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A MULTI-DIMENSIONAL ANALOGUE OF COBHAM'S THEOREM FOR FRACTALS

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Abstract. Let $\mathcal{C}_k$ be the set of hyper-cubes of size $\frac{1}{k} \times \cdots \times \frac{1}{k}$ in $[0,1]^d$ with vertices having coordinates of the form $(a_1/k, a_2/k, \ldots, a_d/k)$. For $c \in \mathcal{C}_k$ we define T_c as the linear expansion map from c to $[0,1]^d$ in the obvious way. We extend the map T_c to a map on all of $[0,1]^d$ by defining $T_c(\overline{\mathbf{x}}) = \emptyset$ for $\overline{\mathbf{x}} \notin c$.

Let X be a compact subset of $[0,1]^d$ and $k > 1$ be an integer. We define the *k-kernel* of X as

$$\{T_c(X) \ \mid \ c \in \mathcal{C}_1 \cup \mathcal{C}_k \cup \mathcal{C}_{k^2} \cup \mathcal{C}_{k^3} \cup \cdots\}.$$

If this set is finite, then we say that X has *finite k-kernel* or, equivalently, that X is *k-self-similar*. Some examples of this are the standard Cantor set, the Sierpiński carpet and the Sierpiński triangle.

Recently Adamczewski and Bell showed an analogue of Cobham's theorem for one-dimensional fractals. Let k and ℓ be multiplicatively independent positive integers. They proved that the compact set $X \subset [0,1]$ is both k- and ℓ-self-similar if and only if X is a union of a finite number of intervals with rational endpoints.

In their paper, Adamczewski and Bell conjectured that a similar result should be true in higher dimensions. We prove their conjecture; in particular we prove:

Theorem. Let k and ℓ be multiplicatively independent positive integers. The compact set $X \subset [0,1]^d$ is both k- and ℓ-self-similar if and only if X is a union of a finite number of rational polyhedra.

1. Introduction and basic definitions

Let $\mathcal{C}_k$ be the set of hyper-cubes of size $\frac{1}{k} \times \cdots \times \frac{1}{k}$ in $[0,1]^d$ with vertices having coordinates of the form $(a_1/k, a_2/k, \ldots, a_d/k)$ with $0 \le a_i \le k$ for $i \in \{1, \ldots, d\}$. In particular,

$$\mathcal{C}_k := \left\{ \left[\frac{i_1}{k}, \frac{i_1+1}{k}\right] \times \cdots \times \left[\frac{i_d}{k}, \frac{i_d+1}{k}\right] \ \middle| \ i_1, i_2, \ldots, i_d \in \{0, 1, \ldots, k-1\} \right\}.$$

For $c \in \mathcal{C}_k$ we define T_c as the linear expansion map from c to $[0,1]^d$ in the obvious way. That is, for $c = \left[\frac{i_1}{k}, \frac{i_1+1}{k}\right] \times \cdots \times \left[\frac{i_d}{k}, \frac{i_d+1}{k}\right]$, define

$$T_c(x_1, x_2, \ldots, x_d) := (kx_1 - i_1,\ kx_2 - i_2,\ \ldots,\ kx_d - i_d).$$

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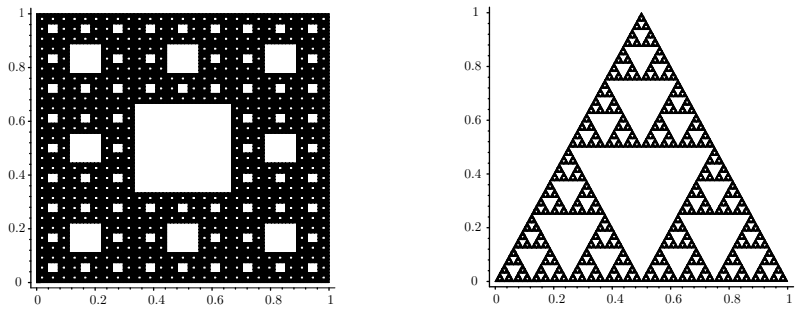

(a) Sierpiński carpet (b) Sierpiński triangle

FIGURE 1.1. Examples of k-self-similar sets

For convenience we will write $\overline{\mathbf{x}} = (x_1, x_2, \ldots, x_d)$. We extend the map T_c to a map on all of $[0,1]^d$ by defining $T_c(\overline{\mathbf{x}}) = \emptyset$ for $\overline{\mathbf{x}} \notin c$.

