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AN OMEGA-RESULT FOR SAITO-KUROKAWA LIFTS

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Abstract. We prove an Omega-result for the Hecke eigenvalues of Hecke operators acting on the space of Siegel modular forms of degree 2 which are Saito–Kurokawa lifts.

1. Introduction

A central problem in the theory of modular forms is the estimation of the Hecke eigenvalues $\lambda(n)$ of the Hecke operators $T(n)$ defined on the space of appropriate modular forms. Especially in the case of cuspidal Siegel modular forms of weight k and degree g, which we denote by S_k^g, the "generalised Ramanujan–Petersson conjecture", meaning that all the Satake p-parameters of F have absolute value 1, implies (see [5]) that

$$\lambda_F(p) \ll_{g,\varepsilon} p^{gk/2-g(g+1)/4+\varepsilon} \quad \text{for all } \epsilon > 0. \tag{1.1}$$

It is well known by the result of Deligne that when $g = 1$ we have

$$|\lambda_f(n)| \leq \tau(n) n^{(k-1)/2},$$

where $\tau(n)$ denotes the number of divisors of n. For two positive arithmetical functions $a(n)$ and $b(n)$, we say that $a(n) = \Omega(b(n))$ when $a(n) = o(b(n))$ does not hold; i.e., there is a subsequence (n_k) and a positive constant c such that $a(n_k) > c \cdot b(n_k)$ for all $k \geq 1$.

We note here that an Omega-result for Ellptic cusp forms was obtained by Rankin [6]. With the Sato-Tate conjecture being settled recently [3], one can now prove the desired conjecture about the Omega-result for eigenvalues for the elliptic cusp forms (see [4]):

$$|\lambda_f(n)| = \Omega\left(n^{(k-1)/2} \exp\{c \log n / \log\log n\}\right) \quad \text{for some } c > 0.$$

However when $g = 2$, (1.1) is known to be false (and also in some cases of higher genus), and the elements of the Maaß space in S_k^2 precisely fail to satisfy it. Nevertheless Weissauer [7] has proved the above conjecture for $g = 2$ for F not a Saito–Kurokawa lift, i.e., not belonging to the Maaß space of degree 2.

In the present paper, we concentrate on the space of Maaß cusp forms denoted by S_k^* (i.e., Saito–Kurokawa lifts of elliptic modular forms) in S_k^2. The aim of this paper is to establish an Omega-result for S_k^*. By a result of Breulman [1], we know

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that $F \in S_k^*$ if and only if for all n, $\lambda_F(n) > 0$. Also, it is known that for $F \in S_k^*$,

$$\lambda_F(n) \ll_\epsilon n^{k-1+\varepsilon} \quad \text{for all } \varepsilon > 0.$$

It is then natural to ask whether the above estimate is optimal, and more precisely we prove the following.

Theorem 1.1. *Let $F \in S_k^*$ be a Hecke eigenform. Then*

$$\lambda_F(n) = \Omega\left(n^{k-1} \log^{1/2} n / \log\log n\right).$$

It might be possible to prove the above theorem using the corresponding Omega-result for the elliptic modular forms. Here, we prove the theorem directly using the the fact that for a nonzero Hecke eigenform f in $S_k := S_k^1$, the primes p such that $a_f(p) > c \cdot p^{(k-1)/2}$ for some $c > 0$ are of positive proportion among all primes. See Lemma 3.1. Of course, by the Sato-Tate conjecture (now a theorem; see [3]) the above density can be easily computed. The point of Lemma 3.1 is to prove such a statement without using Sato-Tate.

2. Notation and preliminaries

We use the standard notation $e(z) := e^{2\pi i z}$ along with $q := e(\tau)$, where $\tau \in \mathbb{H}$, the complex upper half-plane. For a subset A of the set of all primes $\mathcal{P}$, we denote the lower natural density of A by $\underline{\delta}(A)$, i.e.,

$$\underline{\delta}(A) = \varliminf_{x \mapsto \infty} \frac{\#\{p \le x \mid p \in A\}}{\#\{p \le x\}}.$$

The letter p will always denote a prime in this paper and $\pi(x)$ the number of primes up to x. For two functions $\alpha(x), \beta(x)$, $\beta(x) \neq 0$ for all x, we write $\alpha \sim \beta$ if $\lim_{x \mapsto \infty} \alpha(x)/\beta(x) = 1$.

3. A lemma

We first prove a lemma concerning the density of primes at which the Fourier coefficients are not too small. This will be used in the proof of Theorem 1.1. The proof we give below is elementary in the sense that we do not use Sato-Tate; however, we use Deligne's estimate.

Lemma 3.1. *Let $f = \sum_{n \ge 1} a_f(n) q^n \in S_k$ be a normalised Hecke eigenform. Then there exists an absolute constant $2 > c > 0$ such that*

$$\underline{\delta}(\{p \in \mathcal{P} \mid a_f(p) > c \cdot p^{(k-1)/2}\}) > 0.$$

Proof. Let $\lambda_f(p) := a_f(p)/p^{(k-1)/2}$. Then by Deligne, $|\lambda_f(p)| \le 2$. We consider the following weighted sums:

$$S(x) := \sum_{p \le x} (\lambda_f(p) - c)(\lambda_f(p) + 2) \log p; \tag{3.1}$$

$$S^{\mp}(x) := \sum_{p \le x, \lambda_f(p) \lessgtr c} (\lambda_f(p) - c)(\lambda_f(p) + 2) \log p,$$

where $c > 0$ is an appropriate constant to be chosen later.

