PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 2, February 2014, Pages 547–558
S 0002-9939(2013)11842-3
Article electronically published on October 24, 2013

BANACH-SAKS PROPERTIES OF MUSIELAK-ORLICZ AND NAKANO SEQUENCE SPACES

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(Communicated by Thomas Schlumprecht)

Abstract. In this paper Banach-Saks properties of Musielak-Orlicz sequence space ℓ_Φ are studied. It is shown that ℓ_Φ has the weak Banach-Saks property if and only if it is separable. Moreover it is proved that in ℓ_Φ both Banach-Saks type p-properties, (BS_p) and (S_p), are equivalent and that the Schur property and (BS_∞) also coincide in these spaces. As applications, we give characterizations of the weak Banach-Saks property and the (BS_p) property in the Nakano sequence space $\ell^{(p_n)}$ and weighted Orlicz sequence space $\ell^\phi(w)$, in terms of the sequence (p_n), and the Orlicz function ϕ and the weight sequence w, respectively.

1. Introduction

In the geometry of Banach spaces an important role is played by the (weak) Banach-Saks property and its stronger versions such as the Banach-Saks p-property (BS_p) and property (S_p). It is said that a Banach space X satisfies the *Banach-Saks property* (BS) if for every bounded sequence $\{x_n\}$ in X there is a subsequence $\{y_j\}$ such that its Cesaro means converge; that is, the sequence $\{\frac{1}{m}\sum_{j=1}^m y_j\}$ is convergent in norm. A Banach space X is said to satisfy the *weak Banach-Saks property* (wBS) if every weakly null sequence in X has a subsequence such that its Cesaro means converge in norm. It is well known that a Banach space has the (BS)-property if and only if it is reflexive and it has the (wBS)-property [17].

W.B. Johnson introduced the following notion in [8]. Given $1 < p \le \infty$, a Banach space $(X, \|\cdot\|)$ has *Banach-Saks type p-property* (BS_p) if every weakly null sequence $\{x_n\}$ in X has a subsequence $\{y_j\}$ such that for some constant $C > 0$ and for all $m \in \mathbb{N}$,

$$\left\| \sum_{j=1}^m y_j \right\| \le C m^{1/p}.$$

Here $m^{1/\infty} = 1$ for all $m \in \mathbb{N}$. Clearly if X has the (BS_p) property, then it has the (BS_r) property for any $1 < r < p$. The stronger property (S_p) was introduced by H. Knaust and T. Odell in [15]. It is said that X has property (S_p), $1 < p \le \infty$, if

Received by the editors November 14, 2011 and, in revised form, March 19, 2012.

2010 *Mathematics Subject Classification.* Primary 46B03, 46B25, 46B45.

Key words and phrases. Banach-Saks properties, Schur property, Musielak-Orlicz space, Nakano space, variable exponent space, weighted Orlicz space.

The second author is the corresponding author. He was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF), funded by the Ministry of Education, Science and Technology (20121A1A1006869).

every weakly null sequence in X has a subsequence $\{x_k\}$ so that there is a constant $C > 0$ such that for all $m \in \mathbb{N}$ and all real sequences $a = (a_n) \in \ell_p$,

$$\left\| \sum_{j=1}^{m} a_j x_j \right\| \leq C \|a\|_p,$$

where $\|a\|_p$ is the ℓ_p norm of a.

It is clear that $(S_p) \Rightarrow (BS_p) \Rightarrow (wBS)$ for all $1 < p \leq \infty$. The Elton c_0-theorem ([2], Theorem III.3.5) states that $(BS_\infty) \Leftrightarrow (S_\infty)$ and a Banach space X has (BS_∞) if and only if every normalized weakly null sequence contains a subsequence which is equivalent to the unit vector basis of c_0, which is equivalent to the fact that every subspace of X has the Dunford-Pettis property (for details see [3]).

In general, the two properties (BS_p) and (S_p) are not equivalent if $1 < p < \infty$ [14]; however, Rakov ([21] Theorem 3) showed that if $1 < q < p < \infty$, then (BS_p) implies (S_q).

For a Banach space X we define the following set:

$$\Gamma(X) = \{p \in (1, \infty] : X \text{ satisfies } (BS_p)\text{-property}\}.$$

Banach-Saks properties and in particular the set $\Gamma(X)$ have been studied in general rearrangement invariant spaces as well as in specific symmetric spaces such as Orlicz, Lorentz or Marcinkiewicz spaces (e.g. [1, 5, 21–24]). Recall that $\Gamma(\ell_p) = (1, p]$ for $1 < p < \infty$, and $\Gamma(c_0) = \Gamma(\ell_1) = (1, \infty]$. The space ℓ_∞ does not have (wBS) property, so $\Gamma(\ell_\infty) = \emptyset$. Any separable sequence Orlicz space ℓ_ϕ or a separable part h_ϕ of a nonseparable Orlicz space ℓ_ϕ has (wBS) [22]. For a sequence Orlicz space ℓ_ϕ, whenever ℓ_ϕ is reflexive, we have that $(1, \alpha_\phi^0) \subset \Gamma(\ell_\phi) \subset (1, \alpha_\phi^0]$, where α_ϕ^0 is the lower Matuszewska-Orlicz index around zero. Moreover (BS_p) and (S_p) are equivalent in ℓ_ϕ for any $1 < p \leq \infty$ [14].

Here we shall consider Musielak-Orlicz sequence spaces that are not rearrangement invariant in general. We will show that Musielak-Orlicz sequence spaces ℓ_Φ have (wBS) if and only if they are separable. We shall characterize in terms of generating function $\Phi = (\phi_n)$ when ℓ_Φ has (BS_p) property for any $1 < p < \infty$. It follows that in ℓ_Φ the properties (BS_p) and (S_p) are equivalent and that ℓ_Φ has (BS_∞) property if and only if it has the Schur property. We further apply these results to Nakano sequence spaces $\ell^{(p_n)}$ (also called variable exponent spaces) and to weighted sequence Orlicz spaces $\ell^\phi(w)$. For the Nakano space $\ell^{(p_n)}$ we obtain a precise characterization of the set $\Gamma(\ell^{(p_n)})$. In the case when the weight w belongs to a particular class Λ, we characterize the set $\Gamma(\ell^\phi(w))$.