Let X be a compact subset of $[0,1]^d$, and $k > 1$ be an integer. We define the *k-kernel* of X as

$$\{T_c(X) \mid c \in \mathcal{C}_1 \cup \mathcal{C}_k \cup \mathcal{C}_{k^2} \cup \mathcal{C}_{k^3} \cup \cdots\}.$$

If this set is finite, then we say that X has *finite k-kernel* or, equivalently, that X is *k-self-similar*.

Some examples of such sets are the standard Cantor set, the Sierpiński carpet and the Sierpiński triangle. See Figure 1.1 for the Sierpiński carpet and triangle. The Cantor set, and the Sierpiński carpet are both 3-self-similar, while the Sierpiński triangle is 2-self-similar. See for example Figure 1.2 for the 2-kernel of the Sierpiński triangle.

The definition of the k-kernel above is analogous to a definition coming from automata theory, and is connected to the study of k-automatic sequences. For more on k-automatic sequences see [3, Chapter 5]. The analogous construction for k-automatic fractals is studied in [1].

Recently Adamczewski and Bell [1] showed an analogue of Cobham's theorem for one-dimensional fractals, namely:

Theorem (Adamczewski and Bell [1])**.** *Let k and ℓ be multiplicatively independent positive integers. The compact set $X \subset [0,1]$ is both k- and ℓ-self-similar if and only if X is a union of a finite number of intervals with rational endpoints.*

For more on Cobham's theorem, as it relates to k-automatic sequences, see [3, 4, 6]. Adamczewski and Bell conjectured that a similar result should be true in higher dimensions.

In this paper we prove their conjecture.

Theorem 1.1. *Let k and ℓ be multiplicatively independent positive integers. The compact set $X \subset [0,1]^d$ is both k- and ℓ-self-similar if and only if X is a union of a finite number of rational polyhedra.*

