

MULTIPLE SOLUTIONS TO THE BAHRI-CORON PROBLEM IN SOME DOMAINS WITH NONTRIVIAL TOPOLOGY

MÓNICA CLAPP AND JORGE FAYA

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ABSTRACT. We show that in every dimension $N \geq 3$ there are many bounded domains $\Omega \subset \mathbb{R}^N$, having only finite symmetries, in which the Bahri-Coron problem

$$-\Delta u = |u|^{4/(N-2)} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

has a prescribed number of solutions, one of them being positive and the rest sign-changing.

1. INTRODUCTION

We consider the problem

$$(\varphi_\Omega) \quad \begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in \mathbb{R}^N and $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent. Equations of this type arise in fundamental questions in differential geometry such as the Yamabe problem or the scalar curvature problem.

Problem (φ_Ω) has a rich geometric structure: it is invariant under the group of Möbius transformations. This fact causes a lack of compactness of the associated variational functional, which prevents the straightforward application of standard variational methods.

It is well known that the existence of a solution depends on the domain. Pohozaev's identity [19], together with the unique continuation of solutions [13], implies that (φ_Ω) does not have a nontrivial solution if Ω is strictly starshaped. On the other hand, if the domain is an annulus,

$$A = A_{a,b} := \{x \in \mathbb{R}^N : 0 < a < |x| < b\},$$

Kazdan and Warner [12] showed that (φ_A) has infinitely many radial solutions. Moreover, if Ω is invariant under the action of a group G of linear isometries of \mathbb{R}^N and every G -orbit of Ω is infinite, compactness is restored (cf. Theorem 2 below), and standard variational methods provide infinitely many G -invariant solutions to problem (φ_Ω) .

The first nontrivial existence result is due to Coron [10]. He showed that if $0 \notin \Omega$ and Ω contains an annulus $A_{a,b}$ with b/a large enough, then problem (φ_Ω) has at

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least one positive solution. A few years later a remarkable result was obtained by Bahri and Coron [2], who showed that problem (φ_Ω) has at least one positive solution in every domain Ω having nontrivial reduced homology with $\mathbb{Z}/2$ -coefficients.

Concerning multiplicity, as we already mentioned, (φ_Ω) has infinitely many solutions if Ω is G -invariant and every G -orbit of Ω is infinite. On the other hand, several multiplicity results have been established for domains which are obtained by deleting a thin enough neighborhood of a certain subset from a given domain; see e.g. [5, 6, 8, 9, 14–16, 18, 20]. In particular, for the type of domains considered by Coron, the authors of [11] recently obtained a strong multiplicity result: they basically showed that if $0 \notin \Omega$ and Ω contains an annulus $A_{\varepsilon,b}$, then the number of solutions to (φ_Ω) becomes arbitrarily large as $\varepsilon \rightarrow 0$. This result requires no symmetries on the domain Ω . Its proof is based on the Lyapunov-Schmidt reduction method.

But for domains which are neither highly symmetric nor small perturbations of a given domain, multiplicity remains largely open. A first result in this direction was recently established in [7]. Here we shall extend the main result in [7] in a way which provides many new examples of domains Ω in which problem (φ_Ω) has a prescribed number of solutions.

We need some notation. Let $O(N)$ be the group of linear isometries of \mathbb{R}^N . If G is a closed subgroup of $O(N)$, we denote by

$$Gx := \{gx : g \in G\}$$

the G -orbit of $x \in \mathbb{R}^N$ and by $\#Gx$ its cardinality. A domain $\Omega \subset \mathbb{R}^N$ is called G -invariant if $Gx \subset \Omega$ for all $x \in \Omega$, and a function $u : \Omega \rightarrow \mathbb{R}$ is called G -invariant if u is constant on every Gx .

Fix a closed subgroup Γ of $O(N)$ and a nonempty Γ -invariant bounded smooth domain D in \mathbb{R}^N such that $\#\Gamma x = \infty$ for all $x \in D$. We prove the following result.

