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ON THE REGULARITY OF SOLUTIONS OF THE INHOMOGENEOUS INFINITY LAPLACE EQUATION

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Abstract. We study the inhomogeneous infinity Laplace equation and prove that for bounded and continuous inhomogeneities, any blow-up is linear but not necessarily unique. If, in addition, the inhomogeneity is assumed to be C^1, then we prove that any solution is differentiable, i.e., that any blow-up is unique.

1. Introduction

1.1. **The problem.** We study the inhomogeneous infinity Laplace equation

$$\begin{cases} \Delta_\infty u = f \text{ in } B_1, \\ u = g \text{ on } \partial B_1, \end{cases} \tag{1.1}$$

for $f \in L^\infty(B_1) \cap C(B_1)$ and $g \in C(\partial B_1)$. Here

$$\Delta_\infty u = u_{x_i} u_{x_j} u_{x_i x_j},$$

as it was introduced by Aronsson in [Aro67]. This equation and its homogeneous sibling arise in many different contexts. The operator Δ_∞ has a natural connection to the problem of finding optimal Lipschitz extensions. One way to see this is to regard the equation as a limit of the p-Laplace equation, as $p \to \infty$. There is also a "tug-of-war" game approach to the infinity Laplace equation mentioned in, among others, [PSSW09]. The operator is also suggested to be used in image processing; see [CMS98].

The infinity Laplace operator has recently attracted much attention. In [ACJ04], various properties are discussed. In particular, it is settled that solutions to the homogeneous infinity Laplace equation enjoy the so-called comparison with cones property. It also turns out that this is equivalent to being a solution of the infinity Laplace equation; see [CEG01]. Until quite recently, the best regularity known for infinity harmonic functions was that they can be approximated arbitrarily well by linear functions at small scales; see for instance [CE00]. This is not in general enough to be differentiable. But then in [Sav05] and [ES08], the C^1 and $C^{1,\alpha}$ regularity of solutions were proved in dimension two. Later, in [ES11], it was proved in higher dimensions that the solution is differentiable, but without any sort of continuity estimate. For the inhomogeneous equation, some uniqueness and

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stability results are obtained in [LW08]. The main contribution of this paper is to provide results similar to those in [CE00] and [ES11] for the inhomogeneous equation.

1.2. **Main result.** In order to state our result we need to define the notion of blow-ups. We say that u_0 is a blow-up of u at $x_0 \in B_1$ if for some subsequence $r_k \to 0$ there holds

$$u_0(x) = \lim_{r_k \to 0} u_{r_k,x_0}(x),$$

where

$$u_{r_k,x_0}(x) = \frac{u(r_k x + x_0) - u(x_0)}{r_k},$$

and when $|x_0| = 0$, we will simply write u_r for $u_{r,0}$.

The first result states that if $f \in C(B_1) \cap L^\infty(B_1)$ and $g \in C(\partial B_1)$, then all blow-ups of u are linear.

Theorem 1. *Let u be a solution of* (1.1) *and assume $f \in C(B_1) \cap L^\infty(B_1)$ and $g \in C(\partial B_1)$. Then any subsequential limit of u_{r,x_0} is a linear function, for $x_0 \in B_1$.*

The second result states that if in addition $f \in C^1(B_1) \cap L^\infty(B_1)$, then there is a unique linear blow-up at all points; i.e., u is differentiable.

Theorem 2. *Let u be a solution of* (1.1) *and assume $f \in C^1(B_1) \cap L^\infty(B_1)$ and $g \in C(\partial B_1)$. Then for each $x_0 \in B_1$ there is a unique vector $a \in \mathbb{R}^n$ so that $u_{r,x_0} \to a \cdot x$. In particular, u is differentiable.*

We remark that even though Theorems 1 and 2 are stated for solutions in the unit ball, this is not a restriction, since one can by rescaling a solution in an arbitrary open set obtain a solution in the unit ball.

2. Assumptions and preliminaries

In this section we state precisely under which assumptions we are working and also some basic results that are by now quite well known.

Due to the structure of the equation it is necessary to consider solutions in the sense of viscosity solutions. We state the definition below. For more details concerning viscosity solutions, see for instance [CIL92].

