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ON SUMS OF ADMISSIBLE COADJOINT ORBITS

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Abstract. Given a quasi-Hermitian semisimple Lie algebra, we describe possible spectra of the sum of two admissible elements from its dual vector space.

1. Introduction

For a compact Lie algebra $\mathfrak{k}$, the question of finding possible spectra of the sum of two elements from $\mathfrak{k}^*$ has been answered in great detail and turned out to be related to many different areas of mathematics, including representation theory, combinatorics, symplectic geometry, geometric invariant theory and others; see [2] and the references therein. However, the non-compact case seems to remain untouched, for the reasons that the non-compact coadjoint orbits do not possess the necessary nice properties and the sum of two non-compact orbits can in general cover pretty much all arbitrary spectra.

However, there is a class of coadjoint orbits in a quasi-Hermitian semisimple Lie algebra $\mathfrak{g}^*$ which we call admissible and which shares certain properties of the compact case. For example, the moment map for the maximal torus action on such orbits is proper and the image is semibounded. The set of admissible orbits forms a double cone (i.e. the union of a cone and its negative), and we denote by $\mathfrak{g}^*_{\mathrm{adm}}$ the interior of one of its halves. In this setup, it now makes sense to pose the question about the possible spectra of the sum of the orbits of two chosen elements from the dual space of a maximal torus $\mathfrak{t}^*$. For example, if a and b are positive real numbers and A and B are two matrices which are $\mathrm{SU}(1,1)$-conjugate to $\mathrm{diag}(a,-a)$ and $\mathrm{diag}(b,-b)$ respectively, then possible eigenvalues $(c,-c)$ of their sum $A+B$ must necessarily satisfy the reversed triangle inequality $c \geq a+b$.

In general, given two admissible $A, B \in \mathfrak{g}^*_{\mathrm{adm}}$ with prescribed spectra Λ_A and Λ_B respectively, we show that the possible spectra of $A+B$ belong to a convex polyhedral set $(\Pi + \mathcal{C}) \cap \mathfrak{t}^*_+$, where Π is the polytope that solves the problem for the maximal compact subalgebra $\mathfrak{k} \subset \mathfrak{g}$, and $\mathcal{C}$ is the cone defined by the positive non-compact roots. The main ingredients here are Weinstein's generalization of the Kirwan convexity theorem for semisimple Lie groups [11], the Hilgert-Neeb-Plank abelian convexity theorem for non-compact manifolds [5], the Bates-Lerman local normal form [1], and Sjamaar's construction of local cones [10].

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In the last section we describe a relationship with the tensor products of the holomorphic discrete series representations. More precisely, we will see that the irreducible components of such tensor products are parameterized by the lattice points inside the convex polyhedral sets described above.

2. Admissible coadjoint orbits

Let $G^{\mathbb{C}}$ be a complex semisimple Lie group and $\mathfrak{g}^{\mathbb{C}}$ its Lie algebra. Recall the construction of real forms of $\mathfrak{g}^{\mathbb{C}}$ using Vogan diagrams [6, Theorem 6.88]. First, we need to fix some data for $\mathfrak{g}^{\mathbb{C}}$. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ and let Δ be the root system for $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h})$. Fix a choice of positive roots Δ^+ and let Σ be the basis of simple roots. Let $\ll, \gg$ be the Killing form of $\mathfrak{g}^{\mathbb{C}}$ and let root vectors $\{E_\alpha : \alpha \in \Delta\}$ be chosen such that $[E_\alpha, E_{-\alpha}] = H_\alpha$ for each $\alpha \in \Delta^+$, where H_α is the unique element of $\mathfrak{h}$ defined by $\ll H, H_\alpha \gg = \alpha(H)$ for all $H \in \mathfrak{h}$, and such that the numbers $m_{\alpha,\beta}$ given by $[E_\alpha, E_\beta] = m_{\alpha,\beta} E_{\alpha+\beta}$ when $\alpha + \beta \in \Delta$ are real. Define a compact real form $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{C}}$ as

$$\mathfrak{u} = \operatorname{span}_{\mathbb{R}}\{\sqrt{-1}H_\alpha,\, X_\alpha := E_\alpha - E_{-\alpha},\, Y_\alpha := \sqrt{-1}(E_\alpha + E_{-\alpha})\}\ ,$$

and let θ be the complex conjugation of $\mathfrak{g}$ defining $\mathfrak{u}$.

