distinct eigenvalues of multiplicities \( l_1, \ldots, l_s \), then we obtain \( l_i \times k \) and \( k \times l_i \) submatrices \( G^i, F^i \). The stabiliser of \( Y \) is isomorphic to \( \prod_{i=1}^{r} GL_{l_i}(C) \) and the moment map is

(4.13) \[ \mu_Y(F, G) = (G^1F^1, \ldots, G^sF^s). \]

Therefore, an orbit \( \Lambda \) corresponds to \( r + s \) conjugacy classes \( \pi_1, \ldots, \pi_r, \rho_1, \ldots, \rho_s \) of \( k_i \times k_i \) matrices for the \( \pi_i \) and \( l_j \times l_j \) matrices for the \( \rho_j \). The leaf \( L \) will be contained in \( M(k, l)^0/K \) if and only if each conjugacy class consists of matrices of maximal rank (\( k_i \) or \( l_j \)). From the discussion in the previous subsection, we immediately obtain:

**Proposition 4.8.** Let \( L \) be a symplectic leaf of the Poisson manifold \( M(k, l)^0/K \), defined as in (4.11), with semisimple \( X \) and \( Y \). Then the image of \( L \) under the bijection (4.3) consists of isomorphism classes of sheaves \( F \) in \( S(k, l) \) such that the isomorphism classes of \( F|_{\zeta^2=\infty} \) and of \( F|_{\eta^2=\infty} \) are fixed (and determined by \( L \)).

Spelling things out, \( X \) determines \( F|_{\eta=\infty} \simeq \bigoplus_{i=1}^{r} C^{k_i}|_{(\zeta_i, \infty)} \), and each \( \pi_i, i = 1, \ldots, r \), determines \( F \) restricted to a neighbourhood of \( (\zeta_i, \infty) \) in \( \{ \eta^2 = \infty \} \) via (4.10). Similarly, \( Y \) and the \( \rho_j \) determine \( F|_{\zeta^2=\infty} \).

**Remark 4.9.** Symplectic leaves of \( M(k, l)^0/K \) can also be identified with reduced orbits (cf. Remark 4.5) of \( GL(l)^+ \) in \( \mathfrak{gl}(l)^- \). Therefore, the last proposition describes sheaves corresponding to a reduced orbit with \( Y \) semisimple. Furthermore, if we view \( M(k, l)^0/K \) as an open subset of the moduli space of semistable sheaves with Hilbert polynomial \( lx + ky \), then this map is a symplectomorphism between the Mukai-Tyurin-Bottacin symplectic structure, described in the introduction, and the Kostant-Kirillov form on a reduced orbit of a Lie group. For an open dense set where \( F \) is a line bundle on a smooth curve, this follows from results in [24]. Since both symplectic structures extend everywhere, they must be isomorphic everywhere.

**Example 4.10.** If we want \( F \) to be a line bundle over its support, then we must require that all \( k_i \) and all \( l_j \) be equal to 1. A symplectic leaf in \( M(k, l)^0/K \) is now given by fixing diagonal matrices \( X = \text{diag}(\zeta_1, \ldots, \zeta_k) \) and \( Y = \text{diag}(\eta_1, \ldots, \eta_l) \) with all \( \zeta_i \) and all \( \eta_j \) distinct, as well as the diagonal entries of \( FG \) and \( GF \), and quotienting by the group of \( (k+l) \times (k+l) \) diagonal matrices (acting as in (4.3)). If the diagonal entries of \( FG \) are fixed to be \( \alpha_1, \ldots, \alpha_k \), and the diagonal entries of \( GF \) are \( \beta_1, \ldots, \beta_l \), then the corresponding subset of \( S(k, l) \) consists of sheaves \( F \) supported on a 1-dimensional scheme \( S \) such that

\[ S \cap \{ \eta^2 = \infty \} = \bigcup_{i=1}^{k} \left\{ \zeta - \zeta_i = \frac{\alpha_i}{\eta} \right\}, \quad S \cap \{ \zeta^2 = \infty \} = \bigcup_{j=1}^{l} \left\{ \eta - \eta_j = \frac{\beta_j}{\zeta} \right\} \]

and the rank of \( F \) restricted to \( S \cap \{ \eta^2 = \infty \} \) and \( S \cap \{ \eta^2 = \infty \} \) is everywhere 1.

**Remark 4.11.** We expect that Proposition 4.8 remains true if \( X \) or \( Y \) are not semisimple.
5. Rank $k$ perturbations

Let us now assume that $k \leq l$. In [1], the authors consider Hamiltonian flows on a subset $\mathcal{M}$ of $\mathcal{M}^0(k,l)/K$, where rank $F = \text{rank } G = k$. It is clear from the previous section that a generic symplectic leaf of $\mathcal{M}^0(k,l)/K$ is not contained in $\mathcal{M}$. Therefore a flow may leave $\mathcal{M}$ without becoming singular. Since such Hamiltonian flows on a particular symplectic leaf can be linearised on the Jacobian of a spectral curve, it is interesting to know which points of the (affine) Jacobian are outside $\mathcal{M}$. We are going to give a very satisfactory answer to this in terms of cohomology of line bundles.