Definition 1. A lower semi-continuous function $u : B_1 \to (-\infty, \infty]$ is a viscosity supersolution of (1.1) if, whenever $x_0 \in B_1$ and $\phi \in C^2(B_1)$ are such that

$$\phi(x_0) = u(x_0), \quad \phi(x) < u(x) \text{ for } x \neq x_0,$$

then

$$\Delta_\infty \phi(x_0) \leq 0.$$

Similarly, but with test functions from above, one can define viscosity subsolutions. A function is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.

However, nowhere in this paper is it necessary to dig into the theory of viscosity solutions. It is known (see for instance [LW08]) that there is a bounded and locally Lipschitz viscosity solution of (1.1) whenever f and g are continuous and bounded. Moreover, in the case that f does not change sign, the solution is known to be unique. Often we will use the expression "infinity harmonic function", which simply

means that the function solves the equation $\Delta_\infty u = 0$. We remind the reader of the notation

$$L_r^+(u,x) = \sup_{y\in\partial B_r(x)} \frac{u(y)-u(x)}{r}, \quad L_r^-(u,x) = \inf_{y\in\partial B_r(x)} \frac{u(y)-u(x)}{r},$$

where we will also write $L_r^\pm(u) = L_r^\pm(u,0)$ when there is no possible confusion. It is by now well known that for infinity harmonic functions, $\pm L_r^\pm(u,x)$ is non-negative and non-decreasing in r.

2.1. **Fundamental assumptions.** Now we state two assumptions under which we will work throughout the whole paper. We assume that our solution of (1.1) satisfies:

(1) $\pm L_r^\pm(u,x) \geq 5$ for all $x \in B_1$ and $r < \operatorname{dist}(x, \partial B_1)$,
(2) f is bounded away from zero.

We remark that by putting

$$\tilde{u}(x_1,\dots,x_{n+2}) = u(x_1,\dots,x_n) + 5x_{n+1} + C|x_{n+2}|^{\frac{4}{3}}$$

for some appropriate $C = C(f)$, we readily obtain that $\tilde{u}$ satisfies the assumptions above. Now, proving any regularity result for u (up to $C^{1,\frac{1}{3}}$ regularity) is equivalent to proving it for $\tilde{u}$. Hence, we can without loss of generality assume that the solution we are dealing with satisfies the above assumptions. This means that in order to prove any regularity result for the solutions of (1.1) that is valid in any dimensions, it is enough to prove it for the case when f has a sign and when $|\nabla u|$ is bounded from below by, for instance, 5. We remark that whenever (2) is satisfied, u is the unique solution to the problem given its boundary values, due to Theorem 5 in [LW08]. This property is crucial for the proof of Proposition 1.

3. Linearity of blow-ups

In this section we prove the linearity of the blow-ups. This is done using the semi-monotonicity of $L_r^\pm(u,x)$. We first recall the following result, which can be found in [CE00].

Lemma 1. *Suppose u solves $\Delta_\infty u = 0$ in $\mathbb{R}^n$ and that*

$$L_r^\pm(u,0) = L_0^\pm(u,0), \quad \pm L_r^\pm(u,y) \leq \pm L_0^\pm(u,0)$$

for all $r > 0$ and all $y \in \mathbb{R}^n$. Then u is linear.

Proof. This is exactly (8) and (9) in the proof given on pages 125-126 in [CE00]. It is proved on page 126 and the following pages that this implies linearity. □

Now we claim that we can compare solutions with functions of the type $|x|^\beta$, for $\beta < 1$, and at small scales the power β can be chosen arbitrarily close to 1. We compute

$$\Delta_\infty |x-x_0|^\beta = \beta^3(\beta-1)|x-x_0|^{3\beta-4}.$$

So for $x_0, x \in B_1$ we have

$$\Delta_\infty |x-x_0|^\beta < \beta^3(\beta-1). \tag{3.1}$$

With this in mind, we can prove the lemma below.

Lemma 2. *Suppose*

$$\Delta_\infty u = f \text{ in } B_1,$$

with $f \in C(B_1) \cap L^\infty(B_1)$ *such that* $|f| < \frac{1}{2}$. *Let* $C \geq 5$ *and* $r < 1$. *Then*

$$\sup_{\partial B_1} u_r(x) = C|x|^{\beta(r)} = C \text{ on } \partial B_1$$

implies

$$u_r(x) \leq C|x|^{\beta(r)} \text{ in } B_1.$$

Here $\beta(r) = 1 - r/C$ *and*

$$u_r(x) = \frac{u(rx) - u(0)}{r},$$

as before.