Given a Vogan diagram v for $\mathfrak{g}^{\mathbb{C}}$, normalized (i.e. at most one painted root in each connected component) and with the trivial automorphism, let t_v be the unique element in the adjoint group of $\mathfrak{g}^{\mathbb{C}}$ such that

$$\operatorname{Ad}_{t_v}(E_\alpha) = \begin{cases} E_\alpha \text{ if } \alpha \text{ is a blank vertex in } v, \\ -E_\alpha \text{ if } \alpha \text{ is the painted vertex in } v. \end{cases}$$

Define a complex conjugate linear involution

$$\tau := \operatorname{Ad}_{t_v} \circ \theta.$$

We use $\mathfrak{g} = (\mathfrak{g}^{\mathbb{C}})^\tau$ to denote the real form of $\mathfrak{g}^{\mathbb{C}}$ defined by τ. Then θ restricts to a Cartan involution of $\mathfrak{g}$, and $\mathfrak{h}^\tau = \mathfrak{t}$ is a compact Cartan subalgebra of $\mathfrak{g}$. The complexification of τ is

$$\gamma := \tau\theta = \theta\tau = \operatorname{Ad}_{t_v}. \tag{2.1}$$

Since $\gamma(\Delta^+) = \Delta^+$, the Vogan diagram of $\mathfrak{g}^{\mathbb{C}}$ associated to the triple $(\mathfrak{g}, \mathfrak{t}, \Delta^+)$ is v. Moreover, every semisimple real Lie algebra of inner type (i.e. those that have a compact Cartan subalgebra) can be obtained this way [6].

We naturally call the roots from Δ^+, as well as their negatives, *compact* if they do not have a painted root in their decomposition into the sum of simple roots from Σ and *non-compact* otherwise. In what follows, we assume that the Lie algebra $\mathfrak{g}$ is quasi-Hermitian (i.e. maximal compact subalgebras in every simple factor have non-trivial centers) and that the system of positive non-compact roots Δ^+_{nc} is adapted [7], i.e. is invariant under the baby Weyl group $W_{\mathfrak{k}}$, the Weyl group of the pair $(\mathfrak{k}, \mathfrak{t})$, where $\mathfrak{k}$ is the maximal compact subalgebra $\mathfrak{k} = (\mathfrak{g})^\theta$.

Consider the dual vector space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$, which can be identified using the positive definite inner product $-\ll \cdot, \theta\cdot \gg$. Now we recall the definition of two invariant cones in $\mathfrak{t}$:

$$C_{\min} = \operatorname{Cone}\{\sqrt{-1}[E_\alpha, \theta E_\alpha], \alpha \in \Delta^+_{\mathrm{nc}}\} \ \text{ and } \ C_{\max} = \sqrt{-1}(\Delta^+_{\mathrm{nc}})^*\ .$$

Now we define the open convex cone of admissible elements $\mathfrak{t}^*_{\mathrm{adm}} \subset \mathfrak{t}^*$ as the relative interior of the dual to the minimal cone, $C^*_{\min}$. Using the above pairing, we can think of $\mathfrak{t}^*$ as a subspace of $\mathfrak{g}^*$ and define the open cone of admissible elements in the latter as $\mathfrak{g}^*_{\mathrm{adm}} = \mathrm{Ad}^*(\mathfrak{t}^*_{\mathrm{adm}})$. (What we call admissible in this paper is usually called strictly admissible in the literature, but this should not lead to confusion.)