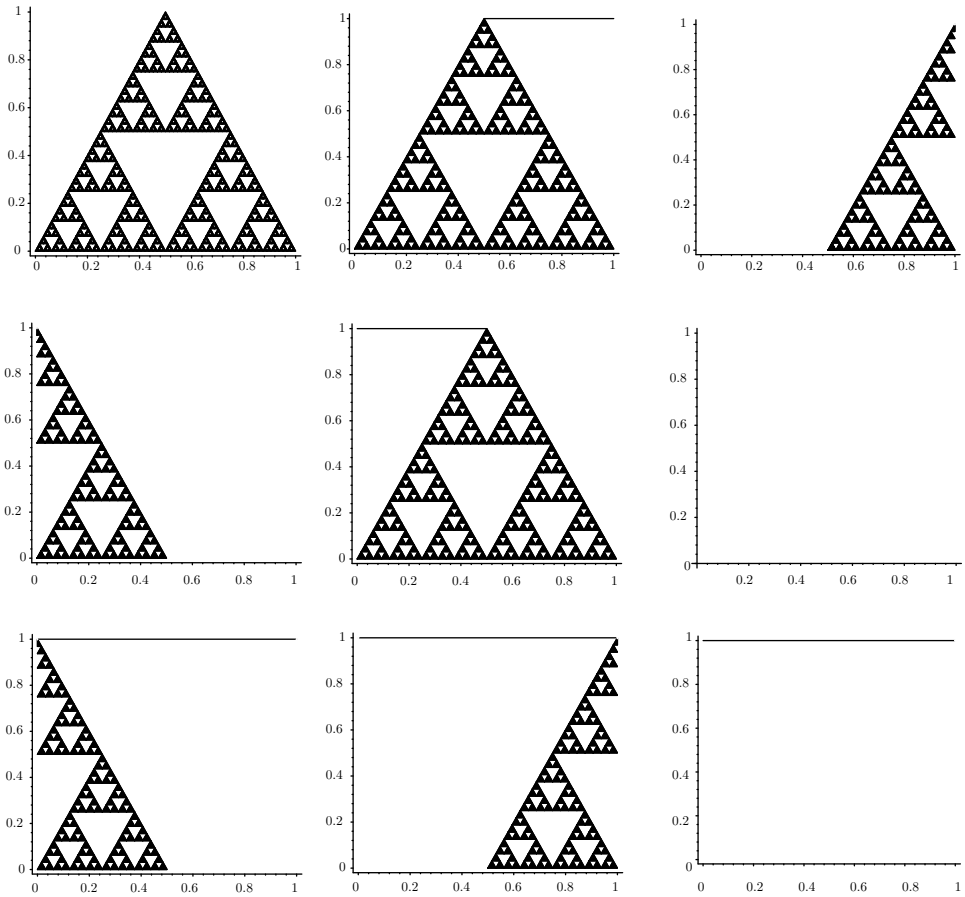

FIGURE 1.2. 2-kernel of the Sierpiński triangle

2. Polyhedral theory

Definition 2.1. We say a set $P \subset \mathbb{R}^d$ is a convex polyhedron if there exists $n \geq 1$, $A \in \mathbb{R}^{n\times d}$ and $\overline{\mathbf{b}} \in \mathbb{R}^n$ such that $P = \{\overline{\mathbf{x}} \in \mathbb{R}^d | \, A\overline{\mathbf{x}} \leq \overline{\mathbf{b}}\}$. We say that P is a polyhedron if it is a connected union of a finite number of convex polyhedra.

If $n = 1$, then we call P a *half-space*. Clearly any intersection of a finite number of convex polyhedra is again a convex polyhedron. For the purposes of this paper, we will assume that polyhedra are always bounded.

In the above definition, if we can find $A \in \mathbb{Q}^{n\times d}$ and $\overline{\mathbf{b}} \in \mathbb{Q}^n$, then we say that P is a *rational polyhedron*. We similarly define a *rational half-space*.

Definition 2.2. Let $S = \{\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2, \ldots, \overline{\mathbf{x}}_n\}$ be a finite collection of points in $\mathbb{R}^d$. We define the convex hull of S as

$$\operatorname{conv}(S) = \left\{ \sum \lambda_i \overline{\mathbf{x}}_i \middle| \sum \lambda_i = 1, \ \lambda_i \geq 0 \right\}.$$

The convex hull can be defined for S where S is not a finite set of points. For our purposes the above definition suffices.

There is a nice connection between bounded convex polyhedra and convex hulls. See for example [2, Section 1.3].

Lemma 2.1. *A bounded convex polyhedron is the convex hull of a finite number of points. A bounded convex rational polyhedron is the convex hull of a finite number of points in $\mathbb{Q}^d$.*

3. Proof of Theorem 1.1

We first prove the easier direction of the main result.

Theorem 3.1. *Suppose $X \subset [0,1]^d$ is a union of a finite number of rational polyhedra. Then X is k-self-similar for any integer $k \geq 2$.*

Proof. The union of a finite number of rational polyhedra can be described using unions and intersections of a finite number of rational half-spaces. Further, the k-kernel of a rational half-space is finite, as rational numbers are eventually periodic base k. Hence, the k-kernel of a union of a finite number of rational polyhedra is made up of unions and intersections of a finite number of rational half-spaces and hence is finite. Hence X is k-self-similar. □

As before, let $\mathcal{C}_k$ be the collection of k^d hyper-cubes of size $\frac{1}{k} \times \cdots \times \frac{1}{k}$. If $\overline{\mathbf{x}} \in [0,1]^d$, then we denote $T_{\overline{\mathbf{x}},k} := T_{c(\overline{\mathbf{x}},k)}$, where $\overline{\mathbf{x}} \in c(\overline{\mathbf{x}},k)$ and $c(\overline{\mathbf{x}},k) \in \mathcal{C}_k$. In the case where $\overline{\mathbf{x}}$ is in more than one hyper-cube, we take the hyper-cube furthest from $(0,0,\ldots,0)$. Later on, in Lemma 3.5, we will modify this to choose the hyper-cube with a particular property, but the details are not relevant at this time.