We first recall the following splitting theorem, which will be used later.

Proposition 1.1 (Proposition 1.a.12 in [18]). *Let X be a Banach space with Schauder basis $\{e_n\}$. If $\{x_n\}$ is a weakly null sequence in X, then there is a subsequence $\{y_k\}$ of $\{x_n\}$ such that $y_k = d_k + z_k$ for all $k \in \mathbb{N}$, where $\{d_k\}$ is a block basic sequence of $\{e_n\}$ and* $\lim_{k\to\infty} \|z_k\| = 0$.

A convex function $\phi : \mathbb{R}_+ \to [0, \infty)$ is said to be an *Orlicz function* if $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. A sequence $\Phi = (\phi_n)$ of Orlicz functions ϕ_n is called a *Musielak-Orlicz function*. The function $\Phi^* = (\phi_n^*)$ is called a conjugate function of Φ whenever each ϕ_n^* is a conjugate function of ϕ_n, that is, $\phi_n^*(t) = \sup_{s\geq 0}\{st - \phi_n(s)\}$, $t \geq 0$. The lower and upper Matuszewska-Orlicz indices of ϕ around zero

[18] are defined by

$$\alpha_\phi^0 = \sup\{q; \sup_{\substack{0<t\le 1\\ 0<\lambda\le 1}} \frac{\phi(\lambda t)}{\phi(\lambda)t^q} < \infty\}, \quad \beta_\phi^0 = \inf\{q; \sup_{\substack{0<t\le 1\\ 0<\lambda\le 1}} \frac{\phi(\lambda t)}{\phi(\lambda)t^q} > 0\},$$

respectively.

The Musielak-Orlicz space ℓ_Φ is the collection of all sequences $x = (a_n)$ of real numbers such that for some $\lambda > 0$,

$$I_\Phi(\lambda x) = \sum_{n=1}^\infty \phi_n(\lambda |a_n|) < \infty.$$

The space ℓ_Φ equipped with the Luxemburg norm

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \le 1\}$$

is a Banach space [18, 25].

For technical reasons, for any Musielak-Orlicz function $\Phi = (\phi_n)$, without loss of generality, we shall always assume that $\phi_n(1) = 1$ for all $n \in \mathbb{N}$. In fact if $\Phi = (\phi_n)$ is a Musielak-Orlicz function, then for each $n \in \mathbb{N}$, there is $a_n > 0$ such that $\phi_n(a_n) = 1$. Letting $\psi_n(t) = \phi_n(ta_n)$ $(t \in \mathbb{R}_+, n \in N)$, the space ℓ_Ψ where $\Psi = (\psi_n)$ is isometrically isomorphic to ℓ_Φ. The order continuous subspace h_Φ of the Musielak-Orlicz space ℓ_Φ is the subspace consisting of all elements x in ℓ_Φ satisfying $I_\Phi(\lambda x) < \infty$ for all $\lambda > 0$. It is well known that the standard unit vectors $\{e_n\}$ form an unconditional basis of h_Φ [18, 25].

Two Musielak-Orlicz functions Φ and Ψ are said to be *equivalent* if the two spaces ℓ_Φ and ℓ_Ψ coincide as sets. Using the closed graph theorem one can show that Φ and Ψ are equivalent if and only if $\ell_\Phi = \ell_\Psi$ as sets and the norms in ℓ_Φ and ℓ_Ψ are equivalent. It is also well known [9] that Φ and Ψ are equivalent if and only if there exist $K > 0$ and $\delta > 0$ and an ℓ_1-sequence (c_n) of nonnegative real numbers such that for all $n \in \mathbb{N}$,

$$\phi_n(Ku) \le \psi_n(u) + c_n \quad \text{if} \quad \psi_n(u) \le \delta,$$

$$\psi_n(Ku) \le \phi_n(u) + c_n \quad \text{if} \quad \phi_n(u) \le \delta.$$

Let $\Phi = (\phi_n)$ be a Musielak-Orlicz function such that for all $n \in \mathbb{N}$, $\phi_n = \phi$, where ϕ is an Orlicz function. Then ℓ_Φ (resp. h_Φ) is an Orlicz sequence space and is denoted by ℓ_ϕ (resp. h_ϕ). If Ψ is a Musielak-Orlicz function equivalent to $\Phi = (\phi_n)$ with $\phi_n = \phi$ for all $n \in \mathbb{N}$, then we say that Ψ is equivalent to ϕ. Also, in this case $\ell_\Psi = \ell_\phi$ as sets and their norms are equivalent.

A Musielak-Orlicz function Φ is said to satisfy condition δ_2 ($\Phi \in \delta_2$) if there exist $K, \delta > 0$ and an ℓ_1-sequence (c_n) of nonnegative numbers such that

$$\phi_n(2u) \le K\phi_n(u) + c_n$$

for every $u \ge 0$ and $n \in \mathbb{N}$ satisfying $\phi_n(u) \le \delta$. The condition δ_2 is equivalent to the notion of the uniform Δ_2-condition introduced in [25] (for details, see [10] and [25]). It is well known that if $\Phi \in \delta_2$, then the dual space ℓ_Φ^* is isomorphic to the space ℓ_{Φ^*}. Moreover, ℓ_Φ is reflexive if and only if both Φ and Φ^* satisfy the δ_2 condition [18].

Given $1 \le p < \infty$ and $\Phi = (\phi_n)$, it is said that Φ satisfies condition δ^p (respectively, δ^{*p}) if there exist $K, \delta > 0$ and an ℓ_1-sequence (c_n) of nonnegative real numbers such that

$$\phi_n(\lambda u) \le K\lambda^p(\phi_n(u) + c_n) \quad \text{(respectively, } \phi_n(\lambda u) \ge K\lambda^p(\phi_n(u) - c_n))$$

for all $\lambda \ge 1$, $n \in \mathbb{N}$ and $u \ge 0$ satisfying $\phi_n(\lambda u) \le \delta$. Provided that Φ and Ψ are equivalent, $\Phi \in \delta_2$ implies $\Psi \in \delta_2$; and if Φ satisfies both conditions δ_2 and δ^{*p} for some $1 \le p < \infty$, then Ψ satisfies condition δ^{*p} [10].

