RESULTS ON WITT KERNELS OF QUADRATIC FORMS
FOR MULTI-QUADRATIC EXTENSIONS

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ABSTRACT. In this paper we compute the Witt kernel of quadratic forms for the composition of a purely inseparable multi-quadratic extension with a separable quadratic extension. We also include the case of a multi-quadratic purely inseparable extension by completing the proof given before by the second author for such an extension.

Throughout this paper $F$ denotes a field of characteristic 2. Let $W_q(F)$ (resp. $W(F)$) be the Witt group of nonsingular quadratic forms over $F$ (resp. the Witt ring of regular bilinear forms over $F$). The group $W_q(F)$ is endowed with a $W(F)$-module structure as follows: To a bilinear form $(V, B)$ and a nonsingular quadratic form $(W, \varphi)$ with the polar form $B \otimes \varphi$ defined on $V \otimes_F W$ by: $B \otimes \varphi(v \otimes w) = B(v, v)\varphi(w)$ for $(v, w) \in V \times W$, whose polar form is $B \otimes B\varphi$ [B].

For an integer $n \geq 1$, let $I^n F$ denote the $n$-th power of the fundamental ideal $IF$ of $W(F)$ (set $I^0 F = W(F)$), and let $I^n W_q(F)$ denote the group $I^n F \otimes W_q(F)$.

We denote by $\langle a_1, \ldots, a_n \rangle_b$ the $n$-dimensional diagonal bilinear form $\sum_{i=1}^n a_i x_i y_i$ for $a_1, \ldots, a_n \in F^* := F \setminus \{0\}$. The 2-dimensional quadratic form $ax^2 + xy + by^2$ will be denoted by $[a, b]$.

Given field extension $K/F$ induces natural homomorphisms $i_K : W_q(F) \rightarrow W_q(K)$ and $j_K : W(F) \rightarrow W(K)$. Let $W_q(K/F)$ and $W(K/F)$ denote the kernels of $i_K$ and $j_K$, respectively.

An important problem in the algebraic theory of quadratic (bilinear) forms is the computation of $W_q(K/F)$ and $W(K/F)$. These kernels are known for some field extensions. For example, for $K/F$ purely transcendental, these kernels are trivial. If $K/F$ is separable, then $W(K/F)$ is trivial since an anisotropic $F$-bilinear form remains anisotropic over a separable extension of $F$. For $K/F$ inseparable, the kernel $W(K/F)$ was computed by Hoffmann in [H]. For $K$ the function field of a hypersurface (including algebraic simple extensions), the kernel $W(K/F)$ was computed by Dolphin and Hoffmann in [DH]. Before this result the second author computed $W(K/F)$ when $K$ is the function field of a quadric [L1]. Still for $K$ the function field of a quadric, some partial results on $W_q(K/F)$ are given in [L1]. Also, for $\varphi$ a bilinear Pfister form, the kernel $I^m(F(\varphi)/F)$ was computed by Baeza and the first author in [AB].

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In this paper we are interested in the kernel $W_q(K/F)$ when $K/F$ is a multi-
quadratic extension. As the characteristic is 2, there are two kinds of quadratic 
extensions: the inseparable quadratic extension $F[t]/(t^2 - a)$ with $a \in F \setminus F^2$, which we denote by $F(\sqrt{a})$, and the separable quadratic extension $F[t]/(t^2 + t + b)$ with $b \not\in \wp(F)$, which we denote by $F(\wp^{-1}(b))$ (here $\wp : F \rightarrow F$ is the Artin-Schreier operator given by: $x \mapsto x^2 - x$).

In the case of a purely inseparable multi-quadratic extension $K/F$, the kernel $W_q(K/F)$ was computed in [L2], but it turns out that the proof contains a mistake. So our first aim in this paper is to give a complete proof of the following theorem (see comments after Proposition 1):

**Theorem 1** ([L2] Th. 1). For any scalars $a_1, \cdots, a_n \in F$, we have the following:

$$W_q(F(\sqrt{a_1}, \cdots, \sqrt{a_n})/F) = \sum_{i=1}^n (1, a_i)_b \otimes W_q(F).$$

After that, based on Theorem 1 we will compute the kernel of the composition of a purely inseparable extension with a quadratic separable extension:

