

EXPLICIT RECONSTRUCTION OF HOMOGENEOUS ISOLATED HYPERSURFACE SINGULARITIES FROM THEIR MILNOR ALGEBRAS

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ABSTRACT. By the Mather-Yau theorem, a complex hypersurface germ \mathcal{V} with isolated singularity is completely determined by its moduli algebra $\mathcal{A}(\mathcal{V})$. The proof of the theorem does not provide an explicit procedure for recovering \mathcal{V} from $\mathcal{A}(\mathcal{V})$, and finding such a procedure is a long-standing open problem. In this paper, we present an explicit way for reconstructing \mathcal{V} from $\mathcal{A}(\mathcal{V})$ up to biholomorphic equivalence under the assumption that the singularity of \mathcal{V} is homogeneous, in which case $\mathcal{A}(\mathcal{V})$ coincides with the Milnor algebra of \mathcal{V} .

1. INTRODUCTION

Let \mathcal{O}_n be the local algebra of holomorphic function germs at the origin in \mathbb{C}^n with $n \geq 2$. For every hypersurface germ \mathcal{V} at the origin (considered with its reduced complex structure) denote by $I(\mathcal{V})$ the ideal of elements of \mathcal{O}_n vanishing on \mathcal{V} . Let f be a generator of $I(\mathcal{V})$, and consider the complex associative commutative algebra $\mathcal{A}(\mathcal{V})$ defined as the quotient of \mathcal{O}_n by the ideal generated by f and all its first-order partial derivatives. The algebra $\mathcal{A}(\mathcal{V})$, called the *moduli algebra* or *Tjurina algebra* of \mathcal{V} , is independent of the choice of f as well as the coordinate system near the origin, and the moduli algebras of biholomorphically equivalent hypersurface germs are isomorphic. Clearly, $\mathcal{A}(\mathcal{V})$ is trivial if and only if \mathcal{V} is non-singular. Furthermore, it is well-known that $0 < \dim_{\mathbb{C}} \mathcal{A}(\mathcal{V}) < \infty$ if and only if the germ \mathcal{V} has an isolated singularity (see, e.g. Chapter 1 in [GLS]).

By a theorem due to Mather and Yau (see [MY]), two hypersurface germs $\mathcal{V}_1, \mathcal{V}_2$ in \mathbb{C}^n with isolated singularities are biholomorphically equivalent if their moduli algebras $\mathcal{A}(\mathcal{V}_1), \mathcal{A}(\mathcal{V}_2)$ are isomorphic. Thus, given the dimension n , the moduli algebra $\mathcal{A}(\mathcal{V})$ determines \mathcal{V} up to biholomorphism. In particular, if $\dim_{\mathbb{C}} \mathcal{A}(\mathcal{V}) = 1$, then \mathcal{V} is biholomorphic to the germ of the hypersurface $\{z_1^2 + \cdots + z_n^2 = 0\}$, and if $\dim_{\mathbb{C}} \mathcal{A}(\mathcal{V}) = 2$, then \mathcal{V} is biholomorphic to the germ of the hypersurface $\{z_1^2 + \cdots + z_{n-1}^2 + z_n^3 = 0\}$. The proof of the Mather-Yau theorem does not provide an explicit procedure for recovering the germ \mathcal{V} from the algebra $\mathcal{A}(\mathcal{V})$ in general, and finding a way for reconstructing \mathcal{V} (or at least some invariants of \mathcal{V}) from $\mathcal{A}(\mathcal{V})$ is an interesting open problem (cf. [Y1], [Y2], [Sch], [EI]). In this paper we present an explicit method for restoring \mathcal{V} from $\mathcal{A}(\mathcal{V})$ up to biholomorphic equivalence under the assumption that the singularity of \mathcal{V} is homogeneous.

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Let \mathcal{V} be a hypersurface germ having an isolated singularity. The singularity of \mathcal{V} is said to be *homogeneous* if for some (hence for every) generator f of $I(\mathcal{V})$ there is a coordinate system near the origin in which f becomes the germ of a homogeneous polynomial. In this case f lies in the Jacobian ideal $\mathcal{J}(f)$ in \mathcal{O}_n , which is the ideal generated by all first-order partial derivatives of f . Hence, for a homogeneous singularity, $\mathcal{A}(\mathcal{V})$ coincides with the *Milnor algebra* $\mathcal{O}_n/\mathcal{J}(f)$ for any generator f of $I(\mathcal{V})$.