Remark 1.2 ([18, 25]). The following conditions are equivalent:

(1) The Musielak-Orlicz function Φ satisfies the δ_2 condition.
(2) $\ell_\Phi = h_\Phi$.
(3) The standard unit vectors $\{e_n\}$ form a Schauder basis in ℓ_Φ.
(4) ℓ_Φ is separable.
(5) h_Φ does not contain a c_0 isomorphic copy.
(6) ℓ_Φ does not contain an ℓ_∞ isomorphic copy.

Theorem 1.3. *The Musielak-Orlicz space ℓ_Φ does not contain an isomorphic copy of ℓ_1 if and only if it is reflexive and its reflexivity is equivalent to the fact that both Φ and Φ^* satisfy condition δ_2.*

Proof. ($\Rightarrow$) If $\Phi \notin \delta_2$, then ℓ_Φ contains an isomorphic copy of ℓ_∞, and so it contains a subspace isomorphic to ℓ_1. If $\Phi^* \notin \delta_2$ and $\Phi \in \delta_2$, then ℓ_{Φ^*} is isomorphic to the dual space ℓ_Φ^*, and thus it contains an isomorphic copy of ℓ_∞. Hence by the well known fact [18, Proposition 2.e.8], ℓ_Φ contains an isomorphic copy of ℓ_1.

($\Leftarrow$) Notice that ℓ_Φ is reflexive if and only if both Φ and Φ^* satisfy condition δ_2 [18]. Hence it is clear that if ℓ_Φ is reflexive, then ℓ_1 is not an isomorphic subspace of ℓ_Φ. □

The next result states that any space h_Φ has a subspace which is isomorphic either to c_0 or to some Orlicz space h_ψ. It can be deduced from the facts concerning subspaces of Musielak-Orlicz sequence spaces (e.g. Proposition 2.3 in [20]); however, we provide a direct proof here for the sake of completeness.

Theorem 1.4. *Let $\Phi = (\phi_n)$ be a Musielak-Orlicz function. Then there is a subsequence $\tilde{\Phi} = (\phi_{n_k})$ such that either $h_{\tilde{\Phi}} = c_0$ with equivalent norms, or there is an Orlicz function ψ equivalent to $\tilde{\Phi}$ that is, $h_{\tilde{\Phi}} = h_\psi$ as sets and the norms are equivalent.*

Proof. Let ϕ be an Orlicz function and ϕ' be its right-derivative. Then for all $\lambda > 0$, we have

$$\phi(\lambda) = \int_0^\lambda \phi'(s)\,ds \ge \int_{\lambda/2}^\lambda \phi'(s)\,ds \ge \frac{\lambda}{2}\phi'\Big(\frac{\lambda}{2}\Big).$$

Hence for each $n \in \mathbb{N}$ and $0 \le t_1 < t_2 \le 1/2$, we have

$$\begin{aligned}|\phi_n(t_2) - \phi_n(t_1)| &= \left|\int_{t_1}^{t_2} \phi_n'(s)\,ds\right| \le \phi_n'(t_2)|t_2 - t_1| \le \phi_n'(1/2)|t_2 - t_1| \\ &= \frac{\phi_n'(1/2)}{\phi_n(1)}|t_2 - t_1| \le 2|t_2 - t_1|.\end{aligned}$$

So (ϕ_n) are equicontinuous on $[0,1/2]$, and since $\phi_n(1)=1$, $n\in\mathbb{N}$, (ϕ_n) is uniformly bounded in $[0,1/2]$. Using the Arzela-Ascoli theorem, there is a subsequence $\tilde{\Phi}=(\phi_{n_k})$ and $\phi\in C[0,1/2]$ such that

$$\sup\{|\phi_{n_k}(t)-\phi(t)| : t\in[0,1/2]\}\leq 1/2^k$$

for all $k\in\mathbb{N}$.

Now we have two cases to deal with. First, suppose that there is $u_0\in(0,1/2]$ such that $\phi(u_0)=0$. We claim that $h_{\tilde{\Phi}}$ is isomorphic to c_0. Let $x=(a_n)$ be an element of c_0 with $\|x\|_\infty\leq u_0$. Then for each $k\in\mathbb{N}$, we have $|\phi_{n_k}(u_0)|\leq 1/2^k$, and $\sum_{i=1}^\infty \phi_{n_i}(|a_i|)\leq 1$. Hence $\|\sum_{i=1}^\infty a_ie_i\|_{\tilde{\Phi}}\leq 1$, and thus for each $x=(a_n)\in c_0$, we have $\|\sum_{i=1}^\infty a_ie_i\|_{\tilde{\Phi}}\leq \frac{1}{u_0}\|x\|_\infty$. On the other hand, if $x=(a_n)$ is an element of $h_{\tilde{\Phi}}$ with $\|x\|_{\tilde{\Phi}}\leq 1$, then $\sum_{k=1}^\infty \phi_{n_k}(|a_k|)\leq 1$, and $|a_k|\leq 1$ for each $k\in\mathbb{N}$. This implies that $\|x\|_\infty\leq\|x\|_{\tilde{\Phi}}$. Therefore, both norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\tilde{\Phi}}$ are equivalent.