Consider $X \in \mathfrak{t}^*_{\mathrm{adm}}$ and its coadjoint orbit $\mathcal{O}_X$. The coadjoint action of G on $\mathcal{O}_X$ is proper, as well as the corresponding T-moment map $\mathcal{O}_X \to \mathfrak{t}^*$, given by the projection $\mathfrak{g}^* \to \mathfrak{t}^*$ dual to the inclusion. The image of this moment map is the sum of the polytope $\mathrm{Conv}(W_{\mathfrak{k}}.X)$ and the cone spanned by $\sqrt{-1}\Delta^+_{\mathrm{nc}}$; see [5].

Example. Consider $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$ identified with the space of traceless complex matrices, $\mathfrak{g} = \mathfrak{su}(p, q)$, the subspace of matrices B satisfying $BJ_{pq} + J_{pq}B^* = 0$; and let $\mathfrak{g}^*$ be its dual vector space, which is identified with the space $\sqrt{-1} \cdot \mathfrak{g}$ of pseudo-Hermitian matrices A satisfying $AJ_{pq} = J_{pq}A^*$. Here $J_{pq} = \mathrm{diag}(\underbrace{1, ..., 1}_{p}, \underbrace{-1, ..., -1}_{q})$ and A^* is the conjugate transpose. In block form,

$$A = \begin{pmatrix} H_p & B \\ -\bar{B}^T & H_q \end{pmatrix} ,$$

where H_p and H_q are $p \times p$ and $q \times q$ Hermitian symmetric matrices respectively and B is a complex $p \times q$ matrix. If we, as usual, take the upper-triangular matrices as the Borel subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ defined by the positive roots, then the cone of admissible elements $\mathfrak{g}^*_{\mathrm{adm}}$ would consist of matrices which are $\mathrm{SU}(p, q)$-conjugate to the diagonal (and thus real) matrices of the form $\mathrm{diag}(\lambda_1, ..., \lambda_p, \mu_1, ..., \mu_q)$ such that $\lambda_i > \mu_j$ for all pairs i, j. We can certainly assume that λ's are arranged in the non-decreasing order $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$ and μ's are in the non-increasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_q$ (this is done for convenience), and thus the condition of admissibility becomes rather simple: $\lambda_1 > \mu_1$.

3. Non-abelian convexity

Let $\Lambda_A, \Lambda_B \in \mathfrak{t}^*_{\mathrm{adm}}$ and let $\mathcal{O}_A$ and $\mathcal{O}_B$ be the corresponding coadjoint orbits. We would like to describe the intersection $(\mathcal{O}_A + \mathcal{O}_B) \cap \mathfrak{t}^*_+$. In other words, given admissible A and B with fixed spectra, we would like to know possible values of the spectrum of their sum $A + B$.

Both orbits $\mathcal{O}_A$ and $\mathcal{O}_B$ are Hamiltonian G-spaces with the moment maps simply given by their inclusions into $\mathfrak{g}^*$. Their product $(\mathcal{O}_A \times \mathcal{O}_B)$ is a Hamiltonian G-space as well, with the moment map Φ equal to the sum of the two inclusions. We note that according to a generalization due to Weinstein [11] of the Kirwan convexity theorem, the intersection $\Phi(\mathcal{O}_A \times \mathcal{O}_B) \cap \mathfrak{t}^*_+$ is a convex polyhedral set $\mathcal{S}_{AB}$ describing exactly all possible spectra of such sums $A + B$. In this section we will give a more detailed description of this set.

We will start by describing the local convexity data. For brevity, denote $M = \mathcal{O}_A \times \mathcal{O}_B$ and ω the product symplectic form on M. For any subset $Q \subset M$ denote by $\mathcal{S}(Q)$ the image of $\Phi(Q)$ in $\mathfrak{t}^*_{\mathrm{adm}} = \mathfrak{g}^*_{\mathrm{adm}}/G \subset \mathfrak{t}^*_+$. For a given point $m \in M$, let G_m be its stabilizer, and let G_y be the stabilizer of $y = \Phi(m)$ with respect to the coadjoint action of G. Note that since $\Phi(M)$ is in the admissible cone, where the action of G is proper, G_y and its subgroup G_m are both compact. Let $\mathfrak{g}_m$ and $\mathfrak{g}_y$ denote the corresponding subalgebras of $\mathfrak{g}$. Using the Killing form as before, we