Proof. First observe that

$$\Delta_\infty u_r(x) = rf(rx)$$

with $|rf(rx)| < \frac{r}{2}$. Assume that the assertion is not true, i.e., that $u_r - C|x|^{\beta(r)}$ attains a positive maximum M at $x_0 \in B_1$. Clearly, $|x_0| \neq 0$. Then $C|x|^{\beta(r)} + M$ is a test function for u_r at x_0. Hence,

$$\Delta_\infty C|x|^{\beta(r)} > -r/2.$$

But at the same time, we have from (3.1) that

$$\Delta_\infty C|x|^{\beta(r)} \leq C^3\beta^3(\beta - 1) = -C^2 r\beta^3 < -25r(1 - 1/5)^3 < -\frac{r}{2},$$

a contradiction. □

The following result shows that the functions $L_r^\pm(u, x)$ are not monotone, but they are almost monotone, which is sufficient for our needs. The liaison can be made to the obstacle problem, where a similar monotonicity formula holds, and when you introduce a non-constant homogeneity, one obtains an almost monotonicity formula; see for instance [Wei99]. In that case, the monotonicity formula implies that all blow-ups are homogeneous of degree two. In our problem, as we will see, the almost monotonicity will imply that all blow-ups are linear.

Corollary 1. *Under the assumptions of the lemma above, the function* $L_r^+(u, x_0) + r$ *is non-decreasing for all* $x_0 \in B_1$ *and* $r < \operatorname{dist}(x_0, \partial B_1)$. *In particular,*

$$\lim_{r\to 0} L_r^+(u, x_0) + r = \lim_{r\to 0} L_r^+(u, x_0)$$

exists for all $x_0 \in B_1$. *The same holds for* $-L_r^-(u, x)$.

Proof. We treat only the case $|x_0| = 0$. By the assumptions on u, $L_r^+(u) \geq 5$. By the previous lemma

$$u_r(x) \leq L_1^+(u_r)|x|^{1-r/L_1^+(u_r)}$$

for $|x| \leq 1$ and $r \leq 1$. Therefore, we have for $s < 1$,

$$L_s^+(u_r) = \sup_{x\in\partial B_s} \frac{u_r(x)}{s} \leq \sup_{x\in\partial B_s} \frac{L_1^+(u_r)s^{1-r/L_1^+(u_r)}}{s} = L_1^+(u_r)s^{-r/L_1^+(u_r)}.$$

Hence,

$$\liminf_{s\to 1} \frac{L_r^+(u) - L_{rs}^+(u)}{r(1-s)} \geq \liminf_{s\to 1} \frac{L_1^+(u_r)(1 - s^{-r/L_1^+(u_r)})}{r(1-s)} = -1.$$

□

The corollary above provides a simple proof of the local Lipschitz regularity.

Corollary 2. *Let u be a solution of* (1.1). *Then u is locally Lipschitz, and, in particular,*

$$\|u\|_{C^{0,1}(B_{\frac{1}{2}})} \leq C(\|f\|_{L^\infty(B_1)}, \|g\|_{L^\infty(\partial B_1)}).$$

Proof. Take $x_0 \in B_{\frac{1}{2}}$. Let

$$v = \frac{u(x+x_0) - u(x_0)}{4^{\frac{1}{3}}\|f\|^{\frac{1}{3}}_{L^\infty(B_1)}}$$

so that v satisfies the assumptions of Lemma 2. We remark that we can still assume that v satisfies our fundamental assumptions, since if necessary we could have made the extension suggested in subsection 2.1 with even larger constants. Then, by Corollary 1 we have

$$L^+_r(v, x_0) + r \leq L^+_{\frac{1}{2}}(v, x_0) + \frac{1}{2}$$

for all $r \in (0, \frac{1}{2})$. Hence,

$$\sup_{\partial B_r} \frac{v(x)}{r} \leq L^+_{\frac{1}{2}}(v, x_0) + \frac{1}{2} \leq 4 \sup_{B_{\frac{1}{2}}} v + \frac{1}{2}.$$

Rescaling back to u we have

$$\sup_{\partial B_r} \frac{u(x+x_0) - u(x_0)}{r} \leq 4 \sup_{B_1} u + \frac{1}{2} 4^{\frac{1}{3}} \|f\|^{\frac{1}{3}}_{L^\infty(B_1)} \leq C(\|f\|_{L^\infty(B_1)}, \|g\|_{L^\infty(\partial B_1)}),$$

using the L^∞ bound on u from [LW08], which can be obtained by comparing it with functions of the type $C|x|^{\frac{4}{3}}$. The bound from below can be obtained similarly using L^-. The result follows. □

We are now ready to prove that all blow-ups are linear.