Let us therefore define the following set:

\begin{equation}
\mathcal{M}(k,l)^1 = \{ M \in \mathcal{M}(k,l) \ ; \ \text{rank } F = \text{rank } G = k \} .
\end{equation}

**Remark 5.1.** The manifold of $GL_k(\mathbb{C})$-orbits in $\mathcal{M}(k,l)^1$ with $X = 0$ and fixed $Y$ can be identified with the set $\{ Y + GF \}$, i.e. with the space of rank $k$ perturbations of the matrix $Y$, as considered first by Moser [13] ($k = 2$) and then by many other authors, in particular Adams, Harnad, Hurtubise, Previato [15].

We now ask which acyclic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ correspond to orbits of $K = GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $\mathcal{M}(k,l)^1$. We have:

**Proposition 5.2.** Let $k \leq l$. The bijection of Corollary [2,6] induces a bijection between:

(i) orbits of $GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $\mathcal{M}(k,l)^1$ and

(ii) isomorphism classes of acyclic sheaves $\mathcal{F}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert polynomial $P_F(x,y) = lx + ky$, which satisfy, in addition, [11] and

\[ H^0(\mathcal{F}(1,-1)) = 0 \text{ and } H^1(\mathcal{F}(1,-1)) = 0. \]

**Proof.** Consider the short exact sequences

\[ 0 \rightarrow \mathcal{O}(-1)^{\oplus k} \xrightarrow{(X - \zeta, G)^T} \mathcal{O}^{\oplus (k+l)} \rightarrow W_1 \rightarrow 0, \]

\[ 0 \rightarrow \mathcal{O}(-1)^{\oplus l} \xrightarrow{(F, Y - \eta)^T} \mathcal{O}^{\oplus (k+l)} \rightarrow W_2 \rightarrow 0. \]

The condition that $G$ has rank $k$ is equivalent to $W_1$ being a vector bundle, isomorphic to $\mathcal{O}(1)^{\oplus k} \oplus \mathcal{O}^{\oplus (l-k)}$. This is equivalent to $H^0(W_1 \otimes \mathcal{O}(-2)) = 0$. On the other hand, we claim that the condition that $F$ has rank $k$ is equivalent to $H^1(W_2 \otimes \mathcal{O}(-2)) = 0$. Indeed, any coherent sheaf on $\mathbb{P}^1$ splits into the sum of line bundles $\mathcal{O}(i)$ and a torsion sheaf [16]. Since $W_2$ has a resolution as above, we know that all degrees $i$ in the splitting are nonnegative and $F$ has rank $k$ if and only if all $i$ are strictly positive, which is equivalent to $H^1(W_2 \otimes \mathcal{O}(-2)) = 0$.

We can use the above exact sequences to obtain two further resolutions of $\mathcal{E} = \mathcal{F}(1,1)$:

\begin{align*}
(5.2) & \quad 0 \rightarrow \mathcal{O}(-1,0)^{\oplus k} \rightarrow \pi_2^* W_2 \rightarrow \mathcal{E} \rightarrow 0, \\
(5.3) & \quad 0 \rightarrow \mathcal{O}(0,-1)^{\oplus l} \rightarrow \pi_1^* W_1 \rightarrow \mathcal{E} \rightarrow 0,
\end{align*}

where the maps between the first two terms are given by the embedding in $\mathcal{O}^{\oplus (k+l)}$ followed by the projection onto the quotients $W_2, W_1$. Tensoring [5,2] with $\mathcal{O}(0,-2)$ shows that $H^1(W_2(-2)) = 0$ if and only if $H^1(\mathcal{E}(0,-2)) = 0$, i.e. $H^1(\mathcal{F}(1,-1)) = 0$. Similarly, tensoring [5,3] with $\mathcal{O}(-2,0)$ shows that $H^0(W_1(-2)) = 0$ if and only if $H^0(\mathcal{E}(-2,0)) = 0$, i.e. $H^0(\mathcal{F}(-1,1)) = 0$. \(\square\)
Remark 5.3. In the case \( k = l \), \( H^0(\mathcal{E}(-2, 0)) = 0 \) implies that \( \mathcal{E}(-2, 0) \) is acyclic (and similarly, \( H^1(\mathcal{E}(0, -2)) = 0 \) implies that \( \mathcal{E}(0, -2) \) is acyclic). In other words \( \mathcal{G} = \mathcal{E}(-1, 0) \) satisfies \( H^*(\mathcal{G}(-1, 0)) = 0 \). Furthermore, the resolution (5.3) becomes the following resolution of \( \mathcal{G} \):

\[
0 \to \mathcal{O}(-1, -1)^{\oplus k} \to \mathcal{O}^k \to \mathcal{G} \to 0.
\]

In the case when \( S = \text{supp} \mathcal{G} \) is smooth and \( \mathcal{G} \) is a line bundle, the corresponding part of \( \text{Jac}^g + 1(S) \) and the resolution (5.4) have been considered by Murray and Singer in [15].

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**References**


School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom

E-mail address: R.Bielawski@ed.ac.uk

Current address: Institut für Differentialgeometrie, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany

Fakultät für Mathematik, TU Dortmund, D-44221 Dortmund, Germany