Let $Q(z)$, with $z := (z_1, \dots, z_n)$, be a holomorphic $(m+1)$ -form on \mathbb{C}^n , i.e. a homogeneous polynomial of degree $m+1$ in the variables z_1, \dots, z_n , where $m \geq 2$. Consider the germ \mathcal{V}_Q of the hypersurface $\{Q(z) = 0\}$ and assume that: (i) the singularity of \mathcal{V}_Q is isolated, and (ii) the germ of Q generates $I(\mathcal{V}_Q)$. These two conditions are equivalent to the non-vanishing of the discriminant $\Delta(Q)$ of Q (see Chapter 13 in [GKZ]). Next, consider the gradient map $\mathbf{Q} : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $z \mapsto \text{grad } Q(z)$. The condition $\Delta(Q) \neq 0$ means that the fiber $\mathbf{Q}^{-1}(0)$ consists of 0 alone; in particular, the map \mathbf{Q} is finite at the origin. The main content of this paper is a procedure for recovering \mathbf{Q} from $\mathcal{A}(\mathcal{V}_Q)$ up to linear equivalence, where we say that two maps $\Phi_1, \Phi_2 : \mathbb{C}^n \rightarrow \mathbb{C}^n$ are linearly equivalent if there exist non-degenerate linear transformations L_1, L_2 of \mathbb{C}^n such that $\Phi_2 = L_1 \circ \Phi_1 \circ L_2$.

In fact, we consider a more general situation. Let p_r , $r = 1, \dots, n$, be holomorphic m -forms on \mathbb{C}^n and I the ideal in \mathcal{O}_n generated by the germs of these forms at the origin. Define $\mathcal{A} := \mathcal{O}_n/I$ and assume that $\dim_{\mathbb{C}} \mathcal{A} < \infty$, which is equivalent to the finiteness of the map $\mathbf{P} : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $z \mapsto (p_1(z), \dots, p_n(z))$ at the origin (see Chapter 1 in [GLS]). Observe that since the components of \mathbf{P} are homogeneous polynomials, \mathbf{P} is finite at 0 if and only if $\mathbf{P}^{-1}(0) = \{0\}$; the latter occurs if and only if the resultant of p_r , $r = 1, \dots, n$, does not vanish (see Chapter 13 in [GKZ]). In this paper we propose a procedure (which requires only linear-algebraic manipulations) for explicitly recovering the map \mathbf{P} from \mathcal{A} up to linear equivalence. As explained in Remark 2.1, this procedure also helps decide whether a given complex finite-dimensional associative algebra is isomorphic to an algebra arising from a finite homogeneous polynomial map as above.

The paper is organized as follows. Reconstruction of \mathbf{P} from \mathcal{A} is done in Section 2. In Section 3 we apply our method to the algebra $\mathcal{A}(\mathcal{V}_Q)$ arising from Q to obtain a map \mathbf{Q}' linearly equivalent to \mathbf{Q} . It is then not hard to derive from \mathbf{Q}' an $(m+1)$ -form Q' linearly equivalent to Q , where two forms Q_1, Q_2 on \mathbb{C}^n are called linearly equivalent if there exists a non-degenerate linear transformation L of \mathbb{C}^n such that $Q_2 = Q_1 \circ L$. Then the germ of the hypersurface $\{Q'(z) = 0\}$ is the sought-after reconstruction of \mathcal{V}_Q from $\mathcal{A}(\mathcal{V}_Q)$ up to biholomorphic equivalence. We conclude the paper by illustrating our reconstruction procedure with the example of simple elliptic singularities of type \widetilde{E}_6 .

2. RECONSTRUCTION OF FINITE POLYNOMIAL MAPS

Recapping the setup outlined in the introduction, let p_r , $r = 1, \dots, n$, be holomorphic m -forms on \mathbb{C}^n and I the ideal in \mathcal{O}_n generated by the germs of these forms at the origin, where $m, n \geq 2$. Define $\mathcal{A} := \mathcal{O}_n/I$ and assume that $\dim_{\mathbb{C}} \mathcal{A} < \infty$ (observe that $\dim_{\mathbb{C}} \mathcal{A} \geq m+1$). In this section we present a method for recovering the map $\mathbf{P} : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $z \mapsto (p_1(z), \dots, p_n(z))$ from \mathcal{A} up to linear equivalence. Everywhere below we suppose that \mathcal{A} is given as an abstract associative algebra, i.e. by a multiplication table with respect to some basis e_1, \dots, e_N , with $N := \dim_{\mathbb{C}} \mathcal{A}$.