can also think of $\mathfrak{g}_m^*$ as a subspace of $\mathfrak{g}_y^*$ and the latter as a subspace of $\mathfrak{g}^*$. Let $\mathfrak{g}_m^\perp$ be the annihilator of $\mathfrak{g}_m$ in $\mathfrak{g}_y^*$ and $\mathfrak{g}_y^\perp$ be the annihilator of $\mathfrak{g}_y$ in $\mathfrak{g}^*$. Then we have a G_m-equivariant splitting $\mathfrak{g}^* = \mathfrak{g}_m^* \oplus \mathfrak{g}_m^\perp \oplus \mathfrak{g}_y^\perp$, and we denote $i : \mathfrak{g}_m^* \to \mathfrak{g}_y^*$ and $j : \mathfrak{g}_y^* \to \mathfrak{g}^*$ as the corresponding injections. Also let $\mathcal{O}_m$ be the G-orbit through m and $\mathcal{O}_y$, the coadjoint orbit of y. Consider the symplectic vector space

$$V = (T_m\mathcal{O}_m)^\perp/((T_m\mathcal{O}_m)^\perp \cap T_m\mathcal{O}_m), \tag{3.1}$$

where $\perp$ stands for the symplectic perp. This space has a natural linear symplectic action of G_m, with moment map $\Psi_V : V \to \mathfrak{g}_m^*$. A theorem of Bates and Lerman [1] extending the results of Guillemin-Sternberg and Marle asserts that:

Proposition 3.1. *There exists a G-invariant neighbourhood U of $\mathcal{O}_m$ in M and a G-invariant neighbourhood U_0 of the zero section of the vector bundle $G \times_{G_m} (\mathfrak{g}_m^\perp \times V) \to G/G_m$, and a G-equivariant symplectomorphism $\eta : U_0 \to U$ such that*

$$\Phi(\eta(g, X, v)) = \mathrm{Ad}_g^*(y + j(X + i(\Psi_V(v)))) \ .$$

Next, recall the constructive proof of the non-abelian convexity theorem, due to Sjamaar, as explained in [3]. This result gives a concrete description of the local structure of the Kirwan polytope, and the convexity is explained in terms of "T to K induction". Denote by Π a convex polytope in $\mathfrak{t}_+^*$, which appears in the Kirwan convexity theorem for the maximal compact subgroup $K \subset G$:

$$\Pi := \mathcal{S}\left(\Phi(\mathrm{Ad}_K^*(\Lambda_A) \times \mathrm{Ad}_K^*(\Lambda_B))\right).$$

Sjamaar's proof of the Kirwan convexity theorem readily extends to our current setup, mainly due to the fact that the actions are proper and all the stabilizers are compact.

Theorem 3.2. *The image $\mathcal{S}_{AB} = \mathcal{S}(M)$ in $\mathfrak{t}_{\mathrm{adm}}^*$ is the intersection of local moment cones and is given by a convex polyhedral set $(\Pi + \mathrm{Cone}(\sqrt{-1}\Delta_{\mathrm{nc}}^+)) \cap \mathfrak{t}_+^*$. A point y is an extremal point of $\mathcal{S}(M)$ if and only if $\mathfrak{g}_y = [\mathfrak{g}_y, \mathfrak{g}_y] + \mathfrak{g}_m$, where $m \in \Phi^{-1}(y)$.*

Proof. The proof of the theorem consists of several steps.

First we note that $\Phi(M) \subset \mathfrak{g}_{\mathrm{adm}}^*$; therefore (M, Φ) is a Hamiltonian $(G, \mathfrak{g}_{\mathrm{adm}}^*)$-space. By a theorem of Weinstein [11, Theorem 3.2] a set $\Phi(M) \cap \mathfrak{t}_+^* = \mathcal{S}(M)$ is a closed, convex, locally polyhedral subset of $\mathfrak{g}_{\mathrm{adm}}^* \cap \mathfrak{t}_+^* = \mathfrak{t}_{\mathrm{adm}}^*$.