Suppose now that $\phi(t)>0$ for all $t\in(0,1/2]$. We will show that ϕ can be extended to an Orlicz function which is equivalent to $\tilde{\Phi}$. Notice that for each $k\in\mathbb{N}$ and for each $t\in[0,1/2)$, we get

$$\frac{\phi_{n_k}(1/2)-\phi_{n_k}(t)}{1/2-t}\leq\frac{\phi_{n_k}(1)-\phi_{n_k}(1/2)}{1-1/2}=\frac{1-\phi_{n_k}(1/2)}{1-1/2}.$$

By taking the limit as $k\to\infty$, we get for all $t\in[0,1/2)$,

$$\frac{\phi(1/2)-\phi(t)}{1/2-t}\leq 2(1-\phi(1/2)).$$

Hence $\phi'(s)\leq 2(1-\phi(1/2))$ for all $s\in(0,1/2)$, and by defining

$$\psi(t)=\begin{cases}\phi(t), & 0\leq t<1/2,\\ \phi(1/2)+\int_{1/2}^t 2(1-\phi(1/2))\,ds, & 1/2\leq t,\end{cases}$$

we get an Orlicz function ψ such that $\psi(1)=1$. We shall show that ψ is equivalent to $\tilde{\Phi}$. Let k_0 be a natural number such that $\psi(1/2)-1/2^{k_0}\geq(1/2)\psi(1/2)$. Then for all $k\geq k_0$, we have $\phi_{n_k}(1/2)\geq\psi(1/2)-1/2^k\geq(1/2)\psi(1/2)$. Let $\delta=(1/2)\psi(1/2)$ and choose the sequence (c_k) to be $c_k=1/2^k$ for $k\geq k_0$, and $c_k=1$ if $1\leq k<k_0$. Then for each $k\geq 1$,

$$\begin{aligned}\phi_{n_k}(u)&\leq\psi(u)+c_k \quad\text{if}\quad \psi(u)\leq\delta,\\ \psi(u)&\leq\phi_{n_k}(u)+c_k \quad\text{if}\quad \phi_{n_k}(u)\leq\delta,\end{aligned}$$

which shows that $\tilde{\Phi}$ is equivalent to ψ, and the proof is complete. □

2. Weak Banach-Saks property

The following notion is useful for studying block basic sequences in h_Φ [13]. Given a Musielak-Orlicz function $\Phi=(\phi_n)$, a Musielak-Orlicz function $\Psi=(\psi_n)$ is said to be a Φ-*convex block* if there exist a sequence $\{F_n\}$ of nonempty finite subsets of $\mathbb{N}$ and a sequence (α_n) of real numbers such that $\max F_n<\min F_{n+1}$, $\sum_{j\in F_n}\phi_j(|\alpha_j|)=1$ and $\psi_n(t)=\sum_{j\in F_n}\phi_j(t|\alpha_j|)$ for all $t\in[0,\infty)$ and all $n\in\mathbb{N}$. It is clear that if Φ satisfies the δ_2 condition, then the Φ-convex block Ψ also satisfies condition δ_2.

Theorem 2.1. *Let Φ be a Musielak-Orlicz function. Then h_Φ has the weak Banach-Saks property.*

Proof. Let $\{x_n\}$ be a weak null sequence in h_Φ. If $\liminf_{n\to\infty} \|x_n\|_\Phi = 0$, then we may choose a subsequence $\{y_n\}$ with $\|y_n\|_\Phi < 1/2^n$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \frac{1}{n} \|\sum_{k=1}^n y_k\|_\Phi = 0$. So we may assume that $C = \inf_n \|x_n\|_\Phi > 0$.

Applying Proposition 1.1, there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that $y_n = d_n + z_n$, where $\{d_n\}$ is a block basic sequence of the standard unit vectors $\{e_n\}$ and $\|z_n\|_\Phi \le C/2^n$ for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, setting $d_n = \sum_{j\in F_n} a_j e_j$ and $\psi_n(t) = \sum_{j\in F_n} \phi_j(t|a_j|/\|d_n\|_\Phi)$, where F_n is the support of d_n, we get the Φ-convex block $\Psi = (\psi_n)$. By Theorem 1.4, there is a subsequence $\tilde{\Psi} = (\psi_{n_k})$ such that $h_{\tilde{\Psi}}$ is isomorphic to either c_0 or h_ψ for some Orlicz function ψ. Notice that the standard basis $\{e_n\}$ in $h_{\tilde{\Psi}}$ and the basic sequence $\{d_{n_k}\}$ in h_Φ are equivalent. Since c_0 and the Orlicz space h_ψ have the weak Banach-Saks property [22], so do both $h_{\tilde{\Psi}}$ and $\overline{\operatorname{span}}\{d_{n_k} : k \in \mathbb{N}\}$. It follows by the fact that $\{d_{n_k}\}$ is weakly null that there is a further subsequence $\{d'_k\}$ of $\{d_{n_k}\}$ such that $\lim_{m\to\infty} \frac{1}{m} \|\sum_{k=1}^m d'_k\|_\Phi = 0$. So for the subsequence $\{y'_k = d'_k + z'_k\}$ of $\{x_n\}$, we get $\lim_{m\to\infty} \frac{1}{m} \|\sum_{k=1}^m y'_k\|_\Phi \le \lim_{m\to\infty} \frac{1}{m} \left(\|\sum_{k=1}^m d'_k\|_\Phi + C\right) = 0$ and the proof is done. □

Corollary 2.2. *Let Φ be a Musielak-Orlicz function. Then ℓ_Φ has the weak Banach-Saks property if and only if it is separable.*

Proof. The sufficiency is clear by Remark 1.2 and Theorem 2.1. For the converse, suppose that ℓ_Φ is not separable. Then it contains an ℓ_∞ isomorphic copy and ℓ_∞ does not have the weak Banach-Saks property, so neither does ℓ_Φ. □

3. (BS_p) and (S_p) properties in ℓ_Φ

Let's begin with an observation relating the δ^{*p} condition to the (S_p) property of ℓ_Φ.