First of all, we find the unit $\mathbf{1}$ of \mathcal{A} . One has $\mathbf{1} = \sum_{k=1}^N \alpha_k e_k$ where the coefficients $\alpha_k \in \mathbb{C}$ are uniquely determined from the linear system

$$\sum_{k=1}^N \alpha_k (e_k e_\ell) = e_\ell, \quad \ell = 1, \dots, N.$$

Assume now that $e_1 = \mathbf{1}$ and find the maximal ideal \mathfrak{m} of \mathcal{A} . Clearly, \mathfrak{m} is spanned by the vectors $e'_k := e_k - \beta_k \mathbf{1}$, $k = 2, \dots, N$, where $\beta_k \in \mathbb{C}$ are uniquely fixed by the requirement that each e'_k is not invertible in \mathcal{A} . Hence, for each k the number β_k is determined from the condition that the linear system

$$(2.1) \quad \sum_{\ell=1}^N \gamma_\ell (e_k e_\ell - \beta_k e_\ell) = e_1$$

cannot be solved for $\gamma_1, \dots, \gamma_N \in \mathbb{C}$. Since system (2.1) has at most one solution for any β_k , this condition is equivalent to the degeneracy of the coefficient matrix M_k of (2.1). We have $M_k = C_k - \beta_k \text{Id}$, where $C_k := (c_{kj\ell})_{j,\ell=1,\dots,N}$, with $c_{kj\ell}$ given by $e_k e_\ell = \sum_{j=1}^N c_{kj\ell} e_j$. It then follows that the required value of β_k is the (unique) eigenvalue of the matrix C_k .

We are now in a position to find the number of variables n and the degree m for the forms p_r . By Nakayama’s lemma, \mathfrak{m} is a nilpotent algebra, and we denote by ν its nil-index, which is the largest integer μ with $\mathfrak{m}^\mu \neq 0$. Observe that $\nu \leq N - 1$, and therefore to determine ν it is sufficient to compute all products of the basis vectors e'_k of length not exceeding $N - 1$. Further, since \mathcal{A} is finite-dimensional, the forms p_r form a regular sequence in \mathcal{O}_n (see Theorem 2.1.2 in [BH]). Hence \mathcal{A} is a complete intersection ring, which implies that \mathcal{A} is a Gorenstein algebra (see [B]). Recall that a (complex) local commutative associative algebra \mathcal{B} with $1 < \dim_{\mathbb{C}} \mathcal{B} < \infty$ is Gorenstein if and only if for the annihilator $\text{Ann}(\mathfrak{n}) := \{x \in \mathfrak{n} : x \mathfrak{n} = 0\}$ of its maximal ideal \mathfrak{n} one has $\dim_{\mathbb{C}} \text{Ann}(\mathfrak{n}) = 1$ (see e.g. [H]). Lemma 3.4 of [Sa] yields that $\text{Ann}(\mathfrak{m})$ is spanned by the element represented by the germ of $J(\mathbf{P}) := \det(\partial p_r / \partial z_s)_{r,s=1,\dots,n}$.

For every $i > 0$, let \mathcal{P}_i be the vector space of all i -forms on \mathbb{C}^n and \mathcal{L}_i be the linear subspace of \mathcal{A} that consists of all elements represented by germs of forms in \mathcal{P}_i . Since \mathfrak{m} consists of all elements of \mathcal{A} represented by germs in \mathcal{O}_n vanishing at the origin, the subspaces \mathcal{L}_i lie in \mathfrak{m} and yield a grading on \mathfrak{m} :

$$\mathfrak{m} = \bigoplus_{i>0} \mathcal{L}_i, \quad \mathcal{L}_i \mathcal{L}_j \subset \mathcal{L}_{i+j} \text{ for all } i, j.$$