**Theorem 2.** For any scalars $a_1, \ldots, a_n, b \in F$, we have the following:

$$W_q(F(\wp^{-1}(b), \sqrt{a_1}, \cdots, \sqrt{a_n})/F) = W(F) \otimes [1, b] + \sum_{i=1}^n (1, a_i)_b \otimes W_q(F).$$

Previously, Theorem 1 was proved by Mammmone and Moresi [MM Th. 2(i)] for $n = 2$, and by Baeza [B, Lem. 4.3, page 182] for $n = 1$. Theorem 2 is due to Ahmad for $n = 1$ [A]. In [B, Cor. 4.16, page 128], Baeza gave the computation in the case of quadratic and bi-quadratic separable extensions: $W_q(F(\wp^{-1}(a), \wp^{-1}(b))/F) = W(F) \otimes [1, a] + W(F) \otimes [1, b]$. But this computation does not extend to tri-quadratic separable extensions as was proved by Mammmone and Moresi [MM Prop. 1].

Our proofs are reduced to computations on differential forms. This is allowed by the well known result of Kato [K], which gives the relation between nonsingular quadratic forms and differential forms (see below).

Let us recall some facts about differential forms. For any integer $n \geq 1$, we denote by $\Omega^n_F = \wedge^n \Omega^1_F$ the space of $n$-differential forms ($\Omega^0_F = F$), where $\Omega^1_F$ is the $F$-vector space generated by the symbols $dx$, $x \in F$, subject to the relations $d(x + y) = dx + dy$ and $d(xy) = xdy + ydx$ for $x, y \in F$. In particular, there is a $F^2$-linear map $F \rightarrow \Omega^1_F$, given by: $x \mapsto dx$. This map extends to the differential operator $d : \Omega^n_F \rightarrow \Omega^{n+1}_F$ as follows: $d(xdx_1 \wedge \cdots \wedge dx_n) = dx \wedge dx_1 \wedge \cdots \wedge dx_n$. The usual Artin-Schreier operator $\wp : F \rightarrow F$, $\wp(a) = a^2 - a$, extends to the general Artin-Schreier map

$$\wp : \Omega^n_F \rightarrow \Omega^n_F / d\Omega^{n-1}_F$$

given on generators by

$$\wp (x\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}) = (x^2 - x)\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} + d\Omega^{n-1}_F.$$

Let $\nu_F(n) = \text{Ker}(\wp)$ and $H^{n+1}_2(F) = \text{Coker}(\wp)$. By a result of Kato [K], there exists an isomorphism

$$e^n : T^n W_q(F) \rightarrow H^{n+1}_2(F)$$
given on generators as follows:

\[ e^n((\langle a_1, \ldots, a_n \rangle)_b \otimes [1, c] + T^{n+1}W_q(F)) = e^{\frac{da_1}{a_1}} \land \cdots \land \frac{da_n}{a_n} + \phi(\Omega^n_F) + d\Omega^{n-1}_F, \]

where \( T^nW_q(F) \) denotes the quotient \( T^nW_q(F) / T^{n+1}W_q(F) \) and \( \langle a_1, \ldots, a_n \rangle)_b \) denotes the \( n \)-fold bilinear Pfister form \( \langle 1, a_1 \rangle_b \otimes \cdots \otimes \langle 1, a_n \rangle_b \). In [K], Kato also established a relation between bilinear forms and differentials forms, but we will not need this fact here.

Let \( \{ a_i \mid i \in I \} \) be a 2-basis of \( F \); i.e., the set \( \{ \prod_{i \in I} a_i^{\epsilon_i} \mid \epsilon_i \in \{ 0, 1 \}, \text{ and } \epsilon_i = 0 \text{ for almost all } i \in I \} \) is an \( F^2 \)-basis of \( F \). We fix some ordering on \( I \) and consider the set \( \sum_n := \{ \sigma : \{ 1, \ldots, n \} \rightarrow I \mid \sigma(i) < \sigma(j) \text{ whenever } i < j \} \). This set \( \sum_n \) is equipped with the lexicographic ordering. Hence, the set \( \{ \frac{da_\sigma}{a_\sigma} \mid \sigma \in \sum_n \} \) is an \( F \)-basis of \( \Omega^n_F \), where \( \frac{da_\sigma}{a_\sigma} = \frac{da_{\sigma(1)}}{a_{\sigma(1)}} \land \cdots \land \frac{da_{\sigma(n)}}{a_{\sigma(n)}} \) for any \( \sigma \in \sum_n \). For \( \alpha \in \sum_n \), let \( \Omega^n_{F,\alpha} \) (resp. \( \Omega^n_{F,<\alpha} \)) be the subspace of \( \Omega^n_F \) generated by \( \frac{da_\sigma}{a_\sigma} \) with \( \sigma \leq \alpha \) (resp. the subspace of \( \Omega^n_F \) generated by \( \frac{da_\sigma}{a_\sigma} \) with \( \sigma < \alpha \)). Hence, we get a filtration of \( \Omega^n_F \) given by \( \Omega^n_{F,\alpha} \) (resp. \( \Omega^n_{F,<\alpha} \), \( \alpha \in \sum_n \)). For any \( i \in I \), let \( F_i \) be the subfield \( F^2(a_j \mid j \leq i) \).