Proposition 3.1. *Let $1 < p < \infty$ and let $\Phi = (\phi_n)$ be a Musielak-Orlicz function satisfying both the δ_2 and δ^{*p} conditions. Then ℓ_Φ has the (S_p) property.*

Proof. By Lemma 6 in [10], there is a Musielak-Orlicz function $\Psi = (\psi_n)$ such that Φ is equivalent to Ψ, and

$$\psi_n(\lambda t) \le \lambda^p \psi_n(t), \quad \text{for all } t \ge 0,\ \lambda \le 1 \text{ and } n \in \mathbb{N}.$$

Since the (S_p) property is invariant under isomorphism, we have only to show that ℓ_Ψ has the (S_p) property. Let $\{x_n\}$ be a weakly null sequence in ℓ_Ψ with $\sup_n \|x_n\|_\Psi \le 1/2$. By Proposition 1.1, there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that for all $n \in \mathbb{N}$, $y_n = d_n + z_n$, $\|z_n\|_\Psi \le 1/2^n$ and $\{d_n\}$ is a block basic sequence of the unit vector basis $\{e_n\}$. Then for each $n \in \mathbb{N}$, $d_n = \sum_{j\in F_n} a_j e_j$, where F_n is the support of d_n. Hence $\|d_n\|_\Psi \le 1$ and $\sum_{j\in F_n} \psi_j(|a_j|) \le 1$. Let (b_i) be a sequence of real numbers satisfying $\sum_{i=1}^\infty |b_i|^p \le 1$. Then

$$I_\Psi\left(\sum_{i=1}^\infty b_i d_i\right) = \sum_{i=1}^\infty \sum_{j\in F_i} \psi_j(|b_i a_j|) \le \sum_{i=1}^\infty \sum_{j\in F_i} \psi_j(|a_j|)|b_i|^p \le \sum_{i=1}^\infty |b_i|^p \sum_{j\in F_i} \psi_j(|a_j|) \le 1,$$

and so $\|\sum_{i=1}^\infty b_i d_i\|_\Psi \le 1$. Thus $\|\sum_{i=1}^\infty b_i x_i\|_\Psi \le \|\sum_{i=1}^\infty b_i d_i\|_\Psi + \sum_{i=1}^\infty |b_i|\, \|z_i\|_\Psi \le 2$, and the proof is finished. □

Proposition 3.2. *Let Φ be a Musielak-Orlicz function and let $1 < p < \infty$. Suppose that ℓ_Φ does not contain an isomorphic copy of ℓ_1. Then the following conditions are equivalent:*

(1) *ℓ_Φ satisfies the (BS_p) property.*
(2) *Φ satisfies the δ_2 condition and every Φ-convex block has a subsequence which satisfies the δ^{*p} condition.*
(3) *ℓ_Φ satisfies the (S_p) property.*

Proof. The implication (3) $\Rightarrow$ (1) is clear.

Suppose now that (1) holds. Since ℓ_Φ has (BS_p), ℓ_Φ does not contain an isomorphic copy of ℓ_∞, and thus Φ satisfies the δ_2 condition by Remark 1.2. Let $\Psi = (\psi_n)$ be a Φ-convex block. Then there exist a sequence $\{F_n\}$ of nonempty subsets of $\mathbb{N}$ and a sequence (α_n) of real numbers such that for all $n \in \mathbb{N}$, $\max F_n < \min F_{n+1}$ and $\psi_n(t) = \sum_{j \in F_n} \phi_j(t|\alpha_j|)$ for all $t \in [0, \infty)$, where $\sum_{j \in F_n} \phi_j(|\alpha_j|) = 1$. It is clear that Ψ also satisfies the δ_2 condition. Setting $u_n = \sum_{j \in F_n} \alpha_j e_j$ for all $n \in \mathbb{N}$, the sequence $\{u_n\}$ is a normalized block basis of the unit vector basis $\{e_n\}$ in ℓ_Φ. By Theorem 1.4, there is a subsequence $\tilde{\Psi} = (\psi_{n_k})$ such that $h_{\tilde{\Psi}} = \ell_{\tilde{\Psi}}$ is isomorphic to c_0 or $\tilde{\Psi}$ is equivalent to some Orlicz function ψ. Notice that $\ell_{\tilde{\Psi}}$ is isometric to $\overline{\text{span}}\{u_{n_k} : k \in \mathbb{N}\}$ in ℓ_Φ and ℓ_Φ does not contain a c_0 isomorphic copy by Remark 1.2. Thus $\overline{\text{span}}\{u_{n_k} : k \in \mathbb{N}\}$ is isomorphic to ℓ_ψ, and thus ℓ_ψ must have the (BS_p) property. Moreover ψ satisfies the δ_2 condition, since otherwise ℓ_ψ would contain an isomorphic copy of ℓ_∞. Now by Theorem 3 in [14], we have

$$C = \sup_{0<s,t\leq 1} \frac{\psi(st)}{\psi(s)t^p} < \infty.$$

This implies that if $\lambda \geq 1$ and $\psi(\lambda u) \leq 1$, then $\psi(\lambda u) \leq C^{-1}\lambda^p\psi(u)$. Hence ψ satisfies the δ^{*p} condition, and then the equivalent Musielak-Orlicz function $\tilde{\Psi}$ also satisfies the δ^{*p} condition. Therefore, (1) $\Rightarrow$ (2) is proved.

Finally, we show that (2) $\Rightarrow$ (3). Suppose that (2) holds and let $\{x_n\}$ be a weakly null sequence. If $\liminf_{n\to\infty} \|x_n\|_\Phi = 0$, then there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\sum_{n=1}^\infty \|y_n\|_\Phi \leq 1$. Then for every sequence (a_n) of real numbers satisfying $\sum_{n=1}^\infty |a_n|^p \leq 1$, $\|\sum_{n=1}^\infty a_n y_n\|_\Phi \leq \sum_{n=1}^\infty |a_n| \, \|y_n\|_\Phi \leq 1$. So we may assume that $C = \inf_n \|x_n\|_\Phi > 0$. By Proposition 1.1, there is a subsequence $\{y_k\}$ of $\{x_n\}$ such that $y_k = d_k + z_k$ for all $k \in \mathbb{N}$, where $\{d_k\}$ is a block basic sequence of $\{e_n\}$ and $\|z_k\|_\Phi < C/2^k$ for all $k \in \mathbb{N}$. Then $0 < \|d_n\|_\Phi$ for all $n \in \mathbb{N}$. Assume that for each $n \in \mathbb{N}$, $d_n = \sum_{j \in F_n} \alpha_j e_j$, where F_n is the support of d_n. Then letting $\psi_n(t) = \sum_{j \in F_n} \phi_j(t|\alpha_j|/\|d_n\|_\Phi), n \in \mathbb{N}$, we have that $\Psi = (\psi_n)$ is a Φ-convex block and Ψ satisfies the δ_2 condition. By the assumption, there is a subsequence Ψ' of Ψ which satisfies δ^{*p}. Then by Theorem 1.4, there is a further subsequence $\tilde{\Psi} = (\psi_{n_k})$ such that $\tilde{\Psi}$ is equivalent to an Orlicz function ψ. Hence ψ satisfies both the δ_2 and δ^{*p} conditions. It follows that