Since $\dim_{\mathbb{C}} \text{Ann}(\mathfrak{m}) = 1$, it immediately follows that $\text{Ann}(\mathfrak{m}) = \mathfrak{m}^\nu = \mathcal{L}_d$ for $d := \max\{i : \mathcal{L}_i \neq 0\}$. On the other hand, $\text{Ann}(\mathfrak{m})$ is spanned by the element represented by the germ of $J(\mathbf{P})$, which is an $n(m - 1)$ -form. Thus $d = n(m - 1)$. Furthermore, we have

$$(2.2) \quad \begin{aligned} \dim_{\mathbb{C}} \mathcal{L}_i &= \dim_{\mathbb{C}} \mathcal{P}_i \quad \text{for } i = 1, \dots, m - 1, \\ \dim_{\mathbb{C}} \mathcal{L}_m &= \dim_{\mathbb{C}} \mathcal{P}_m - n. \end{aligned}$$

Now, observe that

$$(2.3) \quad \mathcal{L}_i = \mathcal{L}_1^i \text{ for all } i;$$

i.e. the graded algebra \mathcal{A} is *standard* in the terminology of [St]. Hence \mathcal{L}_i is a complement to \mathfrak{m}^{i+1} in \mathfrak{m}^i for all $i > 0$. For $i = 1$ this implies that n can be

recovered from the algebra \mathcal{A} as

$$(2.4) \quad n = \dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2;$$

i.e. n is the *embedding dimension* of \mathcal{A} (see Remark 3.2). Next, for $i = \nu$ we obtain $n(m - 1) = \nu$ (in particular, n divides ν). Thus, the degree m of the forms p_r can be recovered from \mathcal{A} as follows:

$$(2.5) \quad m = \nu/n + 1.$$

Note that since \mathcal{A} is given as an abstract associative algebra, finding the grading $\{\mathcal{L}_i\}$ from the available data may be hard. We stress that determination of this grading is not required for recovering n and m .

Further, choose an arbitrary basis f_1, \dots, f_n in a complement to \mathfrak{m}^2 in \mathfrak{m} . Clearly, for some $C \in \text{GL}(n, \mathbb{C})$ one has

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = C \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} + \begin{pmatrix} W_1 \\ \vdots \\ W_n \end{pmatrix},$$

where Z_j is the element of \mathcal{L}_1 represented by the germ of the coordinate function z_j , and $W_j \in \mathfrak{m}^2$. Set

$$K := \dim_{\mathbb{C}} \mathcal{P}_m = \binom{m+n-1}{m},$$

and let $q_1(z), \dots, q_K(z)$ be the monomial basis of \mathcal{P}_m where $z := (z_1, \dots, z_n)$. Next, fix a complement \mathcal{S} to \mathfrak{m}^{m+1} in \mathfrak{m}^m and let $\pi : \mathfrak{m}^m \rightarrow \mathcal{S}$ be the projection onto \mathcal{S} with kernel \mathfrak{m}^{m+1} . Condition (2.3) for $i = m$ then yields that \mathcal{S} is spanned by $\pi(q_1(f)), \dots, \pi(q_K(f))$, where $f := (f_1, \dots, f_n)$. On the other hand, by (2.2) we have $\dim_{\mathbb{C}} \mathcal{S} = K - n$. Hence one can find n linear relations

$$(2.6) \quad \sum_{\rho=1}^K \gamma_{\sigma\rho} \pi(q_{\rho}(f)) = 0, \quad \sigma = 1, \dots, n,$$

where the vectors $\gamma_{\sigma} := (\gamma_{\sigma 1}, \dots, \gamma_{\sigma K}) \in \mathbb{C}^K$ are linearly independent.

Further, extracting from (2.6) the \mathcal{L}_m -components one obtains

$$(2.7) \quad \sum_{\rho=1}^K \gamma_{\sigma\rho} q_{\rho}(CZ) = 0, \quad \sigma = 1, \dots, n,$$

where $Z := (Z_1, \dots, Z_n)$. Identity (2.7) is equivalent to

$$(2.8) \quad \sum_{\rho=1}^K \gamma_{\sigma\rho} q_{\rho}(Cz) \in I, \quad \sigma = 1, \dots, n,$$

where each $q_{\rho}(z)$ is identified with its germ at the origin. From (2.8) one immediately obtains that for some matrix $D \in \text{GL}(n, \mathbb{C})$ the following holds:

$$\Gamma q(Cz) \equiv D\mathbf{P}(z),$$

where $\Gamma := (\gamma_{\sigma\rho})_{\sigma=1, \dots, n, \rho=1, \dots, K}$, and $q := (q_1, \dots, q_K)$. Thus, the map

$$(2.9) \quad \Phi : \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad z \mapsto \Gamma q(z)$$

is linearly equivalent to \mathbf{P} as required.