A crucial result that we will use in our proofs is due to Kato. We will refer to it as Kato’s lemma:

**Proposition 1 ([K]).** Let \( \alpha \in \sum_n \) and \( y \in F \) be such that \( \phi(y \frac{da_\alpha}{a_\alpha}) \in \Omega^n_{F,<\alpha} + d\Omega^{n-1}_F \). Then, there exist \( v \in \Omega^n_{F,\alpha} \) and \( c_i \in F^{*}_{\alpha(i)}, \, 1 \leq i \leq n \), such that

\[ \frac{y \frac{da_\alpha}{a_\alpha}}{c_1} = \frac{dc_1}{c_1} \land \cdots \land \frac{dc_n}{c_n} + v. \]

Clearly, this proposition implies that any element \( \eta \in \nu_F(n) \) can be written as \( \eta = \sum_{\beta \leq \alpha} u_{\beta} \frac{dl_\beta}{l_\beta} \frac{dl_{\beta(n)}}{l_{\beta(n)}} \) for some \( \alpha \in \sum_n \), \( l_{\beta(i)} \in F^{*}_{\beta(i)} \), \( u_{\beta} = 0, 1 \), and \( \frac{dl_\beta}{l_\beta} = \frac{dl_{\beta(1)}}{l_{\beta(1)}} \land \cdots \land \frac{dl_{\beta(n)}}{l_{\beta(n)}} \).

The following proposition is incorporated in [L2] Prop. 1, and it is the basis of Theorem 1. When trying to prove Theorem 2 we noticed that some details in the proof of [L2] Prop. 1 should be corrected (see the explanations in the following footnote). This is the reason why we reproduce the proof here.

**Proposition 2.** For any integer \( n \geq 1 \) and scalars \( \alpha_1, \ldots, \alpha_m \in F \), we have

\[ \text{Ker}(H^{n+1}_2(F) \rightarrow H^{n+1}_2(F(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_m}))) = \sum_{i=1}^m \frac{da_i}{\sqrt{\alpha_i}} \land \Omega^{n-1}_F(F). \]

**Proof.** Let \( L = F(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_m}) \) and \( M = F(\sqrt{\alpha_1}) \). Without loss of generality, we may suppose that \( [L : F] = 2^m \). So we can choose a 2-basis \( B = \{ c_1, \ldots, c_m, c_{m+1}, \ldots \} \) of \( F \) such that \( c_i = \alpha_i \) for \( i = 1, \ldots, m \).

We proceed by induction on \( m \). The case \( m = 1 \) is done in [AB3] Lem. 2.18]. Suppose that the proposition is true for any field \( F' \) of characteristic 2 and any purely inseparable multi-quadratic extension \( L' / F' \) such that \( [L' : F'] < 2^m \).

Since \( [L : M] = 2^m-1 \), we get by the induction hypothesis that \( \text{Ker}(H^{n+1}_2(M) \rightarrow H^{n+1}_2(L)) = \sum_{i=2}^m \frac{da_i}{\sqrt{\alpha_i}} \land \Omega^{n-1}_M(F) \).
Let $\omega \in \Omega_F^n$ be such that $\omega \in \text{Ker}(H_2^{n+1}(F) \rightarrow H_2^{n+1}(L))$. Then, over $M$, we have $\omega \in \text{Ker}(H_2^{n+1}(M) \rightarrow H_2^{n+1}(L))$. Hence, $\omega \in \sum_{i=2}^m d\alpha_i \wedge \Omega^{-1}_M$. This means that there exist $u \in \Omega_M^n$ and $v \in \Omega^{-1}_M$. Let

\[ w = \sum_{i=2}^m \frac{d\alpha_i}{\alpha_i} \wedge \sigma_i + \varphi(u) + d(v), \]

for suitable $\sigma_2, \cdots, \sigma_m \in \Omega^{-1}_M$. For the sequel we use a descent argument in order to reduce to the case where $\sigma_1, \cdots, \sigma_m, u, v$ are defined over $F$.