$$\sup_{0<s,t\leq 1} \frac{\psi(st)}{\psi(s)t^p} < \infty.$$

Notice that $\overline{\text{span}}\{d_{n_k} : k \in \mathbb{N}\}$ in ℓ_Φ is isomorphic to $\ell_{\tilde{\Psi}}$ and ℓ_ψ. By Theorem 3 in [14], $h_\psi = \ell_\psi$ satisfies (S_p) and $\overline{\text{span}}\{d_{n_k} : k \in \mathbb{N}\}$ has the (S_p) property. Let (a_i) be a sequence of real numbers satisfying $\sum_{i=1}^\infty |a_i|^p \leq 1$. Because $\{d_{n_k}\}$ is weakly null, there exist a constant $K > 0$ and a further subsequence $\{d'_k\}$ of $\{d_{n_k}\}$ such that $\|\sum_{i=1}^\infty a_i d'_i\|_\Phi \le K$. Then for $x'_k = d'_k + z'_k, k \in \mathbb{N}$, we have

$$\left\| \sum_{i=1}^\infty a_i x'_i \right\|_\Phi \le C + K,$$

which completes the proof. □

Remark 3.3. In view of Theorem 1.3, the assumption in the above theorem that the space does not contain an isomorphic copy of ℓ_1 is equivalent to reflexivity of ℓ_Φ.

Theorem 3.4. *The properties* (BS_p) *and* (S_p), $1 < p < \infty$, *are equivalent in* ℓ_Φ.

Proof. Suppose that ℓ_Φ has the (BS_p) property. Then $\Phi \in \delta_2$, since otherwise ℓ_Φ contains an isomorphic copy of ℓ_∞. Let $\{x_n\}$ be a weakly null sequence which is not convergent in norm. By Proposition 1.1 we can assume that $\{x_n\}$ is a basic normalized sequence. Then, following the proof of the implication (1) $\Rightarrow$ (2) of Proposition 3.2, we obtain that for the Φ-convex block $\Psi = (\psi_n)$ induced by $\{x_n\}$ there is a subsequence $\tilde{\Psi} = (\psi_{n_k})$ and there exists an Orlicz space ℓ_ψ such that $\ell_{\tilde{\Psi}} = \ell_\psi$ with equivalent norms. Thus $\{x_n\}$ is equivalent to the unit standard vector basis $\{e_n\}$ in ℓ_ψ. Notice that $\psi \in \delta_2$ and ℓ_ψ has the (BS_p) property. If ℓ_ψ is reflexive, then ℓ_ψ has the (S_p) property and the proof is finished. Now if ℓ_ψ is not reflexive, then ψ^* does not satisfy the δ_2 condition, and thus ℓ_ψ^* contains an isomorphic copy of ℓ_∞. It follows that ℓ_ψ contains an isomorphic copy of ℓ_1 by [18, Proposition 2.e.8]. Now applying Remark 6 and Theorem 4 in [22], ℓ_ψ must be isomorphic to ℓ_1. Consequently $\{\mathrm{e}_n\}$ converges to zero in norm in ℓ_ψ, since it is weakly null in ℓ_ψ. However, the latter is a contradiction in view of the fact that $\{e_n\}$ in ℓ_ψ is equivalent to a normalized sequence $\{x_n\}$ in ℓ_Φ. □

4. Schur property and property (BS_∞)

We start with a result showing that all properties (BS_∞), (BS_p) for all $1 < p < \infty$ and the Schur property are not only equivalent in any Orlicz space ℓ_ψ, but they also imply that ℓ_ψ must be isomorphic to ℓ_1.

Proposition 4.1. *Let ℓ_ψ be an Orlicz sequence space. Then the following properties are equivalent:*

(1) *ℓ_ψ has the (BS_p) property for all $1 < p < \infty$.*
(2) *ℓ_ψ is isomorphic to ℓ_1.*
(3) *ℓ_ψ has the Schur property.*
(4) *ℓ_ψ has (BS_∞).*

Proof. For the proof of (1) $\Rightarrow$ (2), suppose that ℓ_ψ has the (BS_p) property for all $1 < p < \infty$. Then $\ell_\psi = h_\psi$ since otherwise it contains an isomorphic copy of ℓ_∞ and so does not have any (BS_p) property. By Theorem 4.a.9. in [18], the space ℓ_p (or c_0 if $p = \infty$) is isomorphic to a subspace of h_ψ if and only if $p \in [\alpha_\psi^0, \beta_\psi^0]$. Hence $\Gamma(h_\psi) \subset (1, \alpha_\psi^0]$. Assume first that $1 < \alpha_\psi^0$. Then $(1, \infty) \subset \Gamma(h_\psi) \subset (1, \alpha_\psi^0]$. So $\alpha_\psi^0 = \infty$ and h_ψ contains c_0. So $\ell_\psi \ne h_\psi$, and this is a contradiction to equality $\ell_\psi = h_\psi$. Therefore $\alpha_\psi^0 = 1$ and h_ψ contains an isomorphic copy of ℓ_1. Then it is isomorphic to ℓ_1 by Remark 6 and Theorem 4 in [22]. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) are clear, and we conclude the proof. □

Notice that the fact that an Orlicz space ℓ_ψ with the Schur property is isomorphic to ℓ_1 can also be deduced from Corollary 1 in [13].

In the next theorem we partially generalize the above result to the Musielak-Orlicz sequence space ℓ_Φ. In fact we show that properties (1), (3) and (4) are equivalent; however, it is well known that there exists a Musielak-Orlicz sequence space with the Schur property which is not isomorphic to ℓ_1 [7].