We now summarize the main steps of our algorithm for recovering \mathbf{P} from \mathcal{A} up to linear equivalence:

1. Find \mathfrak{m} and its nil-index ν .
2. Determine n from formula (2.4).
3. Determine m from formula (2.5).
4. Choose a complement to \mathfrak{m}^2 in \mathfrak{m} and an arbitrary basis f_1, \dots, f_n in this complement.
5. Calculate $q_1(f), \dots, q_K(f)$, where $f := (f_1, \dots, f_n)$ and $q_1(z), \dots, q_K(z)$ are all monomials of degree m in $z := (z_1, \dots, z_n)$.
6. Choose a complement \mathcal{S} to \mathfrak{m}^{m+1} in \mathfrak{m}^m .
7. Compute $\pi(q_1(f)), \dots, \pi(q_K(f))$, where $\pi : \mathfrak{m}^m \rightarrow \mathcal{S}$ is the projection onto \mathcal{S} with kernel \mathfrak{m}^{m+1} .
8. Find n linearly independent linear relations among the vectors $\pi(q_1(f)), \dots, \pi(q_K(f))$ as in (2.6).
9. Formula (2.9) then gives a map linearly equivalent to \mathbf{P} .

In the next section this algorithm will be applied to the gradient map arising from a form with non-zero discriminant.

Remark 2.1. A natural problem is to characterize the algebras that arise from finite homogeneous polynomial maps as above among all complex finite-dimensional Gorenstein algebras. For gradient maps, this problem is a special case of the well-known *recognition problem* for the moduli algebras of general isolated hypersurface singularities and the corresponding Lie algebras of derivations (see, e.g. [Y1], [Y2], [Sch]). The algorithm presented here can help decide whether a given finite-dimensional Gorenstein algebra \mathcal{B} is isomorphic to an algebra \mathcal{A} of the kind considered in this section. Indeed, one can attempt to formally apply the algorithm to \mathcal{B} . For the algorithm to go through one requires that: (i) the nil-index of the maximal ideal of \mathcal{B} be divisible by the number n found from formula (2.4), which is the embedding dimension of \mathcal{B} , (ii) for some basis f_1, \dots, f_n in some complement to \mathfrak{m}^2 in \mathfrak{m} and for some complement \mathcal{S} to \mathfrak{m}^{m+1} in \mathfrak{m}^m there exist n linearly independent linear relations among the vectors $\pi(q_1(f)), \dots, \pi(q_K(f))$, with m being the number found from formula (2.5), and (iii) the map $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ produced in step 9 be finite at the origin. If the algorithm fails (i.e. some of conditions (i)–(iii) are not satisfied), then \mathcal{B} does not arise from a finite homogeneous polynomial map. If the algorithm successfully finishes, the resulting map Φ is a candidate map from which \mathcal{B} may potentially arise. In order to see if this is indeed the case, one needs to check whether \mathcal{B} is isomorphic to the algebra associated to Φ . For this purpose one can use the criterion for isomorphism of finite-dimensional Gorenstein algebras established in [FIKK].

3. RECONSTRUCTION OF HOMOGENEOUS SINGULARITIES

Suppose now that $\mathbf{P} = \mathbf{Q} := \text{grad } Q$ for a holomorphic $(m + 1)$ -form Q on \mathbb{C}^n with $\Delta(Q) \neq 0$, where Δ is the discriminant. Let Φ be a map linearly equivalent to \mathbf{Q} produced by the procedure described in Section 2 from the algebra $\mathcal{A}(\mathcal{V}_Q)$, where \mathcal{V}_Q is the germ of the hypersurface $\{Q = 0\}$ at the origin. We then have

$$(3.1) \quad \Phi(z) \equiv C_1 \text{grad } Q(C_2 z)$$

for some $C_1, C_2 \in \text{GL}(n, \mathbb{C})$. Our next task is to recover Q from Φ up to linear equivalence.