More formally, if $\overline{\mathbf{x}} = (x_1,\ldots,x_d)$, then we choose $i_1, i_2, \ldots, i_d$ such that $\frac{i_j}{k} \leq x_j < \frac{i_j+1}{k}$. If $x_j = 1$, then we take $i_j = k-1$. We then choose

$$c(\overline{\mathbf{x}},k) := \left[\frac{i_1}{k}, \frac{i_1+1}{k}\right] \times \left[\frac{i_2}{k}, \frac{i_2+1}{k}\right] \times \cdots \times \left[\frac{i_d}{k}, \frac{i_d+1}{k}\right].$$

Lemma 3.1. *Let $k \geq 2$ be an integer, $X \subset [0,1]^d$ be k-self-similar and $\overline{\mathbf{x}} \in \mathbb{Q}^d \cap [0,1]^d$. Then the sequence*

$$X,\ T_{\overline{\mathbf{x}},k}(X),\ T_{\overline{\mathbf{x}},k^2}(X),\ T_{\overline{\mathbf{x}},k^3}(X),\ T_{\overline{\mathbf{x}},k^4}(X),\ \ldots$$

is eventually periodic.

Proof. Let $\overline{\mathbf{x}}_n := T_{\overline{\mathbf{x}},k^n}(\overline{\mathbf{x}}) = T_{\overline{\mathbf{x}}_{n-1},k}(\overline{\mathbf{x}}_{n-1})$. We see that $\overline{\mathbf{x}}_n$ is eventually periodic, as $\overline{\mathbf{x}}$ has rational coefficients. Hence we can find an n and m such that

$$\overline{\mathbf{x}}_n = \overline{\mathbf{x}}_{n+m} = \overline{\mathbf{x}}_{n+2m} = \overline{\mathbf{x}}_{n+3m} = \cdots.$$

Consider the subsequence

$$T_{\overline{\mathbf{x}},k^n}(X),\ T_{\overline{\mathbf{x}},k^{n+m}}(X),\ T_{\overline{\mathbf{x}},k^{n+2m}}(X),\ T_{\overline{\mathbf{x}},k^{n+3m}}(X),\ \ldots.$$

As X has a finite k-kernel, this sequence contains only finitely many distinct sets, and hence there exist k_1 and k_2, with $k_1 < k_2$, such that

$$T_{\overline{\mathbf{x}},k^{n+k_1 m}}(X) = T_{\overline{\mathbf{x}},k^{n+k_2 m}}(X)$$

and $\overline{\mathbf{x}}_{n+k_1 m} = \overline{\mathbf{x}}_{n+k_2 m}$. This then gives that the sequence is eventually periodic, with period $(k_2 - k_1)m$ and pre-period $n + k_1 m$. □

Next we show that both translation by a vector in $\mathbb{Q}^d$ and multiplication by a positive integer preserve the self-similarity of a set.

Lemma 3.2. *Let $X \subset [0,\,1]^d$ be k-self-similar for some $k \geq 2$. Then for any $\overline{\mathbf{q}} \in \mathbb{Q}^d$ the set $(X + \overline{\mathbf{q}}) \cap [0,\,1]^d$ is k-self-similar.*

Proof. The image of $\overline{\mathbf{q}}$ under the maps T_c with $c \in \mathcal{C}_k$ has a finite number of possibilities. This is because $\overline{\mathbf{q}}$ is rational, hence has eventually periodic base k expansion. For each of these images of $\overline{\mathbf{q}}$ we get at most S^{2^d} possible combinations for the new k-kernel (where d is the dimension and S is the size of the original k-kernel). This is finite, as required. □

Lemma 3.3. *Let $X \subset [0,\,1]^d$ be k-self-similar for some $k \geq 2$. Then for any integer $n \geq 1$, the set $nX \cap [0,\,1]^d$ is k-self-similar.*

Proof. Each X_i' in the k-kernel of $nX \cap [0,\,1]^d$ corresponds to some $T_c(X_j)$, where X_j is in the k-kernel of X and $c \in \mathcal{C}_n$. As there are a finite number of X_j and a finite number of c, the result follows. □

Lemma 3.4. *Let k and ℓ be multiplicatively independent positive integers. Let $X \subset [0,1]^d$ be compact, and let $\overline{\mathbf{q}} \in X$ be such that $X = T_{\overline{\mathbf{q}},k}(X) = T_{\overline{\mathbf{q}},\ell}(X)$ and $\overline{\mathbf{q}} = T_{\overline{\mathbf{q}},k}(\overline{\mathbf{q}}) = T_{\overline{\mathbf{q}},\ell}(\overline{\mathbf{q}})$. Then for any $\overline{\mathbf{x}} \in \partial([0,1]^d)$, the boundary of $[0,1]^d$, either*

$$\{\overline{\mathbf{q}} + r(\overline{\mathbf{x}} - \overline{\mathbf{q}}) \ \mid \ 0 \leq r \leq 1\} \subset X$$

or

$$\{\overline{\mathbf{q}} + r(\overline{\mathbf{x}} - \overline{\mathbf{q}}) \ \mid \ 0 < r \leq 1\} \subset [0,1]^d \setminus X.$$