Theorem 4.2. *The following properties are equivalent:*

(1) *ℓ_Φ has the (BS_p) property for all $1 < p < \infty$.*
(2) *ℓ_Φ has the Schur property.*
(3) *ℓ_Φ has the (BS_∞) property.*

Proof. It is enough to show the implication from (1) to (2). Suppose that ℓ_Φ has the (BS_p) property for all $1 < p < \infty$ and ℓ_Φ does not have the Schur property. It follows that $\Phi \in \delta_2$ since otherwise ℓ_Φ contains an isomorphic copy of ℓ_∞. Let $\{x_n\}$ be a weakly null sequence which is not convergent in norm. By Proposition 1.1 we can assume that $\{x_n\}$ is a basic normalized sequence. Then following the proof of the implication (1) $\Rightarrow$ (2) of Proposition 3.2, we obtain that for the Φ-convex block $\Psi = (\psi_n)$ induced by $\{x_n\}$ there is a subsequence $\tilde{\Psi} = (\psi_{n_k})$ and there exists an Orlicz space ℓ_ψ such that $\ell_{\tilde{\Psi}} = \ell_\psi$ with equivalent norms. Thus $\{x_n\}$ is equivalent to the unit standard vector basis $\{e_n\}$ in ℓ_ψ and ℓ_ψ has the (BS_p) property for all $1 < p < \infty$ and is isomorphic to ℓ_1 by Proposition 4.1. Hence the unit vector basis converges to zero in norm, which is a contradiction. $\square$

5. Nakano sequence spaces $\ell^{(p_n)}$

Let (p_n) be a sequence of real numbers in $[1, \infty)$ and let $\Phi = (\phi_n)$, where $\phi_n(t) = t^{p_n}$ for each $n \in \mathbb{N}$. Then the Musielak-Orlicz space ℓ_Φ is said to be a *Nakano sequence space* (called also a variable exponent space) and is denoted by $\ell^{(p_n)}$. It is well known and standard to check that Φ satisfies the δ_2 condition if and only if $\sup_{n\in\mathbb{N}} p_n < \infty$.

We state the first results ((1) and (2) are already known) below which are consequences of Theorem 1.4, Remark 1.2 and Corollary 2.2.

Corollary 5.1 ([20])**.** (1) *The space $\ell^{(p_n)}$ contains an ℓ_p isomorphic copy for $1 \le p < \infty$ (resp. contains a c_0 isomorphic copy) if there is a subsequence (p_{n_k}) with $p = \lim_{k\to\infty} p_{n_k}$ (resp. $\lim_{k\to\infty} p_{n_k} = \infty$).*

(2) [4] *The space $\ell^{(p_n)}$ has a weak Banach-Saks property if and only if it is separable, which is equivalent to $\sup_{n\in\mathbb{N}} p_n < \infty$.*

(3) *Given $1 < p < \infty$, (BS_p) and (S_p) are equivalent in $\ell^{(p_n)}$.*

In the next theorem we find an exact characterization of the set $\Gamma(\ell^{(p_n)})$.

Theorem 5.2. *Let $1 < p = \liminf_{n\to\infty} p_n \le \limsup_{n\to\infty} p_n < \infty$. Then $\Gamma(\ell^{(p_n)}) = (1, p]$.*

Proof. Since $p = \liminf_n p_n$, the Nakano sequence space $\ell^{(p_n)}$ contains an ℓ_p isomorphic copy and $\Gamma(\ell^{(p_n)}) \subset \Gamma(\ell_p) = (1, p]$. So we have only to show that $p \in \Gamma(\ell^{(p_n)})$.

Let $\Psi = (\psi_n)$ be a Φ-convex block; that is, for all $n \in \mathbb{N}$, $\psi_n(t) = \sum_{j\in F_n} t^{p_j} |\alpha_j|^{p_j}$ for all $t \in [0, \infty)$, where $\{F_n\}$ is a sequence of nonempty finite subsets of $\mathbb{N}$ such that $\max F_n < \min F_{n+1}$ and $\sum_{j\in F_n} |\alpha_j|^{p_j} = 1$ for all $n \in \mathbb{N}$. In order to complete the proof, by Proposition 3.2 it is enough to show that there is a subsequence (ψ_{n_k})

which satisfies δ^{*p}. Letting $q_n = \min\{p_i : i \in F_n\}$, there is a subsequence (q_{n_k}) such that $\lim_k q_{n_k} = q$ and $|q_{n_k} - q| < 1/2^k$ for all $k \in \mathbb{N}$. We claim that (ψ_{n_k}) satisfies the δ^{*p} condition. Given $k \in \mathbb{N}$ and $i \in F_{n_k}$, for all $\lambda \geq 1$ and for all $u \geq 0$ satisfying $\lambda u \leq 1$, the following inequality holds:

$$(\lambda u)^{p_i} \geq \lambda^p (u^{p_i} - 1/2^k).$$

Indeed, set $d_k = 1/2^k$ and $c_{k,i} = \sup\{u^{p_i}(1 - \lambda^{-d_k}) : \lambda \geq 1, u \geq 0, u\lambda \leq 1\}$. Then $c_{k,i} = \sup\{u^{p_i}(1 - x^{d_k}) : 0 < x \leq 1, 0 \leq u \leq x\} = \max\{f(u,x) : 0 \leq x \leq 1, 0 \leq u \leq x\}$, where $f(u,x) = u^{p_i}(1 - x^{d_k})$. It is elementary to check that f attains its maximum on the boundary $u = x$ for $0 \leq x \leq 1$. So $c_{k,i} = \max_{0 \leq x \leq 1} x^{p_i}(1 - x^{d_k}) = \left(\frac{p_i}{p_i + d_k}\right)^{\frac{p_i}{d_k}} \frac{d_k}{p_i + d_k} \leq d_k$. Notice that $p - d_k \leq q - d_k \leq q_{n_k} \leq p_i$ for $i \in F_{n_k}$. Then $c_{k,i} \leq d_k$ implies that for all $\lambda \geq 1$ and for all $u \geq 0$ satisfying $\lambda u \leq 1$,

$$(\lambda u)^{p_i} \geq \lambda^{d_k + p_i}(u^{p_i} - d_k) \geq \lambda^p (u^{p_i} - d_k).$$

This proves the desired claim. If $\lambda \geq 1$ and $u \geq 0$ with $\psi_{n_k}(\lambda u) \leq 1$, then $\lambda u \leq 1$ and

$$\begin{aligned}
\psi_{n_k}(\lambda u) &= \sum_{i \in F_{n_k}} (\lambda u)^{p_i} |\alpha_i|^{p_i} \\
&\geq \sum_{i \in F_{n_k}} \lambda^p (u^{p_i} - 1/2^k) |\alpha_i|^{p_i} \\
&= \lambda^p (\psi_{n_k}(u) - 1/2^k).
\end{aligned}$$

This shows that (ψ_{n_k}) satisfies the δ^{*p} condition, and the proof is completed. □

We finish with a corollary which follows from Theorem 4.2 and is the well known criterion for the Schur property of $\ell^{(p_n)}$ [13, 19].