Proof of Theorem 1. We give the proof for $x_0 = 0$ and $u(0) = 0$. Up to rescaling u as in the proof above, we can assume that u satisfies the assumption of Lemma 2 so that $\pm L^\pm_r(u, x)$ are almost monotone in r, possibly with worse constant. With v as in the proof of Corollary 2, we have

$$\Delta_\infty v_r = r f(rx) \to 0,$$

and by Corollary 2 we have for $x, y \in B_{\frac{1}{2r}}$ that

$$|v_r(x)| \leq C|x|, \quad |v_r(x) - v_r(y)| \leq C|x-y|.$$

Therefore, up to a subsequence, v_r converges locally uniformly to a function v_0 that fulfills

$$\Delta_\infty v_0 = 0, \quad v_0(0) = 0.$$

Moreover, since $L^+_r(v)$ has a limit as $r \to 0$, we have

$$L^+_0(v) \leftarrow L^+_{sr}(v) = L^+_s(v_r) \to L^+_s(v_0).$$

Hence, $L^+_r(v_0)$ is constant in r. In addition, again by Corollary 1,

$$L^+_s(v_r, x) = L^+_{sr}(v, rx) \leq L^+_R(v, rx) + C(R - rs)$$

whenever $R > rs$ is small enough. Passing $r \to 0$ we obtain for all $x \in \mathbb{R}^n$ and for $R > 0$ small enough that

$$L_s^+(v_0, x) \leq L_R^+(v, 0) + CR.$$

Hence,

$$L_s^+(v_0, x) \leq L_0^+(v),$$

for all $x \in \mathbb{R}^n$. The corresponding holds also for L_s^-. By Lemma 1, v_0 must be linear. This clearly implies that any blow-up of the original function u must be linear as well. □

4. Differentiability

In this section we prove that all solutions of (1.1), with $f \in C^1(B_1) \cap L^\infty(B_1)$, are differentiable everywhere. The idea is to use the fact that all blow-ups are linear and then use the method of Evans and Smart in [ES11] to prove that any linear blow-up limit must be unique.

First we need the following quite expected result on solutions of the regularized version of $\Delta_\infty u = f$:

$$\begin{cases} \varepsilon \Delta u^\varepsilon + \Delta_\infty u^\varepsilon = f \text{ in } B_1, \\ u^\varepsilon = u \text{ on } \partial B_1. \end{cases} \tag{4.1}$$

Proposition 1. *There is a unique solution of* (4.1)*, and, moreover, the solution satisfies*

$$\sup_{B_1} |u^\varepsilon| + \sup_{B_{\frac{1}{2}}} |\nabla u^\varepsilon| \leq C$$

for some constant $C = C(\|u\|_{L^\infty(B_1)})$ *and*

$$u^\varepsilon \to u$$

uniformly in $\overline{B_1}$.

Proof. The proof is exactly the same as the one of Theorem 2.1 in [ES11] except for two extra terms in the estimates. Let us briefly spell out the details. For simplicity we write u instead of u_ε.

First of all, by standard theory for uniformly elliptic equations, there exists a unique smooth solution u such that

$$\sup_{B_1} |u| \leq \sup_{\partial B_1} |u|.$$

Now we remark that $u_k = \frac{\partial}{\partial x_k} u$ solves

$$(\varepsilon \Delta + L) u_k = \frac{\partial f}{\partial x_k} = f_k,$$

where

$$Lv = u_i u_j v_{ij} + 2 u_j u_{ij} v_i.$$

The differentiation is justified since for all positive ε, u is smooth. We compute

$$\begin{aligned} (\varepsilon \Delta + L)(|\nabla u|^2) &= 2\varepsilon |D^2 u|^2 + 2|D^2 u \nabla u|^2 + 2 \nabla f \cdot \nabla u \\ &\geq 2\varepsilon |D^2 u|^2 + 2|D^2 u \nabla u|^2 - C|\nabla u| \end{aligned} \tag{4.2}$$

and

$$(\varepsilon \Delta + L)(u^2) = 2uf + 2\varepsilon |\nabla u|^2 + 2|\nabla u|^4 + 4u \Delta_\infty u. \tag{4.3}$$

We remark that the only difference in these formulae from the corresponding ones in [ES11] are the terms $2\nabla f \cdot \nabla u$ and $2uf$, which can be estimated by $C|\nabla u|$ and C, since u is bounded.