Let Q' be the $(m + 1)$ -form defined by $Q'(z) := Q(C_2z)$ for all $z \in \mathbb{C}^n$. Then $\text{grad } Q(C_2z) = (C_2^{-1})^T \text{grad } Q'(z)$, and (3.1) implies

$$\Phi(z) \equiv C \text{grad } Q'(z)$$

for some $C \in \text{GL}(n, \mathbb{C})$. For any $n \times n$ -matrix D we now introduce the holomorphic differential 1-form $\omega^D := \sum_{r=1}^n \Psi_r^D dz_r$ on \mathbb{C}^n , where $(\Psi_1^D, \dots, \Psi_n^D)$ are the components of the map $\Psi^D := D\Phi$. Consider the equation

$$(3.2) \quad d\omega^D = 0$$

as a linear system with respect to the entries of the matrix D . Clearly, C^{-1} is a solution of (3.2). Let D_0 be another solution of (3.2) and assume that $D_0 \in \text{GL}(n, \mathbb{C})$. Every closed holomorphic differential form on \mathbb{C}^n is exact, and integrating Ψ^{D_0} one obtains an $(m + 1)$ -form Q'' on \mathbb{C}^n . Then $\text{grad } Q'' = \Psi^{D_0} = D_0 C \text{grad } Q'$, and therefore $\Delta(Q'') \neq 0$. Furthermore, the Milnor algebras of the germs $\mathcal{V}_{Q'}$ and $\mathcal{V}_{Q''}$ coincide. By the Mather-Yau theorem, this implies that $\mathcal{V}_{Q'}$ and $\mathcal{V}_{Q''}$ are biholomorphically equivalent and therefore $\mathcal{V}_{Q''}$ is biholomorphically equivalent to \mathcal{V}_Q , which yields that Q'' is linearly equivalent to Q . Thus, any non-degenerate matrix that solves linear system (3.2) leads to an $(m + 1)$ -form linearly equivalent to Q and a hypersurface germ biholomorphically equivalent to \mathcal{V}_Q .

We will now illustrate our method for recovering \mathcal{V}_Q from $\mathcal{A}(\mathcal{V}_Q)$ by the example of simple elliptic singularities of type \tilde{E}_6 . These singularities form a family parametrized by $t \in \mathbb{C}$ satisfying $t^3 + 27 \neq 0$. Namely, for every such t let $\mathcal{V}_t := \mathcal{V}_{Q_t}$, where Q_t is the following cubic on \mathbb{C}^3 :

$$Q_t(z) := z_1^3 + z_2^3 + z_3^3 + tz_1z_2z_3, \quad z := (z_1, z_2, z_3).$$

Below we explicitly show how Q_t can be recovered from the algebra $\mathcal{A}_t := \mathcal{A}(\mathcal{V}_t)$ up to linear equivalence.

Recall that the starting point of our reconstruction procedure is a multiplication table with respect to some basis. The algebra \mathcal{A}_t has dimension 8 and with respect to a certain basis e_1, \dots, e_8 is given by (see Remark 3.1 below):

$$(3.3) \quad \begin{aligned} e_1e_j &= e_j \text{ for } j = 1, \dots, 8, \quad e_2^2 = -\frac{t}{3}e_3 + \frac{2t}{3}e_6, \\ e_2e_3 &= e_6, \quad e_2e_4 = e_5 - e_6 - e_8, \quad e_2e_5 = e_7, \quad e_2e_6 = 0, \\ e_2e_7 &= 0, \quad e_2e_8 = e_7, \quad e_3e_j = 0 \text{ for } j = 3, \dots, 8, \quad e_4^2 = -\frac{t}{3}e_7, \\ e_4e_5 &= e_3 - 2e_6, \quad e_4e_6 = 0, \quad e_4e_7 = e_6, \quad e_4e_8 = e_3 - 2e_6, \\ e_5^2 &= -\frac{t}{3}e_5 + (2 + t)e_6 + \frac{t}{3}e_8, \quad e_5e_6 = 0, \quad e_5e_7 = 0, \\ e_5e_8 &= -\frac{t}{3}e_5 + (1 + t)e_6 + \frac{t}{3}e_8, \quad e_6e_j = 0 \text{ for } j = 6, 7, 8, \\ e_7e_j &= 0 \text{ for } j = 7, 8, \quad e_8^2 = -\frac{t}{3}e_5 + te_6 + \frac{t}{3}e_8. \end{aligned}$$

It is clear from (3.3) that $e_1 = \mathbf{1}$ and $\mathfrak{m}_t = \langle e_2, \dots, e_8 \rangle$, where $\langle \cdot \rangle$ denotes the linear span and \mathfrak{m}_t is the maximal ideal of \mathcal{A}_t . We then have $\mathfrak{m}_t^2 = \langle e_3, e_6, e_7, e_5 - e_8 \rangle$, $\mathfrak{m}_t^3 = \langle e_6 \rangle$, $\mathfrak{m}_t^4 = 0$; hence $\nu = 3$. Further, by formula (2.4) we obtain $n = 3$, which together with formula (2.5) yields $m = 2$.