The set $B' := \{c_2, \cdots, c_m, c_{m+1}, \cdots, \sqrt{\alpha_1}\}$ is a 2-basis of $M$. We may suppose that $\sqrt{\alpha_1}$ is the last element in $B'$. Let $\delta$ be the maximal multi-index in (I).

Suppose that $\delta(1) > m$. Then we get

\[ w \frac{dc_\delta}{c_\delta} = \varphi(u_\delta \frac{dc_\delta}{c_\delta}) + t \frac{dc_\delta}{c_\delta} \]

such that $t \frac{dc_\delta}{c_\delta} + d(v) := w' \in \Omega_F^n$. \[ \text{(i)} \]

Suppose that $c_{\delta(n)} = \sqrt{\alpha_1}$: In this case, we have $w_\delta = 0$, and thus $\varphi(u_\delta \frac{dc_\delta}{c_\delta}) \in d\Omega^{-1}_M + \Omega^{-1}_{F, < \delta}$. By Kato’s lemma, $u_\delta \frac{dc_\delta}{c_\delta} = \frac{dc_{\delta(1)}}{\alpha_{\delta(1)}} \wedge \cdots \wedge \frac{dc_{\delta(n)}}{\alpha_{\delta(n)}} + u'$, where $e_{\delta(n)} \in M_{\delta(n)}$ and $u' \in \Omega_{F, < \delta}$. By substituting the expression $u_\delta \frac{dc_\delta}{c_\delta}$ in (2) and using the fact that $\varphi(\frac{dc_{\delta(1)}}{\alpha_{\delta(1)}} \wedge \cdots \wedge \frac{dc_{\delta(n)}}{\alpha_{\delta(n)}}) \in d\Omega^{-1}_M$, we are reduced to the case where $c_{\delta(n)} \in F$.

\[ \text{(ii)} \]

Suppose that $c_{\delta(n)} \in F$: In this case, we have $v \in \Omega_{F, < \delta}$. Otherwise, $v = v_0 + \sqrt{\alpha_1} v_1$ with $v_0, v_1 \in \Omega_{F, < \delta}$ and $v_1 \neq 0$. Then, $d(v) = d(v_0) + \sqrt{\alpha_1} d(v_1) + v_1 \wedge d(\sqrt{\alpha_1})$. Consequently, $\frac{dc_\delta}{c_\delta}$ contains the slot $d(\sqrt{\alpha_1})$, which is not possible. Hence, $t \frac{dc_\delta}{c_\delta}$ and $w'$ belong to $\Omega_F^n$. In particular, $u_\delta \frac{dc_\delta}{c_\delta} \in \Omega_F^n$. Now substituting $w_\delta \frac{dc_\delta}{c_\delta}$ in $w$, we get

\[ w = w_\delta \frac{dc_\delta}{c_\delta} + w'' \quad \text{with} \quad w'' \in \Omega_{F, < \delta} \]

\[ = \varphi(u_\delta \frac{dc_\delta}{c_\delta}) + d(v) + w' + w'' \]

\[ = w''' \quad \text{mod} \quad \varphi(\Omega_F^n) + d\Omega_{F, < \delta}^{-1}, \]

with $w''' = w' + w'' \in \Omega_{F, < \delta}$. By iterating the procedure whenever $\delta(1) > m$, we are reduced to an element $w'''$ whose maximal multi-index $\lambda$ satisfies $\lambda(1) \leq m$. This means that $w''' \in \sum_{i=2}^m \frac{d\alpha_i}{\alpha_i} \wedge \Omega_{F, < \delta}^{-1}$. Hence, there exist $\tau_2, \cdots, \tau_m \in \Omega_{F, < \delta}^{-1}$, $r \in \Omega_{F, < \delta}$ and $s \in \Omega_F$, such that, over the field $M$, we have

\[ w = \sum_{i=2}^m \frac{d\alpha_i}{\alpha_i} \wedge \tau_i + \varphi(r) + d(s). \]