Theorem 1. *There exists an increasing sequence (ℓ_m) of positive real numbers, depending only on Γ and D , with the following property: If Ω contains D and if it is invariant under the action of a closed subgroup G of Γ such that*

$$\min_{x \in \Omega} \#Gx > \ell_m,$$

then problem (φ_Ω) has at least m pairs of G -invariant solutions $\pm u_1, \dots, \pm u_m$ such that u_1 is positive, u_2, \dots, u_m change sign, and

$$\int_{\Omega} |\nabla u_k|^2 \leq \ell_k S^{N/2} \quad \text{for every } k = 1, \dots, m,$$

where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^}(\mathbb{R}^N)$.*

The particular case where $\Gamma = O(N)$ and $D = A_{a,b}$ was established in [7]. This situation is, however, quite restrictive, particularly in odd dimensions. For example, if $N = 3$, then $\min_{x \in A_{a,b}} \#Gx \leq 12$ for every subgroup $G \neq SO(3), O(3)$; cf. [3]. As we shall see, the number ℓ_1 goes to infinity as $b/a \rightarrow 1$. Therefore, the main result in [7] will provide solutions in subdomains of \mathbb{R}^3 only if b/a is sufficiently large, which is the case already handled by Coron [10] and by Ge, Musso and Pistoia [11].

Theorem 1, on the other hand, provides examples in every dimension of domains Ω , having only finite symmetries, in which problem (φ_Ω) has a prescribed number of solutions. Specific examples may be obtained as follows: let D_0 be a bounded

smooth domain in \mathbb{R}^{N-1} , $N \geq 3$, with $D_0 \subset \{(x, y) \in \mathbb{R} \times \mathbb{R}^{N-2} : x \geq \varepsilon\}$ for some $\varepsilon > 0$. Set

$$D := \{(z, x') \in \mathbb{C} \times \mathbb{R}^{N-2} \equiv \mathbb{R}^N : (|z|, y) \in D_0\}.$$

Then D is invariant under the action of the group $\Gamma := \mathbb{S}^1$ of unit complex numbers, acting by $e^{i\theta}(z, x') := (e^{i\theta}z, x')$. Note that this action is free on $(\mathbb{C} \setminus \{0\}) \times \mathbb{R}^{N-2}$, so if $G_n := \{e^{2\pi ik/n} : k = 0, \dots, n-1\}$ is the cyclic subgroup of order n , then $\#G_n x = n$ for every $x \in (\mathbb{C} \setminus \{0\}) \times \mathbb{R}^{N-2}$. Therefore, for every $n > \ell_m$ and every G_n -invariant bounded smooth domain Ω in \mathbb{R}^N with

$$D \subset \Omega \subset (\mathbb{C} \setminus \{0\}) \times \mathbb{R}^{N-2},$$

Theorem 1 yields at least m pairs of solutions to problem (φ_Ω) .

This result supports our belief that multiplicity should hold in noncontractible domains, such as those considered in [2]. But the proof of such a general statement is still way out of reach.

The proof of Theorem 1 is variational and is given in the following section.

2. PROOF OF THE MAIN THEOREM

Let G be a closed subgroup of $O(N)$. If Ω is G -invariant, the principle of symmetric criticality [17] asserts that the G -invariant solutions of problem (φ_Ω) are the critical points of the restriction of the functional

$$J(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{2^*} \int_\Omega |u|^{2^*}$$

to the space of G -invariant functions

$$H_0^1(\Omega)^G := \{u \in H_0^1(\Omega) : u(gx) = u(x) \text{ for all } g \in G, x \in \Omega\}.$$

We shall say that J satisfies the Palais-Smale condition $(PS)_c^G$ in $H_0^1(\Omega)$ if every sequence (u_n) such that

$$u_n \in H_0^1(\Omega)^G, \quad J(u_n) \rightarrow c, \quad \nabla J(u_n) \rightarrow 0$$

contains a convergent subsequence.