We now list all monomials of degree 2 in z as follows:

$$\begin{aligned} q_1(z) &:= z_1^2, & q_2(z) &:= z_2^2, & q_3(z) &:= z_3^2, \\ q_4(z) &:= z_1z_2, & q_5(z) &:= z_1z_3, & q_6(z) &:= z_2z_3 \end{aligned}$$

(here $K = 6$). Next, we let $f_1 := e_2, f_2 := e_4, f_3 := e_5$, which for $f := (f_1, f_2, f_3)$ yields

$$\begin{aligned} q_1(f) &= -\frac{t}{3}e_3 + \frac{2t}{3}e_6, & q_2(f) &= -\frac{t}{3}e_7, & q_3(f) &= -\frac{t}{3}e_5 + (2+t)e_6 + \frac{t}{3}e_8, \\ q_4(f) &= e_5 - e_6 - e_8, & q_5(f) &= e_7, & q_6(f) &= e_3 - 2e_6. \end{aligned}$$

Further, define $\mathcal{S} := \langle e_3, e_7, e_5 - e_8 \rangle$. Clearly, \mathcal{S} is a complement to \mathfrak{m}_t^3 in \mathfrak{m}_t^2 . Then for the projection $\pi : \mathfrak{m}_t^2 \rightarrow \mathcal{S}$ with kernel \mathfrak{m}_t^3 one has

$$\begin{aligned} \pi(q_1(f)) &= -\frac{t}{3}e_3, & \pi(q_2(f)) &= -\frac{t}{3}e_7, & \pi(q_3(f)) &= -\frac{t}{3}e_5 + \frac{t}{3}e_8, \\ \pi(q_4(f)) &= e_5 - e_8, & \pi(q_5(f)) &= e_7, & \pi(q_6(f)) &= e_3. \end{aligned}$$

The vectors $\pi(q_1(f)), \dots, \pi(q_6(f))$ satisfy the following three linearly independent linear relations:

$$\begin{aligned} \pi(q_1(f)) + \frac{t}{3}\pi(q_6(f)) &= 0, \\ \pi(q_2(f)) + \frac{t}{3}\pi(q_5(f)) &= 0, \\ \pi(q_3(f)) + \frac{t}{3}\pi(q_4(f)) &= 0. \end{aligned}$$

Hence we have

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t/3 \\ 0 & 1 & 0 & 0 & t/3 & 0 \\ 0 & 0 & 1 & t/3 & 0 & 0 \end{pmatrix},$$

which for $q(z) := (q_1(z), \dots, q_6(z))$ yields

$$\Phi(z) = \Gamma q(z) = \begin{pmatrix} z_1^2 + \frac{t}{3}z_2z_3 \\ z_2^2 + \frac{t}{3}z_1z_3 \\ z_3^2 + \frac{t}{3}z_1z_2 \end{pmatrix}.$$

It remains to recover Q_t from Φ up to linear equivalence. For

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

system (3.2) is equivalent to the following system of equations:

$$(3.4) \quad \begin{aligned} 2d_{12} - \frac{t}{3}d_{23} &= 0, & 2d_{21} - \frac{t}{3}d_{13} &= 0, & td_{11} - td_{22} &= 0, \\ 2d_{13} - \frac{t}{3}d_{32} &= 0, & 2d_{31} - \frac{t}{3}d_{12} &= 0, & td_{11} - td_{33} &= 0, \\ 2d_{23} - \frac{t}{3}d_{31} &= 0, & 2d_{32} - \frac{t}{3}d_{21} &= 0, & td_{22} - td_{33} &= 0. \end{aligned}$$

If $t \neq 0$ and $t^3 \neq 216$, the only non-degenerate solutions of (3.4) are non-zero scalar matrices. Integrating Ψ^D for such a matrix D we obtain a form proportional to Q_t , which is obviously linearly equivalent to Q_t . If $t = 0$, any non-degenerate solution of (3.4) is a diagonal matrix with non-zero d_{11}, d_{22}, d_{33} . Integrating Ψ^D for such a matrix D we obtain the form

$$\frac{1}{3} (d_{11}z_1^3 + d_{22}z_2^3 + d_{33}z_3^3),$$

which is linearly equivalent to $Q_0 = z_1^3 + z_2^3 + z_3^3$ by suitable dilations of the variables.