Consider the function

$$w = \frac{1}{2}\eta^2|\nabla u|^2 + \frac{1}{2}\alpha u^2,$$

where $\eta \in C_0^\infty(B_1)$, $\eta = 1$ in $B_{\frac{1}{2}}$ and α is to be chosen later. First we compute

$$\begin{aligned}
&(\varepsilon\Delta + L)\left(\frac{1}{2}\eta^2|\nabla u|^2\right) \\
&= \frac{\eta^2}{2}(\varepsilon\Delta + L)(|\nabla u|^2) + \frac{|\nabla u|^2}{2}(\varepsilon\Delta + L)\eta^2 + 2\varepsilon\eta\nabla\eta D^2u\nabla u + 4u_iu_j\eta\eta_iu_ku_{kj} \\
&\geq \frac{\eta^2}{2}(\varepsilon\Delta + L)(|\nabla u|^2) - \frac{1}{2}\eta^2|D^2u\nabla u|^2 - C|\nabla u|^4 - C.
\end{aligned} \tag{4.4}$$

Hence, at a point where w attains a maximum, we then have, owing to (4.2), (4.3), (4.4) and Young's inequality,

$$\begin{aligned}
0 \geq (\varepsilon\Delta + L)w \geq \eta^2\left(|D^2u\nabla u|^2 + \varepsilon|D^2u|^2\right) + \alpha\left(|\nabla u|^4 + \varepsilon|\nabla u|^2\right) \\
- C\alpha|D^2u\nabla u||\nabla u| - C|\nabla u|^4 - \frac{1}{2}\eta^2|D^2u\nabla u|^2 - C|\nabla u| - C\alpha.
\end{aligned}$$

Using Young's inequality and taking α large enough, we can conclude that at that point

$$C|\nabla u| + C + C|D^2u\nabla u||\nabla u| \geq \eta^2|D^2u\nabla u|^2 + |\nabla u|^4.$$

Applying Young's inequality with ε on the gradient term and the usual Young's inequality on the term involving D^2u, we arrive at

$$\eta^2|D^2u\nabla u|^2 + C|\nabla u|^4 \leq C + C|D^2u\nabla u|^{\frac{4}{3}}.$$

Multiplying the above inequality with η^4 yields

$$\eta^6|D^2u\nabla u|^2 + C\eta^4|\nabla u|^4 \leq C\eta^4 + C\eta^4|D^2u\nabla u|^{\frac{4}{3}} \leq C\eta^4 + C + \eta^6|D^2u\nabla u|^2,$$

where we have used Young's inequality once more. Thus,

$$C\eta^4|\nabla u|^4 \leq C\eta^4 + C,$$

at a point where w attains its maximum. Hence, ∇u is locally bounded in B_1.

In order to obtain the equicontinuity up to the boundary, one remarks that, due to (3.1), functions of the type

$$w(x) = |x - x_0|^\beta\lambda,$$

with $\beta \in (0,1)$ and $\lambda > 0$ large enough, are supersolutions. This implies that we have the estimate

$$|u(x) - u(x_0)| \leq C|x - x_0|^\beta,$$

for $x \in B_1$ and $x_0 \in \partial B_1$. Thus we can extract a subsequence $u_i = u_{\varepsilon_i}$ so that u_i converges uniformly in $\overline{B}_1$ to a solution of (1.1).

It remains to remark that due to assumption (2) of subsection 2.1, the solution of (1.1) is the unique one with its boundary values. Hence, the whole sequence converges uniformly in $\overline{B}_1$ to the unique solution u. □

The key result is the lemma below.

Lemma 3. *Let u^ε be a solution to*

$$\varepsilon\Delta u^\varepsilon + \Delta_\infty u^\varepsilon = f \text{ in } B_2,$$

with $f \in C^1(B_2) \cap L^\infty(B_2)$, and assume

$$\sup_{B_2} |u^\varepsilon - x_n| \le \lambda$$

for λ small enough. Then

$$|\nabla u^\varepsilon|^2 \le u^\varepsilon_{x_n} + C\left(\lambda^{\frac{1}{4}} + \frac{\varepsilon^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\right) \text{ in } B_1.$$

Here C is a constant depending only on $\|\nabla u^\varepsilon\|_{L^\infty(B_2)}$ and $\|u^\varepsilon\|_{L^\infty(B_2)}$.