Since $\text{Ker}(H_2^{n+1}(F) \rightarrow H_2^{n+1}(M)) = \frac{d\alpha_1}{\alpha_1} \wedge \Omega_{F, < \delta}^{-1}$, there exists $\tau_1 \in \Omega_{F, < \delta}$ such that $\overline{w} = \sum_{i=1}^m \frac{d\alpha_i}{\alpha_i} \wedge \tau_i$. \[ \square \]

The proof of Theorem 2 will be based on the following proposition:

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\[ ^{1} \text{A mistake in the proof of [L2 Prop. 1] has been made when trying to make this descent. More precisely, it happens in [L2] page 2484, lines 7-8} \]

..."
**Proposition 3.** Let $\alpha_1, \ldots, \alpha_m, a \in F$, and let $L = F(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_m})$, $M = F(\phi^{-1}(a))$. Then we have the following:

$$\text{Ker}(H_n^{n+1}(F) \rightarrow H_n^{n+1}(L \cdot M)) = a_{nF}(n) + \sum_{i=1}^{m} d\alpha_i \wedge \Omega_{F}^{n-1}.$$ 

**Proof.** Without loss of generality, we may suppose that $[L : F] = 2^m$, $a \in F \setminus \phi(F)$ and $a \in F^2$. Put $\beta = \phi^{-1}(a)$. Let $B := \{c_1, \ldots, c_m, c_{m+1}, \ldots\}$ be a 2-basis of $F$ such that $c_i = \alpha_i$ for $i = 1, \ldots, m$. Since $M/F$ is separable, $B$ remains a 2-basis of $M$. Then $\Omega_{M, <\delta} = \Omega_{F, <\delta} + B\Omega_{F, <\delta}$.

Let $\omega \in \Omega_{F}$ be such that $\omega \in \text{Ker}(H_n^{n+1}(F) \rightarrow H_n^{n+1}(L \cdot M))$. Then, over $M$, $\omega \in \text{Ker}(H_n^{n+1}(M) \rightarrow H_n^{n+1}(L \cdot M))$. It follows from Proposition 2 that

$$\omega = \sum_{i=1}^{m} \sigma_i \wedge d\alpha_i + \phi(u) + d(v)$$

with $u \in \Omega_{M, <\delta}$, $v \in \Omega_{M}^{n-1}$, and $\sigma_1, \ldots, \sigma_m \in \Omega_{M}^{n-1}$. Let $\delta$ be the maximal multi-index in relation (3).

Suppose that $\delta(1) > m$. Then, as in the relation (2), there exists $w' \in \Omega_{M, <\delta}$ such that

$$w_\delta \frac{dc_\delta}{c_\delta} = \phi(u_\delta \frac{dc_\delta}{c_\delta}) + d(v) + w'.$$

Let $r_\delta, s_\delta \in F$, $v_0, v_1 \in \Omega_{F}^{n-1}$, and $w'_0, w'_1 \in \Omega_{F, <\delta}$ be such that $u_\delta = r_\delta + \beta s_\delta$, $v = v_0 + \beta v_1$ and $w' = w'_0 + \beta w'_1$. A simple computation shows that the relation (4) gives the following:

$$\begin{cases}
  w_\delta \frac{dc_\delta}{c_\delta} = \phi(r_\delta \frac{dc_\delta}{c_\delta}) + d(v_0) + a_{s_\delta}^2 \frac{dc_\delta}{c_\delta} + w'_0, \\
  \phi(s_\delta \frac{dc_\delta}{c_\delta}) = d(v_1) + w'_1.
\end{cases}$$

By applying Kato’s lemma to the second relation in (5), we get

$$s_\delta \frac{dc_\delta}{c_\delta} = \frac{d\alpha_{\delta(1)}}{a_{\delta(1)}} \wedge \cdots \wedge \frac{d\alpha_{\delta(n)}}{a_{\delta(n)}} + w'_2,$$

for some $w'_2 \in \Omega_{F, <\delta}$ and $a_{\delta(j)} \in F_{\delta(j)}$. Again with the second relation in (5), we obtain

$$a_{s_\delta}^2 \frac{dc_\delta}{c_\delta} = a_{s_\delta} \frac{dc_\delta}{c_\delta} + d(v_1) + aw'_1 = a_{s_\delta} \frac{d\alpha_{\delta(1)}}{a_{\delta(1)}} \wedge \cdots \wedge \frac{d\alpha_{\delta(n)}}{a_{\delta(n)}} + d(v_1) + aw'_1 + aw'_2.$$