The remaining case $t^3 = 216$ is more interesting. Writing $t = 6\lambda$ with $\lambda^3 = 1$, we see that D is a solution of (3.4) if and only if

$$d_{11} = d_{22} = d_{33}, \quad d_{12} = \lambda^2 d_{31}, \quad d_{23} = \lambda d_{31}, \quad d_{21} = \lambda^2 d_{32}, \quad d_{13} = \lambda d_{32}.$$

Such a matrix D is non-degenerate if and only if $d_{11}^3 + d_{31}^3 + d_{32}^3 - 3\lambda d_{11}d_{31}d_{32} \neq 0$. For example, letting $d_{11} = 0, d_{31} = 0, d_{32} = 1$ one obtains

$$\Psi^D = \begin{pmatrix} \lambda z_3^2 + 2\lambda^2 z_1 z_2 \\ \lambda^2 z_1^2 + 2z_2 z_3 \\ z_2^2 + 2\lambda z_1 z_3 \end{pmatrix}.$$

Integration of Ψ^D leads to the form $Q_\lambda := \lambda^2 z_1^2 z_2 + \lambda z_1 z_3^2 + z_2^2 z_3$. As we have noted above, the Mather-Yau theorem implies that Q_λ is linearly equivalent to Q_t . Furthermore, the cubic Q_λ is equivalent to Q_1 by the map $(z_1, z_2, z_3) \mapsto (z_1/\lambda, z_2, z_3)$. Hence each of the three cubics Q_λ with $\lambda^3 = 1$ is linearly equivalent to each of the three cubics Q_t with $t^3 = 216$.

This last fact can also be understood without referring to the Mather-Yau theorem, as follows. It is well-known that all non-equivalent ternary cubics with non-vanishing discriminant are distinguished by the invariant

$$J := \frac{I_4^3}{\Delta},$$

where I_4 is a certain classical $SL(3, \mathbb{C})$ -invariant of degree 4 (see, e.g., pp. 381–389 in [El]). For any ternary cubic Q with $\Delta(Q) \neq 0$ one has $J(Q) = j(Z_Q)/110592$, where $j(Z_Q)$ is the value of the j -invariant for the elliptic curve Z_Q in \mathbb{CP}^2 defined by Q . Details on computing $J(Q)$ for any Q can be found, for example, in [Ea]. In particular, $J(Q_1) = 0$, and for the cubic Q_t with any $t \in \mathbb{C}, t^3 + 27 \neq 0$, one has

$$J(Q_t) = -\frac{t^3(t^3 - 216)^3}{110592(t^3 + 27)^3}.$$

It then follows that each of the cubics Q_λ with $\lambda^3 = 1$ is linearly equivalent to each of the cubics Q_t with $t^3 = 216$, as stated above.

Remark 3.1. One basis in which the algebra \mathcal{A}_t is given by multiplication table (3.3) is as follows:

$$\begin{aligned} e_1 &= \mathbf{1}, e_2 = Z_1 + Z_1 Z_3, e_3 = Z_2 Z_3 + 2Z_1 Z_2 Z_3, e_4 = Z_2 + Z_2 Z_3, \\ e_5 &= Z_3 + Z_1 Z_2 + 3Z_1 Z_2 Z_3, e_6 = Z_1 Z_2 Z_3, e_7 = Z_1 Z_3, e_8 = Z_3. \end{aligned}$$

Note that in our reconstruction of Q_t from \mathcal{A}_t above we used only table (3.3), not the explicit form of the basis.

Remark 3.2. As was noted in the introduction, a hypersurface germ \mathcal{V} with isolated singularity is determined, in general, by the algebra $\mathcal{A}(\mathcal{V})$ and the dimension n of the ambient space. We stress that in the case of homogeneous singularities n can be extracted from $\mathcal{A}(\mathcal{V})$ as its embedding dimension (see formula (2.4)).

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