The proof follows the proof of Theorem 2.2 in [ES11] with only minor modifications.

Proof. For simplicity we write u instead of u_ε. First we remark again that $u_k = \frac{\partial}{\partial x_k} u$ solves

$$(\varepsilon\Delta + L)u_k = \frac{\partial f}{\partial x_k} = f_k,$$

where

$$Lv = u_i u_j v_{ij} + 2u_j u_{ij} v_i.$$

Furthermore, as before,

$$(\varepsilon\Delta + L)(|\nabla u|^2) = 2\varepsilon|D^2u|^2 + 2|D^2u\nabla u|^2 + 2\nabla f \cdot \nabla u \tag{4.5}$$

and

$$\begin{aligned}
&(\varepsilon\Delta + L)[(u - x_n)]^2 \\
&= 2(u - x_n)f + 2\varepsilon|\nabla u - e_n|^2 + 2(|\nabla u|^2 - u_n)^2 + 4(u - x_n)(\Delta_\infty u - u_i u_{in}) \\
&\ge 2(|\nabla u|^2 - u_n)^2 - C\lambda|D^2u\nabla u| - C\lambda, \qquad (4.6)
\end{aligned}$$

where we have used that $|u^\varepsilon - x_n| \le \lambda$, $|\nabla u| \le C$ and $|f| \le C$.

Let $\phi(p) = (|p|^2 - p_n)_+^2$, where the subscript $+$ denotes the positive part. At a point where $|\nabla u|^2 - u_n > 0$ we use the equation for u_k, which multiplied by $\frac{\partial\phi}{\partial p_k}(|\nabla u|)$ implies

$$(\varepsilon\Delta + L)u_k \frac{\partial\phi}{\partial p_k}(|\nabla u|) = f_k \frac{\partial\phi}{\partial p_k}(|\nabla u|) = 2f_k(|\nabla u|^2 - u_n)(2u_k - \delta_{kn}) := g. \tag{4.7}$$

We expand the left hand side to be

$$\begin{aligned}
&(\underbrace{2\varepsilon\Delta u_k u_k - \varepsilon\Delta u_n}_{A} + \underbrace{2u_i u_j u_{ijk} u_k - u_i u_j u_{ijn}}_{B} + \underbrace{4u_j u_{ij} u_{ik} u_k - 2u_j u_{ij} u_{in}}_{C}) \\
&\times 2(|\nabla u|^2 - u_n).
\end{aligned}$$

Moreover,

$$
\begin{aligned}
&(\varepsilon\Delta + L)[(|\nabla u|^2 - u_n)^2] \\
&= 2\varepsilon \underbrace{(2u_k u_{ik} - u_{in})(2u_\ell u_{i\ell} - u_{in})}_{\geq 0} + 2\varepsilon(|\nabla u|^2 - u_n)(2u_{i\ell}^2 + \underbrace{2u_\ell \Delta u_\ell - \Delta u_n}_{A}) \\
&+ 2u_i u_j (2u_k u_{ik} - u_{in})(2u_\ell u_{k\ell} - u_{jn}) \\
&+ 2u_i u_j (|\nabla u|^2 - u_n)(2u_{i\ell} u_{j\ell} + \underbrace{2u_\ell u_{ij\ell} - u_{ijn}}_{B}) \\
&+ 2(|\nabla u|^2 - u_n)\underbrace{(4u_j u_{ij} u_\ell u_{i\ell} - 2u_j u_{ij} u_{in})}_{C} \\
&\geq 4(|\nabla u|^2 - u_n)(\varepsilon|D^2 u|^2 + |D^2 u \nabla u|^2) + 2(2\Delta_\infty u - u_i u_{in})^2 + g \\
&\geq 4(|\nabla u|^2 - u_n)|D^2 u \nabla u|^2 - C(|\nabla u|^2 - u_n),
\end{aligned}
\tag{4.8}
$$

whenever $\phi(|\nabla u|) > 0$. Here we used (4.7) and estimated g from below. Now take $\eta \geq 0$, $\eta = 1$ on B_1 and $\eta \in C_0^\infty(B_2)$. Then we can compute using (4.8), $|\nabla u| \leq C$ and Young's inequality, again at a point where $\phi(|\nabla u|) > 0$,