If we substitute the expression $a_{s_\delta}^2 \frac{dc_\delta}{c_\delta}$ in the first relation in (5), we get

$$w_\delta \frac{dc_\delta}{c_\delta} = \phi(r_\delta \frac{dc_\delta}{c_\delta}) + d(v') + a\eta_\delta + w''$$

where $v' = v_0 + av_1$, $\eta_\delta = \frac{d\alpha_{\delta(1)}}{a_{\delta(1)}} \wedge \cdots \wedge \frac{d\alpha_{\delta(n)}}{a_{\delta(n)}}$, and $w'' = aw'_1 + aw'_2 + \eta_\delta \in \Omega_{F, <\delta}$.

Since $a\eta_\delta \in \phi(\Omega_{M}^{n}) \subset \phi(\Omega_{M}^{n}) + d(\Omega_{M}^{n-1})$, the multi-index has been reduced. By iterating the procedure whenever $\delta(1) > m$, we obtain, over $M$, the following:

$$w = \phi(\gamma_1) + d(\gamma_2) + a\gamma_3 + \gamma_4,$$

for $\gamma_1 \in \Omega_{F}^{n}$, $\gamma_2 \in \Omega_{F}^{n-1}$, $\gamma_3 \in \nu_{F}(n)$ and $\gamma_4 \in \Omega_{F}^{n}$, with the condition that the maximal multi-index $\lambda$ in $\gamma_4$ satisfies $\lambda(1) \leq m$. Then $\gamma_4 = \sum_{i=1}^{m} d\alpha_i \wedge \Omega_{F}^{n-1}$. Now,
since the kernel $\operatorname{Ker}(H_2^{n+1}(F) \to H_2^{n+1}(M)) = av_F(n)$ \cite{AB}, we get the desired conclusion. \hfill \Box

As a corollary, we get:

**Corollary 1.** Let $n \geq 1$ be an integer, and let $\alpha_1, \ldots, \alpha_m, a \in F$. Let $L = F(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_m})$ and $M = L(\sqrt{-1}(a))$. Then we have:

(1) $\operatorname{Ker}(I^nW_q(F) \to I^nW_q(L)) = \sum_{i=1}^m \langle 1, \alpha_i \rangle_b \otimes I^{n-1}W_q(F)$.

(2) $\operatorname{Ker}(I^nW_q(F) \to I^nW_q(L \cdot M)) = I^nF \otimes [1, a] + \sum_{i=1}^m \langle 1, \alpha_i \rangle_b \otimes I^{n-1}W_q(F)$.

**Proof.** We use Propositions \ref{prop2} and \ref{prop3} with the commutativity of the diagram

$$
\begin{array}{ccc}
I^nW_q(F) & \longrightarrow & I^nW_q(K) \\
\downarrow & & \downarrow \\
H_2^{n+1}(F) & \longrightarrow & H_2^{n+1}(K)
\end{array}
$$

where $K = L$ (resp. $K = L \cdot M$), the vertical arrows are given by the isomorphism $e^n$, and the horizontal arrows are induced by the inclusion $F \subset K$. \hfill \Box

**Proof of Theorems \ref{thm1} and \ref{thm2}** We use Corollary \ref{cor1} and we follow the same argument used for the proof of \cite[L2, Theorem 1]{L2}.

We also have the following corollary, whose proof is the same as for Theorems \ref{thm1} and \ref{thm2}.

**Corollary 2.** Let $n \geq 1$ be an integer, and let $\alpha_1, \ldots, \alpha_m, a \in F$. Let $L = F(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_m})$ and $M = L(\sqrt{-1}(a))$. Then we have:

(1) $\operatorname{Ker}(I^nW_q(F) \to I^nW_q(L)) = \sum_{i=1}^m \langle 1, \alpha_i \rangle_b \otimes I^{n-1}W_q(F)$.

(2) $\operatorname{Ker}(I^nW_q(F) \to I^nW_q(L \cdot M)) = I^nF \otimes [1, a] + \sum_{i=1}^m \langle 1, \alpha_i \rangle_b \otimes I^{n-1}W_q(F)$.

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