From the fact that $(\lambda_f(p) - c)(\lambda_f(p) + 2) \lessgtr 0$ according as $\lambda_f(p) \lessgtr c$, we have

$$S(x) \le S^+(x) \le 16 \, \#\{p \le x \mid \lambda_f(p) > c\} \log x. \tag{3.2}$$

Therefore we need to find a suitable lower bound for $S(x)$ as x goes to infinity. To this end, we have by [2, p. 135] and [6] respectively that for large x

$$\sum_{p\le x}\lambda_f(p)\log p \ll x\exp(-\kappa\sqrt{\log x}), \quad \sum_{p\le x}\lambda_f^2(p)\log p \sim x,$$

where $\kappa > 0$ is an absolute constant. On the other hand, it is a classical result (see [2, p. 43]) that

$$\sum_{p\le x}\log p \le \kappa_1 x, \quad \text{for all } x \ge 1, \tag{3.3}$$

where $\kappa_1 > 0$ is an absolute constant.

Hence, from (3.1) we get for x large,

$$S(x) \ge x/2 - 2c\kappa_1 x + O\left(x\exp(-\kappa\sqrt{\log x})\right) \gg x,$$

where we choose $c > 0$ small enough so that $1/8 > 2c\kappa_1$.

Therefore, combining the above estimate with (3.2), we get the proof of the lemma. □

Remark 3.2. (i) The above lemma certainly holds for all $0 < c < 2$ satisfying $1/2 > 2c\kappa_1$.

(ii) By the Sato-Tate conjecture, the above result of course holds for any $0 < c < 2$ and we also know the density. We recall the Sato-Tate conjecture for f. First, we write by Deligne,

$$\lambda_f(p) = 2\cos\theta_p; \quad \theta_p \in [0,\pi].$$

If $I \subset [0,\pi]$ is an interval, then

$$\frac{\#\{p \le x \mid \theta_p \in I\}}{\#\{p \le x\}} \sim \int_I \frac{2}{\pi}\sin^2\theta \; d\theta.$$

For example, when $c = 1$, we have $\{p \le x \mid \lambda_f(p) > 1\} = \{p \le x \mid \theta_p \in (0,\pi/3)\}$, and so the density is $1/3 - \sqrt{3}/(2\pi)$.

4. Proof of Theorem 1.1

Let $F \in S_k^*$ be a nonzero Hecke eigenform. Then F is the Saito–Kurokawa lift of a normalised Hecke eigenform f of weight $2k-2$ for $SL(2,\mathbf{Z})$. We know that (see [1]) if $\lambda_F(p)$ is the eigenvalue of the Hecke operator $T(p)$, then

$$\lambda_F(p) = p^{k-1}(1 + 1/p + a_f(p)/p^{k-1}). \tag{4.1}$$

Let us denote the set $\{p \mid a_f(p) > c\cdot p^{k-3/2}\}$ by A (c is as in Lemma 3.1 applied to the situation of weight $2k-2$) and define the integers $P_m := \prod_{p\le m, p\in A} p$.

Then from (4.1) we get

$$\lambda_F(P_m)/P_m^{k-1} > \prod_{p\le m,p\in A}(1 + cp^{-1/2}) > c\sum_{p\le m,p\in A} 1/p^{1/2}. \tag{4.2}$$

We need to have a lower bound for the partial sums (4.2). This is well known for the set of all primes, and we claim that in our case as well

$$\sum_{p\le x,p\in A}\frac{1}{p^{1/2}} \gg \frac{x^{1/2}}{\log x}, \quad \text{as } x \mapsto \infty.$$

This is easily seen by partial summation. We define $b(n) = 1$ if n is a prime in A and 0 otherwise and set $g(t) = 1/t^{1/2}$ for $t > 0$. Further, we define $B(x) := \sum_{n \leq x} b(n)$. Then

$$\sum_{p \leq x, p \in A} \frac{1}{p^{1/2}} = \sum_{n \leq x} b(n)/n^{1/2} = B(x)/x^{1/2} + \frac{1}{2} \int_2^x \frac{B(t)}{t^{3/2}} dt. \tag{4.3}$$

Since A has a positive lower density, we have $\pi(t) \geq B(t) \gg t/\log t$ t large. Hence for large values of x, both the terms in (4.3) are $\gg x^{1/2}/\log x$.

Now the prime number theorem (more precisely, see (3.3)) and (4.2) imply $m \gg \log P_m$. Also from (4.2), $\log P_m \geq \log 2 \, \#\{p \leq m \mid p \in A\} \gg m/\log m$ for large m, since A has positive lower density. Hence $\log\log P_m \gg \log m - \log\log m \gg \log m$, for large m. We finally see that

$$\lambda_F(P_m)/P_m^{k-1} \gg \log^{1/2} P_m / \log\log P_m.$$

This completes the proof of Theorem 1.1.

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