Let $c_\infty := \frac{1}{N} S^{N/2}$ be the energy of the positive solution (unique up to translation and dilation) to the problem

$$-\Delta u = |u|^{2^*-2}u, \quad u \in D^{1,2}(\mathbb{R}^N).$$

We shall make use of the following results.

Theorem 2. *J satisfies condition $(PS)_c^G$ in $H_0^1(\Omega)$ for every*

$$c < \min_{x \in \overline{\Omega}} (\#Gx) c_\infty.$$

In particular, if $\#Gx = \infty$ for all $x \in \overline{\Omega}$, then J satisfies condition $(PS)_c^G$ in $H_0^1(\Omega)$ for every $c \in \mathbb{R}$.

Proof. See [4, Corollary 2]. □

Theorem 3. *Let W be a finite dimensional subspace of $H_0^1(\Omega)^G$. If J satisfies condition $(PS)_c^G$ in $H_0^1(\Omega)$ for all $c \leq \sup_W J$, then J has at least $\dim(W) - 1$ pairs of sign-changing critical points $u \in H_0^1(\Omega)^G$ such that $J(u) \leq \sup_W J$.*

Proof. See [7, Theorem 3.7]. □

Proof of Theorem 1. Let $\mathcal{P}_1(D)$ be the set of all nonempty Γ -invariant bounded smooth domains contained in D , and define

$$\mathcal{P}_k(D) := \{(D_1, \dots, D_k) : D_i \in \mathcal{P}_1(D), D_i \cap D_j = \emptyset \text{ if } i \neq j\}.$$

Note that $\mathcal{P}_k(D) \neq \emptyset$ for every $k \in \mathbb{N}$. Since $\#\Gamma x = \infty$ for all $x \in D_i$, Theorem 2 asserts that J satisfies condition $(PS)_c^\Gamma$ in $H_0^1(D_i)$ for all $c \in \mathbb{R}$. Hence, the mountain pass theorem [1] yields a nontrivial least energy Γ -invariant solution ω_{D_i} to problem (\wp_{D_i}) . We define

$$c_k := \inf \left\{ \sum_{i=1}^k J(\omega_{D_i}) : (D_1, \dots, D_k) \in \mathcal{P}_k(D) \right\} \quad \text{and} \quad \ell_k := c_\infty^{-1} c_k.$$

Note that $c_1 = J(\omega_D)$. Since $J(\omega_{D_i}) \geq c_\infty$, we have that

$$c_{k-1} + c_\infty \leq \sum_{i=1}^k J(\omega_{D_i})$$

for every $(D_1, \dots, D_k) \in \mathcal{P}_k(D)$, $k \geq 2$. It follows that

$$c_{k-1} + c_\infty \leq c_k \quad \text{and} \quad \ell_{k-1} + 1 \leq \ell_k.$$

Let $m \in \mathbb{N}$ and let Ω be a bounded smooth domain containing D , which is invariant under the action of a closed subgroup G of Γ such that

$$(2.1) \quad \min_{x \in \Omega} \#Gx > \ell_m.$$

Given $\varepsilon \in (0, c_\infty)$ with $c_m + \varepsilon < (\min_{x \in \Omega} \#Gx) c_\infty$, we choose $(D_1, \dots, D_m) \in \mathcal{P}_m(D)$ such that

$$c_m \leq \sum_{i=1}^m J(\omega_{D_i}) < c_m + \varepsilon.$$

Observe that $\omega_{D_i} \in H_0^1(\Omega)^G$ and satisfies

$$(2.2) \quad J(\omega_{D_i}) = \max_{t \geq 0} J(t\omega_{D_i}).$$

For each $k = 1, \dots, m$, let W_k be the subspace of $H_0^1(\Omega)^G$ generated by $\{\omega_{D_1}, \dots, \omega_{D_k}\}$ and $d_k := \sup_{W_k} J$. Since $D_i \cap D_j = \emptyset$ if $i \neq j$, the intersection of the supports of ω_{D_i} and ω_{D_j} has measure zero. Therefore ω_{D_i} and ω_{D_j} are orthogonal in $H_0^1(\Omega)^G$ and, consequently, $\dim W_k = k$. Identity (2.2) implies that

$$d_k = \sup_{W_k} J \leq \sum_{i=1}^k J(\omega_{D_i}) < \left(\min_{x \in \Omega} \#Gx \right) c_\infty.$$