In order to obtain the description stated in the theorem we need a local description of this polyhedral set. In what follows we discuss it in detail.

By Proposition 3.1, in a G-invariant neighborhood of point m we have a simple canonical model given by $Y = G \times_{G_m} (\mathfrak{g}_m^\perp \times V)$. The symplectic manifold Y can be realized as a symplectic quotient of $X = G \times \mathfrak{g}_y^* \times T^* G_y \times V$ by the $G_y \times G_m$ action. Now, combining this with the commutativity of reduction in stages, we are able to compute the *local moment cones*. Let us first describe the Hamiltonian structure on X more explicitly. The manifold $G \times \mathfrak{g}_y^*$ carries a natural closed two-form (the minimal-coupling form; see [4]), which is non-degenerate in a G-invariant neighborhood of $G \times \{0\}$. Now, identifying $T^* G_y$ with $G_y \times \mathfrak{g}_y^*$ by means of left translations, the action of $G_y \times G_m$ on X is given by

$$(g, h) \,.\, (l, \xi, k, \mu, v) = (lg^{-1}, \mathrm{Ad}_g^* \xi, gkh^{-1}, \mathrm{Ad}_h^* \mu, hv) \ ,$$

which is Hamiltonian with the moment map

$$\Psi(l, \xi, k, \mu, v) = (-\xi + \mathrm{Ad}_k^* \mu, -\pi(\mu) + \Phi_V(v)) \ ,$$

where Φ_V is the moment map for the linear symplectic action of G_m on V and $\pi : \mathfrak{g}_y^* \to \mathfrak{g}_m^*$ is the natural projection. Note that since both G_y and G_m are compact, Ψ is proper. Since $G_y \times G_m$ acts freely, the symplectic quotient $X_y = \Psi^{-1}(-y,0)/(G_y \times G_m)$ is a smooth manifold. The space X_y carries a natural G-action inherited from X, which is Hamiltonian. Now, in order to obtain the desired description of Y, consider the map ϕ from $G \times \mathfrak{g}_m^\perp \times V$ to $\Psi^{-1}(-y,0)$ defined by

$$\phi(l,\mu,v) \;=\; (l, \mu + \Phi_V(v) + y, 1, \mu + \Phi_V(v), v) \ ,$$

which clearly is G-equivariant. Therefore ϕ descends to the following G-equivariant diffeomorphism:

$$\bar{\phi} : Y \to X_y \ .$$

Now, using the equivariant Darboux theorem, one can see that this map is in fact a symplectomorphism. If we perform the above reduction in stages, namely first with respect to G_m and then by G_y, we can present Y as an iterated bundle:

$$Y \cong G \times_{G_y} (G_y \times_{G_m} (\mathfrak{g}_m^\perp \times V)) = G \times^{G_y} \tilde{Y}$$

over the coadjoint orbit, diffeomorphic to G/G_y, with the fiber $\tilde{Y} = G_y \times_{G_m} (\mathfrak{g}_m^\perp \times V)$. The space $\tilde{Y}$ is a Hamiltonian G_y-space with the moment map $\Phi|_{\tilde{Y}}$, which is a restriction of Φ to $\tilde{Y}$. Therefore we can write Φ as the composition of maps:

$$Y \;=\; G \times_{G_y} \tilde{Y} \xrightarrow{id \times \Phi} G \times_{G_y} \mathfrak{g}_y^* \xrightarrow{\ \iota\ } \mathfrak{g}^*,$$

where $\iota([g,\xi]) = \mathrm{Ad}_g^*(\xi)$. On the other hand, $\mathfrak{g}_y^*$ is a slice at y for a coadjoint action on $\mathfrak{g}^*$. Thus, restricting ι to a sufficiently small G-invariant neighborhood of $[g,\xi]$, we obtain a G-equivariant embedding into an open neighborhood of y. Hence there is a G-invariant neighborhood U of $[1,0,0]$ in Y so that Φ becomes the bundle map of associated bundles over G/G_y. Therefore $U \cap \Phi^{-1}(\mathfrak{g}_y^*) = U \cap Y$, and the image $\Phi(U)$ is a bundle over $G.y$ with the fiber $\Phi(U \cap \tilde{Y})$. If U is small enough, we get