Proof. Consider a point $\overline{\mathbf{x}} \in X$. As $X = T_{\overline{\mathbf{q}},k}(X)$ we see that $T_{\overline{\mathbf{q}},k}^{-1}(\overline{\mathbf{x}}) \in X$. Further, if $T_{\overline{\mathbf{q}},k}(\overline{\mathbf{x}})$ is well defined, then $T_{\overline{\mathbf{q}},k}(\overline{\mathbf{x}}) \in X$. A similar statement is true for ℓ. This gives us that

$$\{\overline{\mathbf{q}} + k^m \ell^n (\overline{\mathbf{x}} - \overline{\mathbf{q}}) \ \mid \ m, n \in \mathbb{Z}\} \cap [0,1]^d \subset X.$$

We notice that $\{k^m \ell^n \ \mid \ m, n \in \mathbb{Z}\}$ is dense in $[0, \infty]$. As X is compact, X is closed, and hence for all $\overline{\mathbf{x}} \in \partial([0,1]^d)$ we have

$$\{\overline{\mathbf{q}} + r(\overline{\mathbf{x}} - \overline{\mathbf{q}}) \ \mid \ 0 \leq r \leq 1\} \subset X$$

or

$$\{\overline{\mathbf{q}} + r(\overline{\mathbf{x}} - \overline{\mathbf{q}}) \ \mid \ 0 < r \leq 1\} \subset [0,1]^d \setminus X,$$

as required. □

Corollary 3.1. *Let $d = 1$. Let X, k, ℓ and $\overline{\mathbf{q}} = (q)$ satisfy the conditions in the statement of Lemma 3.4. Then one of the following is true:*

$$X = [0,1], \quad X = [0,q], \quad X = [q,1], \quad X = \emptyset, \quad \text{or} \quad X = \{q\}.$$

Hence X is a (possibly empty) union of a finite number of rational polyhedra.

Lemma 3.5. *Let $X \subset [0,1]^d$ be both k- and ℓ-self-similar. For any $\overline{\mathbf{x}} \in X$ there exists a neighbourhood N of $\overline{\mathbf{x}}$ such that $\overline{N} \cap X$ is a union of a finite number of rational polyhedra.*

Proof. We proceed by induction on the dimension d.

We wish to show that, locally around $\overline{\mathbf{x}}$, X is a union of a finite number of rational polyhedra. We may assume without loss of generality that $\overline{\mathbf{x}}$ does not have k-ary or ℓ-ary rational coordinates. That is, the coordinates are not of the form $\frac{a}{k^n}$ or $\frac{b}{\ell^m}$. To see this, we note by Lemma 3.2 that a rational translate of X is k- and ℓ-self-similar. Hence we may translate $\overline{\mathbf{x}}$ and X by $\overline{\mathbf{q}}$ such that $\overline{\mathbf{x}} + \overline{\mathbf{q}}$ does not have

k- or ℓ-ary rational coordinates. If we show that, locally around $\overline{\mathbf{x}}+\overline{\mathbf{q}}$, $X+\overline{\mathbf{q}}$ is a union of a finite number of rational polyhedra, then we have shown that, locally around $\overline{\mathbf{x}}$, X is a finite union of rational polyhedra.

Consider the sequence

$$X,\ T_{\overline{\mathbf{x}},k}(X),\ T_{\overline{\mathbf{x}},k^2}(X),\ T_{\overline{\mathbf{x}},k^3}(X),\ T_{\overline{\mathbf{x}},k^4}(X),\ \ldots.$$

As before, we use the notation $\overline{\mathbf{x}}_m = T_{\overline{\mathbf{x}},k^m}(\overline{\mathbf{x}})$. As X is k-self-similar, there exists an infinite subset $N \subset \mathbb{N}$ such that $T_{\overline{\mathbf{x}},k^m}(X) = T_{\overline{\mathbf{x}},k^n}(X)$ for all $m,n \in N$. As $\overline{\mathbf{x}}$ does not have k-ary rational coordinates we can choose $m,n \in N$, with $m<n$ such that the image of $[0,1]^d$ under the map $T^{-1}_{\overline{\mathbf{x}}_m,k^{n-m}}$ is strictly in the interior of $[0,1]^d$.