$$
\begin{aligned}
&(\varepsilon\Delta + L)(\eta^2 \phi(|\nabla u|)) \\
&= \eta^2 (\varepsilon\Delta + L)(\phi(|\nabla u|)) + \phi(|\nabla u|)(\varepsilon\Delta + L)(\eta^2) \\
&\quad + 2\varepsilon \nabla\eta^2 \nabla\phi(|\nabla u|) + 4u_i u_j \eta\eta_i (\phi(|\nabla u|))_j \\
&\geq -C(|\nabla u|^2 - u_n) - C\varepsilon|D^2 u|.
\end{aligned}
\tag{4.9}
$$

Let $v = \eta^2\phi(|\nabla u|) + \alpha(u - x_n)^2 + \lambda|\nabla u|^2$, where $\alpha = \max(2C, \lambda^{-\frac{1}{2}})$. We will now prove that

$$
v \leq C\left(\sqrt{\lambda} + \frac{\varepsilon}{\lambda}\right). \tag{4.10}
$$

This will then imply the desired result.

If v attains a maximum on ∂B_2, then at that point

$$
v = \alpha(u - x_n)^2 + \lambda|\nabla u|^2 \leq \alpha\lambda^2 + C\lambda \leq (2C + \lambda^{-\frac{1}{2}})\lambda^2 + C\lambda \leq C\sqrt{\lambda}.
$$

Moreover, the same estimate holds if the maximum is attained at a point where $\phi(|\nabla u|) \leq 0$. Consequently, we can assume that the maximum is attained at a point where $\phi(|\nabla u|) > 0$. Then $(\varepsilon\Delta + L)v \leq 0$ at that point. Combining (4.5), (4.6) and (4.9) we have

$$
\begin{aligned}
(\varepsilon\Delta + L)v &= (\varepsilon\Delta + L)(\eta^2\phi(|\nabla u|)) + \alpha(\varepsilon\Delta + L)(u - x_n)^2 + \lambda(\varepsilon\Delta + L)(|\nabla u|^2) \\
&\geq -C\varepsilon|D^2 u| - C(|\nabla u|^2 - u_n) + 2\lambda\varepsilon|D^2 u|^2 \\
&\quad + 2\lambda|D^2 u \nabla u|^2 + 2\lambda\nabla f \cdot \nabla u + 2\alpha(|\nabla u|^2 - u_n)^2 - C\alpha\lambda|D^2 u \nabla u| - C\lambda\alpha.
\end{aligned}
$$

Rearranging, and using that $|2\lambda\nabla f \cdot \nabla u| \leq C\lambda$, we obtain

$$
\begin{aligned}
&2\lambda\varepsilon|D^2 u|^2 + 2\lambda|D^2 u \nabla u|^2 + 2\alpha(|\nabla u|^2 - u_n)^2 \\
&\quad \leq C\varepsilon|D^2 u| + C(|\nabla u|^2 - u_n) + C\alpha\lambda|D^2 u \nabla u| + C\lambda\alpha + C\lambda.
\end{aligned}
$$

Now we use Young's inequality to say

$$C\alpha\lambda|D^2u\nabla u| \leq C\alpha^2\lambda + 2\lambda|D^2u\nabla u|^2$$

and

$$C\varepsilon|D^2u| \leq \varepsilon\left(2\lambda|D^2u|^2 + \frac{C}{\lambda}\right),$$

where possibly C is another bigger constant. This implies that

$$2\alpha(|\nabla u|^2 - u_n)^2 \leq C\alpha^2\lambda + \frac{C\varepsilon}{\lambda} + C\alpha\lambda + C\lambda + C(|\nabla u|^2 - u_n)$$

so that from Young's inequality

$$2(|\nabla u|^2 - u_n)^2 \leq C\alpha\lambda + \frac{C\varepsilon}{\alpha\lambda} + C\lambda + \frac{C}{\alpha}\left((|\nabla u|^2 - u_n)^2 + 1\right).$$

Using $\alpha = \max(2C, \lambda^{-\frac{1}{2}})$ finally yields

$$\begin{aligned}(|\nabla u|^2 - u_n)^2 &\leq C\alpha\lambda + \frac{\varepsilon}{2\lambda} + C\lambda + \frac{C}{\alpha}\\ &\leq C(2C + \lambda^{-\frac{1}{2}})\lambda + C\lambda + \frac{\varepsilon}{2\lambda} + C\lambda^{\frac{1}{2}}\\ &\leq C\left(\sqrt{\lambda} + \frac{\varepsilon}{\lambda}\right).\end{aligned}$$

Hence, in all cases, (4.10) holds. From this, the result follows. □

At this point, we need the following lemma from [ES11], more specifically Lemma 3.1.