$$\mathcal{S}(U) = \Phi(U) \cap \mathfrak{t}_+^* = \Phi(U \cap Y) \cap \mathfrak{t}_+^* = \mathcal{S}(U \cap \tilde{Y}) \ ,$$

where $\mathcal{S}(U \cap \tilde{Y})$ is the moment image of $U \cap \tilde{Y}$ as a Hamiltonian G_y-space. Hence the description of a local moment map reduces to computing the local moment map of the Hamiltonian G_y-space $\tilde{Y}$. Note that since G_y is compact and $\Phi|_{U \cap \tilde{Y}}$ is proper, we can use the vertex criterion given in [10, Theorem 6.7], which implies that if y is a vertex, then $\mathfrak{g}_y = [\mathfrak{g}_y, \mathfrak{g}_y] + \mathfrak{g}_m$ or, equivalently, $G_y = [G_y, G_y]G_m$. In particular, if y is an interior point of $\mathfrak{t}_+^*$ and T fixes m, then $m \in (W_{\mathfrak{k}}.A, W_{\mathfrak{k}}.B)$.

Now, in what follows, we give an explicit description of the local moment cone. The space $\tilde{Y}$ is a symplectic quotient of $T^*G_y \times V \cong G_y \times \mathfrak{g}_y^* \times V$ by G_m with the moment map

$$\widetilde{\Psi}(g,\xi,v) = -\pi(\xi) + \Phi_V(v) + \pi(y) \ ,$$

where we shifted the moment map by y. Let us assume that G_m is abelian, which is a case for the open dense set of elements in M. This follows from the fact that the isotropy group G_y which contains G_m is a subgroup of T for the dense open subset of elements $y \in \mathfrak{t}_+^*$. Therefore we can think of G_m as a subgroup of T. The G_m-moment map image of V is the cone

$$\mathcal{C}_m = \{\sum_{i=1}^n t_i \alpha_i \,|\, t_i \geq 0\} \ ,$$

where $\{\alpha_i\}_{i=\overline{1,n}}$ are the weights of the representation of G_m on V. For $\lambda \in \mathfrak{t}_+^*$, consider the coadjoint orbit $G_y \cdot \lambda$ through λ. If we regard $T^*G_y \times V$ as a Hamiltonian G_y-space and reduce it with respect to G_y, we obtain $(-G_y \cdot \lambda) \times V$. It is a Hamiltonian G_m-space with the moment map image

$$-\pi(\mathtt{Conv}(W_y \cdot \lambda)) + \mathcal{C}_m + \pi(y),$$

where $\mathtt{Conv}$ denotes a convex hull of given set. Thus the reduction of $(-G_y \cdot \lambda) \times V$ with respect to $0 \in \mathfrak{g}_m^*$ is non-empty if and only if

$$0 \in -\pi(\mathtt{Conv}(W_y \cdot \lambda)) + \mathcal{C}_m + \pi(y).$$

If we reduce $T^*G_y \times V$ by G_m to obtain $\widetilde{Y}$ and then reduce with respect to G_y, we obtain the very same space, according to the general result for reduction in stages [8]. Hence this space is non-empty if and only if $G_y \cdot \lambda$ is in the moment image of $\widetilde{Y}$. This implies that the local moment cone is the set

$$\{\, \lambda \in \mathfrak{t}_+^* \mid \pi(\mathtt{Conv}(W_y \cdot \lambda)) \cap (\mathcal{C}_m + \pi(y)) \neq \emptyset \}$$

or, equivalently, that there is some neighborhood U of y in $\mathfrak{t}_+^*$ such that

$$U \cap \mathcal{S}(\tilde{Y}) = U \cap (\pi_{\mathfrak{t}}^{-1}(\mathcal{C}_m) + y),$$

where $\pi_{\mathfrak{t}} : \mathfrak{t}^* \to \mathfrak{g}_m^*$ is the natural projection.