Corollary 5.3. *In Nakano sequence space $\ell^{(p_n)}$ the conditions* (i) (BS_∞), (ii) *the Schur property, and* (iii) $\lim_n p_n = 1$ *are equivalent.*

6. Weighted Orlicz sequence space

An important class of Musielak-Orlicz spaces is the class of weighted Orlicz spaces studied among others in [20] and [6]. Let ϕ be an Orlicz function and $w = (w_n)$ be a weight sequence of positive numbers. Then the Musielak-Orlicz function $\Phi = (\phi_n)$, where $\phi_n(u) = \phi(u)w_n$, $n \in \mathbb{N}$, defines a *weighted Orlicz sequence space* ℓ_Φ, which will be denoted by $\ell^\phi(w)$. Following [20, Definition 2.6], a weight sequence $w = (w_n)$ is said to be in the class Λ ($w \in \Lambda$) if there is a subsequence (w_{n_k}) such that $\lim_{k\to\infty} w_{n_k} = 0$ and $\sum_{k=1}^\infty w_{n_k} = \infty$. Recall that an Orlicz function ϕ is said to satisfy the Δ_2 condition (in short $\phi \in \Delta_2$) if there is a constant $K > 0$ such that $\phi(2u) \leq K\phi(u)$ for all $u \geq 0$. The lower Matuszewska-Orlicz index of ϕ on $[0, \infty)$ is defined by

$$\alpha_\phi = \sup\{1 \leq p < \infty : \sup_{\substack{0<t\leq 1 \\ \lambda > 0}} \frac{\phi(\lambda t)}{\phi(\lambda)t^p} < \infty\}.$$

It is well known that $\alpha_\phi > 1$ if and only if $\phi^* \in \Delta_2$ [11, 16]. Moreover, it is standard to show that if $w \in \Lambda$, then $\phi \in \Delta_2$ if and only if $\Phi \in \delta_2$. So, if $w \in \Lambda$, then $\ell^\phi(w)$ is separable if and only if $\phi \in \Delta_2$.

Theorem 6.1. *Let $\ell^\phi(w)$ be a weighted Orlicz sequence space.*

(1) *$\ell^\phi(w)$ has the weak Banach-Saks property if and only if it is separable.*
(2) *Let $\ell^\phi(w)$ be separable and ϕ^* satisfy the Δ_2 condition. If $1 < p < \alpha_\phi$, then $\ell^\phi(w)$ has the (S_p) property. Consequently $(1, \alpha_\phi) \subset \Gamma(\ell^\phi(w))$.*
(3) *Suppose that both ϕ and ϕ^* satisfy the Δ_2 condition. If $w \in \Lambda$, then $(1, \alpha_\phi) \subset \Gamma(\ell^\phi(w)) \subset (1, \alpha_\phi]$.*

Proof. (1) In view of the fact that $\ell^\phi(w)$ is isometrically isomorphic to a Musielak-Orlicz space ℓ_Ψ where $\Psi = (\psi_n)$ and $\psi_n(u) = \phi(\phi^{-1}(1/w_n)u)w_n$, $n \in \mathbb{N}$, and from Theorem 2.1, the statement follows.

(2) By the assumption $\phi^* \in \Delta_2$ we have $\alpha_\phi > 1$. Now letting $1 < p < \alpha_\phi$, there is a constant $C > 0$ such that

$$\sup_{\substack{0<t\leq 1\\ 0>\lambda}} \frac{\phi(\lambda t)}{\phi(\lambda)t^p} \leq C.$$

Let $\{x_n\}$ be a weakly null sequence in $\ell^\phi(w)$. By Proposition 1.1 and Remark 1.2 we may assume that $\{x_n\}$ is a normalized block basic sequence of the standard unit vector basis $\{e_n\}$; that is, $x_n = \sum_{j\in F_n} \beta_j e_j$, where F_n is a support of x_n for each $n \in \mathbb{N}$. For a sequence (a_n) of real numbers with $\sum_{n=1}^\infty |a_n|^p \leq 1/(1+C)$, we have for each $m \in \mathbb{N}$,

$$\begin{aligned}
I_\Phi\Big(\sum_{n=1}^m a_n x_n\Big) &= \sum_{n=1}^m \sum_{j\in F_n} \phi(|\beta_j||a_n|)w_j \\
&\leq C\sum_{n=1}^m \sum_{j\in F_n} \phi(|\beta_j|)|a_n|^p w_j \\
&\leq C\sum_{n=1}^m |a_n|^p \sum_{j\in F_n} \phi(|\beta_j|)w_j \\
&\leq \frac{C}{C+1} < 1.
\end{aligned}$$

Hence $\|\sum_{n=1}^m a_n x_n\|_\Phi \leq 1$, and thus $\ell^\phi(w)$ has the (S_p) property.

(3) Suppose that $w \in \Lambda$. The assumption $\phi \in \Delta_2$ implies that $\Phi \in \delta_2$, and thus by Remark 1.2 the space $\ell^\phi(w)$ is separable. Hence by (2) we get $(1, \alpha_\phi) \subset \Gamma(\ell^\phi(w))$. Now letting $p = \alpha_\phi > 1$, by Theorem 2.7 in [20], there is an ℓ_p isomorphic copy in $\ell^\phi(w)$. Hence it is clear that $\Gamma(\ell^\phi(w)) \subset \Gamma(\ell_p) = (1, p] = (1, \alpha_\phi]$, which completes the proof. □

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