Then, by Theorem 2, J satisfies $(PS)_c^G$ in $H_0^1(\Omega)$ for all $c \leq d_k$, so the mountain pass theorem [1] yields a positive critical point $u_1 \in H_0^1(\Omega)^G$ of J such that $J(u_1) \leq d_1$. Moreover, applying Theorem 3 to each W_k , we obtain $m - 1$ pairs of sign-changing critical points $\pm u_2, \dots, \pm u_m$ such that

$$J(u_k) \leq d_k \quad \text{for every } k = 1, \dots, m.$$

Note that

$$d_k + (m - k)c_\infty \leq \sum_{i=1}^m J(\omega_{D_i}) < c_m + \varepsilon,$$

so, since $\varepsilon \in (0, c_\infty)$, we conclude that

$$(2.3) \quad J(u_k) < c_m \quad \text{for every } k = 1, \dots, m - 1.$$

Next, we prove that we may choose u_m such that

$$(2.4) \quad J(u_m) \leq c_m.$$

Let $\varepsilon_n \in (0, c_\infty)$ be such that $\varepsilon_n \rightarrow 0$, and let $u_{m,n}$ denote the m -th critical point obtained by applying the previous argument with $\varepsilon = \varepsilon_n$. Then $J(u_{m,n}) < c_m + \varepsilon_n$. If $J(u_{m,n_0}) \leq c_m$ for some n_0 , we are done. If $J(u_{m,n}) > c_m$ for all n , then $J(u_{m,n}) \rightarrow c_m$. Since $\nabla J(u_{m,n}) = 0$ and J satisfies $(PS)_{c_m}^G$, there exists a $u_m \in H_0^1(\Omega)^G$ such that, after passing to a subsequence, $u_{m,n} \rightarrow u_m$. Therefore, u_m is a critical point of J with $J(u_m) = c_m$. Note that u_m is positive if $m = 1$ and it is sign-changing if $m \geq 2$. Moreover, (2.3) implies that $u_m \neq u_k$ for every $k = 1, \dots, m - 1$. This proves (2.4).

Finally, note that if Ω satisfies (2.1), then it also satisfies

$$\min_{x \in \Omega} \#Gx > \ell_k \quad \text{for each } k = 1, \dots, m.$$

So, applying the previous argument to each k , we obtain k pairs of G -invariant solutions $\pm u_1^k, \dots, \pm u_k^k$ to (φ_Ω) such that u_1^k is positive, u_2^k, \dots, u_k^k change sign, and

$$J(u_i^k) \leq c_k \quad \text{for every } i = 1, \dots, k.$$

Setting $u_1 := u_1^1$ and choosing $u_k \in \{u_2^k, \dots, u_k^k\}$ with $k \geq 2$ inductively such that $u_k \neq u_i$ for every $i = 1, \dots, k - 1$, we obtain m pairs of G -invariant solutions $\pm u_1, \dots, \pm u_m$ such that u_1 is positive, u_2, \dots, u_m change sign, and

$$J(u_k) \leq c_k \quad \text{for every } k = 1, \dots, m,$$

as claimed. □

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INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, CIRCUITO EXTERIOR C.U., 04510 MÉXICO D.F., MEXICO
E-mail address: monica.clapp@im.unam.mx

INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, CIRCUITO EXTERIOR C.U., 04510 MÉXICO D.F., MEXICO
E-mail address: jorgefaya@gmail.com