Lemma 4. *Assume $|b| = 1, b \in \mathbb{R}^n$ and v is smooth. If*

$$\sup_{B_1} |v - b \cdot x| \leq \eta,$$

then there is $y \in B_1$ so that $|\nabla v(y) - b| \leq 4\eta$.

We are now ready to prove the differentiability. This is exactly as in the proof in [ES11], except that in (4.11) we choose $2C\lambda^{\frac{1}{4}} = \theta/4$ instead of $\lambda^{\frac{1}{2}}$.

Proof of Theorem 2. We do the proof only at the origin. First, re-scale u so that it is a solution of (1.1) in B_4. By abuse of notation, we denote our new solution by u as well. The argument is by contradiction. Assume that we have two sequences $r_k, s_k \to 0$ so that

$$u_{s_k} \to a \cdot x, \quad u_{r_k} \to b \cdot x,$$

where $a \neq b$. Then we know

$$|a| = |b| = \lim_{r\to 0} \pm L_r^{\pm}(u).$$

Assume for simplicity that $a = e_n$, $b \neq e_n$ and

$$\theta = 1 - b_n > 0.$$

Take C as the constant appearing in the preceding lemma and choose λ so that

$$2C\lambda^{\frac{1}{4}} = \frac{\theta}{4} \tag{4.11}$$

and also put

$$\varepsilon_1 = \lambda^2.$$

Now take $r = r(\lambda)$ so that

$$\sup_{B_1} |u_r - x_n| = \sup_{B_r} \frac{|u - x_n|}{r} \leq \frac{\lambda}{4}.$$

Take $\varepsilon_2 = \varepsilon_2(r, \lambda)$ so small that

$$\sup_{B_2} |v_r^\varepsilon - x_n| \leq \lambda,$$

where v^ε is the solution to the regularized equation in B_2, i.e., with $\varepsilon\Delta + \Delta_\infty$. This is possible due to the uniform convergence $v^\varepsilon \to u$ in $\overline{B_2}$.

Now pick η so that $12\eta = \theta/4$, choose $s = s(\eta)$ so that

$$\sup_{B_s} \frac{|u - b \cdot x|}{s} = \sup_{B_1} |u_s - b \cdot x| \leq \eta/2$$

and choose $\varepsilon_3 = \varepsilon_3(s, \eta)$ so that

$$\sup_{B_s} \frac{|v^\varepsilon - b \cdot x|}{s} \leq \eta, \text{ for } \varepsilon \leq \varepsilon_3.$$

If we apply Lemma 4, we obtain $y \in B_s$ so that $|\nabla v(y) - b| \leq 4\eta$, which implies

$$|v_n^\varepsilon(y) - b_n| \leq 4\eta. \tag{4.12}$$

Since $|b| = 1$, we have as a consequence

$$|\nabla v^\varepsilon(y)| \geq 1 - 4\eta. \tag{4.13}$$

Here is the point where we apply Lemma 3 to v_s^ε with $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ to obtain

$$\begin{aligned} |\nabla v^\varepsilon(y)|^2 &\leq v_n^\varepsilon(y) + C\left(\lambda^{\frac{1}{4}} + \frac{\varepsilon^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\right) \\ &\leq v_n^\varepsilon(y) + 2C\lambda^{\frac{1}{4}} \leq v_n^\varepsilon(y) + \frac{\theta}{4}. \end{aligned}$$

But from (4.12) and (4.13) we have

$$\begin{aligned} (1 - 4\eta)^2 = 1 + 16\eta^2 - 8\eta \leq |\nabla v^\varepsilon(y)|^2 &\leq v_n^\varepsilon(y) + \frac{\theta}{4} \\ &\leq |v_n^\varepsilon(y) - b_n| + b_n + \frac{\theta}{4} \leq 4\eta + b_n + \frac{\theta}{4} \end{aligned}$$

so that

$$\theta = 1 - b_n \leq 12\eta + \frac{\theta}{4} = \frac{\theta}{2},$$

which clearly is a contradiction. □

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