We recall a simple result about closed convex sets [5, Proposition 5]. Let $\mathcal{C}$ be a closed convex set in a vector space V. Then

$$\mathcal{C} = \mathtt{Conv}(\mathrm{Ext}(\mathcal{C})) + \lim(\mathcal{C}) \ ,$$

where $\mathrm{Ext}(\mathcal{C})$ is the set of extremal points of the cone, and $\lim(\mathcal{C}) := \{v \in V : \mathcal{C} + v \subseteq \mathcal{C}\}$. Now using the above vertex criterion $\mathrm{Ext}(\mathcal{S}(M)) \cap (\mathfrak{t}_+^*)^\circ = \Phi(W_{\mathfrak{k}}.A, W_{\mathfrak{k}}.B) \cap (\mathfrak{t}_+^*)^\circ$ where $(\mathfrak{t}_+^*)^\circ$ is the interior of $\mathfrak{t}_+^*$, we deduce that $\mathtt{Conv}(\mathrm{Ext}(\mathcal{S}(M))) \cap (\mathfrak{t}_+^*)^\circ = \Pi \cap (\mathfrak{t}_+^*)^\circ$. Hence we have

$$\mathcal{S}(M) \cap (\mathfrak{t}_+^*)^\circ = (\Pi + \lim \mathcal{S}(M)) \cap (\mathfrak{t}_+^*)^\circ \ .$$

On the other hand, the cones at the extremal points have a very simple form. Using [5, Remark 5.18] for the moment map image of coadjoint orbits, the cone at those points is given by $\mathrm{Cone}(\sqrt{-1}\Delta_{\mathrm{nc}}^+)$. Therefore we have

$$\mathcal{S}(M) \cap (\mathfrak{t}_+^*)^\circ = (\Pi + \mathrm{Cone}(\sqrt{-1}\Delta_{\mathrm{nc}}^+)) \cap (\mathfrak{t}_+^*)^\circ,$$

and by continuity we conclude that

$$\mathcal{S}(M) = (\Pi + \mathrm{Cone}(\sqrt{-1}\Delta_{\mathrm{nc}}^+)) \cap \mathfrak{t}_+^* \ . \qquad \square$$

Example 1. Let $G = \mathrm{SU}(2,1)$. Identify, as before, $\mathfrak{g}^*$ with the space of pseudo-Hermitian matrices of signature $(2,1)$ and take $\Lambda_A = \mathrm{diag}(4,1,-5)$ and $\Lambda_B = \mathrm{diag}(2,1,-3)$. Then the possible eigenvalues $(\lambda_1, \lambda_2, \mu)$ of $A + B$, taken in the order $\lambda_1 \geq \lambda_2 > \mu$, are given by

$$\lambda_1 \geq 5, \quad \lambda_2 \geq 2, \quad \lambda_1 + \lambda_2 \geq 8, \quad \text{and} \quad \mu = -\lambda_1 - \lambda_2.$$

See Figure 1.

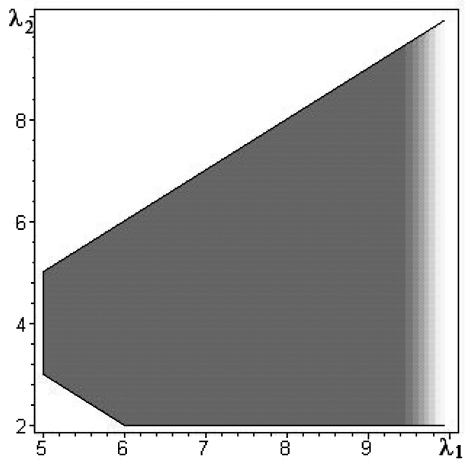

FIGURE 1. Moment polyhedron for Example 1.