Denote $Y := T_{\overline{\mathbf{x}},k^m}(X)$, $K := k^{n-m}$ and $\overline{\mathbf{y}} := T_{\overline{\mathbf{x}},k^m}(\overline{\mathbf{x}})$. By the preceding paragraph $Y = T_{\overline{\mathbf{y}},K}(Y) \subset [0,1]^d$. By Lemmas 3.2 and 3.3, Y is both k- and ℓ-self-similar. The map $T^{-1}_{\overline{\mathbf{y}},K}$ is a rational contraction map, hence has a rational fixed point $\overline{\mathbf{q}}$. By our choice of m and n, this contraction map will map into the interior of $[0,1]^d$; hence $\overline{\mathbf{q}}$ is in the interior of $[0,1]^d$. Hence $T_{\overline{\mathbf{q}},K}(\overline{\mathbf{q}}) = \overline{\mathbf{q}}$ and $T_{\overline{\mathbf{q}},K}(Y) = Y$. This gives us that the sequence

$$Y, T_{\overline{\mathbf{q}},K}(Y),\ T_{\overline{\mathbf{q}},K^2}(Y),\ T_{\overline{\mathbf{q}},K^3}(Y),\ \ldots$$

is trivially purely periodic, with period 1. As $\overline{\mathbf{q}}$ is in the interior, the coordinates of $\overline{\mathbf{q}}$ are of the form $\frac{a_i}{K-1}$ for some $1 \le a_i \le K-2$.

Consider the sequence

$$Y, T_{\overline{\mathbf{q}},\ell}(Y),\ T_{\overline{\mathbf{q}},\ell^2}(Y),\ T_{\overline{\mathbf{q}},\ell^3}(Y),\ T_{\overline{\mathbf{q}},\ell^4}(Y),\ \ldots.$$

As $\overline{\mathbf{q}}$ is rational and Y is ℓ-self-similar, this sequence is eventually periodic, by Lemma 3.1. By the proof of Lemma 3.1 we can choose m' and n' with $m'<n'$ such that $T_{\overline{\mathbf{q}},\ell^{m'}}(Y) = T_{\overline{\mathbf{q}},\ell^{n'}}(Y)$ and $T_{\overline{\mathbf{q}},l^{m'}}(\overline{\mathbf{q}}) = T_{\overline{\mathbf{q}},l^{n'}}(\overline{\mathbf{q}})$. Let $Z := T_{\overline{\mathbf{q}},\ell^{m'}}(Y)$, $L := \ell^{n'-m'}$ and $\overline{\mathbf{q}}' := T_{\overline{\mathbf{q}},\ell^{m'}}(\overline{\mathbf{q}})$. For now, assume that $\overline{\mathbf{q}}'$ is in the interior of $[0,1]^d$. At the end of this proof we will discuss the case where $\overline{\mathbf{q}}'$ is on the boundary. By construction, Z, m' and n' were chosen so that $\overline{\mathbf{q}}'$ is the fixed point of $T_{\overline{\mathbf{q}}',L}$ and $T_{\overline{\mathbf{q}}',L}(Z) = Z$.

The map $T_{\overline{\mathbf{q}},\ell^{m'}}$ will take coordinates of the form $\frac{a_i}{K-1}$ to $\frac{a_i'}{K-1}$ for some $0 \le a_i' \le K-1$. Therefore $\overline{\mathbf{q}}' = T_{\overline{\mathbf{q}},\ell^{m'}}(\overline{\mathbf{q}})$ is the fixed point of $T_{\overline{\mathbf{q}}',K}$. Since $T_{\overline{\mathbf{q}},K}(Y) = Y$, we have $T_{\overline{\mathbf{q}}',K}(Z) = Z$.

By Lemmas 3.2 and 3.3, Z is both k- and ℓ-self-similar. By construction, both K and L are multiplicatively independent.

If $d=1$, then by Corollary 3.1 Z is the union of a finite number of rational polyhedra.

If $d>1$, then

$$Z \cap ([0,1] \times \cdots \times \{0\} \times \cdots \times [0,1])$$

and

$$Z \cap ([0,1] \times \cdots \times \{1\} \times \cdots \times [0,1])$$

are k-self-similar, and ℓ-self-similar sets of dimension $d-1$, and hence by the induction hypothesis these $d-1$-dimensional sets are a union of a finite number of rational polyhedra. Hence by Lemmas 2.1 and 3.4, Z is a union of a finite number of rational polyhedra.