Example 2. Let $G = \mathrm{SU}(2,2)$. Take $\Lambda_A = \mathrm{diag}(4,2,1,-7)$ and $\Lambda_B = \mathrm{diag}(3,2,1,-6)$. Then the possible eigenvalues $(\lambda_1, \lambda_2, \mu_1, \mu_2)$ of $A + B$, taken in the order $\lambda_1 \geq \lambda_2 > \mu_1 \geq \mu_2$, are given by

$$\lambda_1 \geq 6, \quad \lambda_2 \geq 4, \quad \lambda_1 + \lambda_2 \geq 11, \quad \lambda_1 + \lambda_2 + \mu_1 \geq 6, \quad \lambda_1 + \lambda_2 + 2\mu_1 \geq 0,$$

$$\mu_1 \leq 2 \ \text{ and } \ \mu_2 = -\lambda_1 - \lambda_2 - \mu_1.$$

Now projecting our picture to the first three coordinates with $\mathfrak{t}^*_{\mathrm{adm}}$, we get the $x \geq y > z > 0$ polyhedron presented in Figure 2.

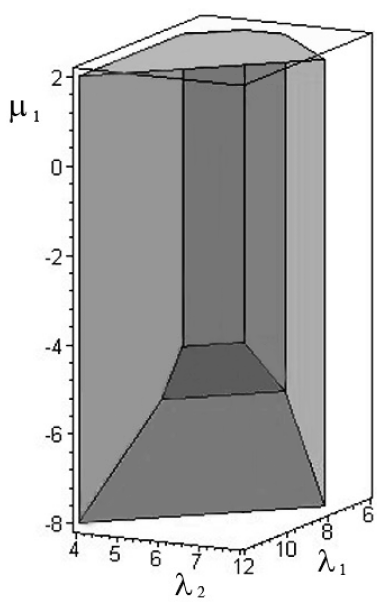

FIGURE 2. Moment polyhedron for Example 2.

4. Relationship with representation theory

Let $\Lambda \in \mathfrak{t}^*_{\mathrm{adm}}$ be a dominant integral weight with respect to the compact positive roots and let V be an irreducible unitary K-module with highest weight Λ. Following Harish-Chandra, one can construct a unique unitary irreducible representation ρ_Λ of G such that the corresponding representation of $G^{\mathbb{C}}$ has highest weight Λ. The underlying space of ρ_Λ is $V \otimes \mathcal{U}_+$, where $\mathcal{U}_+$ is the universal enveloping algebra of the nilradical $\mathfrak{n}_+$ spanned over $\mathbb{C}$ by the non-compact positive roots. (Note that we have a bit different convention and take $\mathfrak{n}_+$ instead of $\mathfrak{n}_-$, but also our Λ is in $\mathfrak{t}^*_{\mathrm{adm}}$ and not in its negative.) Such a ρ_Λ is called a holomorphic discrete series representation and is a generalized Verma module as a representation of $\mathcal{U}_{\mathfrak{g}}$.

The weights of ρ_Λ have the form $\lambda + \mathbb{N}.\Delta^+_{\mathrm{nc}}$, where λ is a weight of V and $\mathbb{N}$ is the semigroup of non-negative integers.

Moreover, a theorem of Repka [9] says that for two such representations ρ_A and ρ_B, their tensor product decomposes into the sum of subspaces, whose weights are sums of weights of ρ_A and ρ_B, and all are of the form $\lambda_{AB} + \mathbb{N}.\Delta^+_{\mathrm{nc}}$, where λ_{AB} is a weight of $V_A \otimes V_B$.

Therefore, our results translate into the following statement.

Proposition 4.1. *The tensor product of holomorphic discrete series representations of G with highest weights Λ_A and Λ_B decomposes into the direct sum of representations with highest weights given by the lattice points in the convex polyhedral set $\mathcal{S}(\mathcal{O}_A \times \mathcal{O}_B)$, with finite multiplicities.*

It would be interesting to find a direct approach to proving this result, as well as to establish that the (finite) multiplicities of the representations appearing as summands in such tensor products correspond to counting the lattice points in certain polyhedral sets, similar to the compact case.

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