Let $Y' := T^{-1}_{\overline{\mathbf{q}},\ell^{m'}}(Z) \subset Y$. We see that Y' is a union of a finite number of rational polyhedra. Further, as q' is in the interior of $[0,1]^d$, we can find an n such that $T_{\overline{\mathbf{q}},K^n}(Y') = T_{\overline{\mathbf{q}},K^n}(Y) = Y$. This implies that Y is a union of a finite number of rational polyhedra.

Let $X' := T^{-1}_{\overline{\mathbf{q}},k^m}(Y) \subset X$. We see that X' is a union of a finite number of rational polyhedra. By our choice of m, we have that $\overline{\mathbf{x}}$ is in the interior of $T^{-1}_{\overline{\mathbf{q}},k^m}([0,1]^d)$.

Taking $N := T^{-1}_{\overline{\mathbf{q}},k^m}\left((0,1) \times \cdots \times (0,1)\right)$ gives the desired result.

If $\overline{\mathbf{q}}'$ is not in the interior, but instead is on the boundary, then we modify our choice of hyper-cube around which we expand $T_{\overline{\mathbf{q}},\ell^{m'}}(Y)$. In particular, when we originally defined $T_{\overline{\mathbf{q}},\ell^{m'}}(Y)$, we always chose the hyper-cube furthest from $(0,\ldots,0)$. Instead, we now choose the hyper-cube where the image of $\overline{\mathbf{x}}$ under the contraction map $T^{-1}_{\overline{\mathbf{y}},K}$ is found. As $\overline{\mathbf{x}}$ does not have k- or ℓ-ary rational coordinates, this choice is always unambiguous. This is done so that when we pull back to our neighbourhood N for which X is locally a union of finite number of rational polyhedra, N will contain $\overline{\mathbf{x}}$. □

Proof of Theorem 1.1. For each $\overline{\mathbf{x}}$ in X, find an open neighbourhood $N_{\overline{\mathbf{x}}}$, such that $\overline{N_{\overline{\mathbf{x}}}} \cap X$ is a union of a finite number of rational polyhedra. As X is compact, there exists a finite set of $\overline{\mathbf{x}}_i$ such that

$$X \subset N_{\overline{\mathbf{x}}_1} \cup N_{\overline{\mathbf{x}}_2} \cup \cdots \cup N_{\overline{\mathbf{x}}_k}.$$

Thus

$$X = \left(\overline{N_{\overline{\mathbf{x}}_1}} \cap X\right) \cup \left(\overline{N_{\overline{\mathbf{x}}_2}} \cap X\right) \cup \cdots \cup \left(\overline{N_{\overline{\mathbf{x}}_k}} \cap X\right);$$

hence X is a union of a finite number of rational polyhedra. □

4. Further comments

There are many obvious questions that are raised by this research. One question is: which generalizations of, and results about, k-automatic sequences can be translated into this new language of k-automatic fractals? This was started in [1], but there are many directions still unexplored.

In another direction, many k-self-similar fractals are special cases of fractals arising from iterated function systems [5, Chapter 9]. What are the minimal conditions needed for the resulting compact set to be "nice"? For example, consider the Cantor middle fifth set satisfying $X = f_1(X) \cup f_2(X)$ where $f_1(x) = \frac{2}{5}x$ and $f_2(x) = \frac{2}{5}x + \frac{3}{5}$. This is a very simple fractal, but it is not covered by the theory in this paper. In this case we could think of X as being $\mathcal{C}$-self-similar for the partition $\mathcal{C} = \{[0,2/5],[2/5,3/5],[3/5,1]\}$. What conditions would we need on two partitions $\mathcal{C}$ and $\mathcal{C}'$ to ensure that X being $\mathcal{C}$- and $\mathcal{C}'$-self-similar would imply that X is "nice"? What is the correct generalization of "nice"? For the example above, with $\mathcal{C} = \{[0,2/5],[2/5,3/5],[3/5,1]\}$, one can show that if a point x is eventually periodic under these expansion maps, then x is rational. The converse is not true, and there exist rationals that are not eventually periodic under these expansion maps.

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