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GENERALIZATIONS OF A LAPLACIAN-TYPE EQUATION IN THE HEISENBERG GROUP AND A CLASS OF GRUSHIN-TYPE SPACES

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Abstract. In their 1996 paper, Beals, Gaveau and Greiner found the fundamental solution to a 2-Laplace-type equation in a class of sub-Riemannian spaces. This solution is related to the well-known fundamental solution to the $\mathtt{p}$-Laplace equation in Grushin-type spaces and the Heisenberg group. We extend the 2-Laplace-type equation to a $\mathtt{p}$-Laplace-type equation. We show that the obvious generalization does not have desired properties, but rather, our generalization preserves some natural properties.

1. Introduction and motivation

In [2], fundamental solutions to a generalization of the 2-Laplace equation were found in a wide class of sub-Riemannian spaces. This class includes some of the spaces in [5, 6, 11]. The methodology of [2] mixes the geometric properties of the space with the linearity of the 2-Laplace operator. In this article, we study the generalization of [2] and look to extend it to an equation based on the $\mathtt{p}$-Laplace equation for $1 < \mathtt{p} < \infty$. Because the $\mathtt{p}$-Laplace equation is nonlinear, we face some technical issues, the first of which is the proper way to generalize the original equation. In Section 3, we discuss the original equations of [2], and in Section 4, we find that a seemingly "natural" generalization is not optimal. In Section 5, we will find a generalization that extends the 2-Laplace equation while maintaining the connection to the fundamental solutions of the $\mathtt{p}$-Laplace equation, and in Section 6, we explore the limiting case as $\mathtt{p} \to \infty$. We focus on two specific classes of sub-Riemannian spaces, namely, Grushin-type planes, which are two-dimensional sub-Riemannian spaces lacking a group law, and the Heisenberg group, a sub-Riemannian space possessing a group law. For the sake of completeness, we highlight the key properties of our environments in Section 2.

2. The environments

We concern ourselves with two sub-Riemannian environments, the Heisenberg group and Grushin-type planes, which are 2-dimensional Grushin-type spaces. We will recall the construction of these spaces and then highlight their main properties.

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2.1. **The Heisenberg group.** We begin with $\mathbb{R}^{2n+1}$ using the coordinates $(x_1, x_2, \ldots, x_{2n}, z)$ and consider the linearly independent vector fields $\{X_i, Z\}$, where the index i ranges from 1 to $2n$, defined by

$$\begin{aligned} X_i &= \begin{cases} \dfrac{\partial}{\partial x_i} - \dfrac{x_{n+i}}{2}\dfrac{\partial}{\partial z} & \text{if } 1 \le i \le n, \\[2ex] \dfrac{\partial}{\partial x_i} + \dfrac{x_{i-n}}{2}\dfrac{\partial}{\partial z} & \text{if } n < i \le 2n, \end{cases} \\ Z &= \frac{\partial}{\partial z}\cdot \end{aligned}$$

For $i \le j$, these vector fields obey the relations

$$[X_i, X_j] = \begin{cases} Z & \text{if } j = i + n, \\ 0 & \text{otherwise,} \end{cases}$$

and for all i, $[X_i, Z] = 0$. We then have a Lie algebra denoted by h_n that decomposes as a direct sum $h_n = V_1 \oplus V_2$ where V_1 is spanned by the X_i's and V_2 is spanned by Z. We endow h_n with an inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and related norm $\| \cdot \|_{\mathbb{H}}$ so that this basis is orthonormal. The corresponding Lie group is called the general Heisenberg group of dimension n and is denoted by $\mathbb{H}^n$. With this choice of vector field the exponential map is the identity map, so that for any p, q in $\mathbb{H}^n$, written as $p = (x_1, x_2, \ldots, x_{2n}, z_1)$ and $q = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{2n}, z_2)$, the group multiplication law is given by

$$p \cdot q = (x_1 + \hat{x}_1, x_2 + \hat{x}_2, \ldots, x_{2n} + \hat{x}_{2n}, z_1 + z_2 + \frac{1}{2}\sum_{i=1}^{n}(x_i\hat{x}_{n+i} - x_{n+i}\hat{x}_i)).$$

The natural metric on $\mathbb{H}^n$ is the Carnot-Carathéodory metric given by

$$d_C(p, q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\|_{\mathbb{H}} dt,$$

where the set Γ is the set of all curves γ such that $\gamma(0) = p, \gamma(1) = q$ and $\gamma'(t) \in V_1$. By Chow's theorem (see, for example, [1]) any two points can be connected by such a curve, which makes $d_C(p, q)$ a left-invariant metric on $\mathbb{H}^n$.

Given a smooth function $u : \mathbb{H}^n \to \mathbb{R}$, we define the horizontal gradient by

$$\nabla_0 u = (X_1 u, X_2 u, \ldots, X_{2n} u).$$

Additionally, given a vector field $F = \sum_{i=1}^{2n} f_i X_i + f_{2n+1} Z$, we define the Heisenberg divergence of F, denoted by $\mathrm{div} F$, by

$$\mathrm{div} F = \sum_{i=1}^{2n} X_i f_i.$$

A quick calculation shows that when $f_{2n+1} = 0$, we have

$$\mathrm{div} F = \mathrm{div}_{\mathrm{eucl}} F,$$

where $\mathrm{div}_{\mathrm{eucl}}$ is the standard Euclidean divergence. The main operator we are concerned with is the horizontal $\mathtt{p}$-Laplacian for $1 < \mathtt{p} < \infty$ defined by

$$\begin{aligned} \Delta_{\mathtt{p}} u &= \mathrm{div}(\|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-2}\nabla_0 u) = \sum_{i=1}^{2n} X_i\big(\|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-2} X_i u\big) \\ &= \sum_{i=1}^{2n} \frac{1}{2}(\mathtt{p}-2)\|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-4} X_i \|\nabla_0 u\|_{\mathbb{H}}^2 X_i u + \sum_{i=1}^{2n} \|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-2} X_i X_i u. \end{aligned} \tag{2.1}$$

For a more complete treatment of the Heisenberg group, the interested reader is directed to [1], [4], [7], [8] [9], [10], [12], [13] and the references therein.

2.2. **Grushin-type planes.** The Grushin-type planes differ from the Heisenberg group in that Grushin-type planes lack an algebraic group law. We begin with $\mathbb{R}^2$, possessing coordinates (y_1, y_2), $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$. We use them to make the vector fields

$$Y_1 = \frac{\partial}{\partial y_1} \text{ and } Y_2 = c(y_1 - a)^n \frac{\partial}{\partial y_2}.$$

For these vector fields, the only (possibly) nonzero Lie bracket is

$$[Y_1, Y_2] = cn(y_1 - a)^{n-1} \frac{\partial}{\partial y_2}.$$

Because $n \in \mathbb{N}$, it follows that Hörmander's condition is satisfied by these vector fields.

We will put a (singular) inner product on $\mathbb{R}^2$, denoted $\langle \cdot, \cdot \rangle_{\mathbb{G}}$, with related norm $\|\cdot\|_{\mathbb{G}}$, so that the collection $\{Y_1, Y_2\}$ forms an orthonormal basis. We then have a sub-Riemannian space that we will call g_n, which is also the tangent space to a generalized Grushin-type plane $\mathbb{G}_n$. Points in $\mathbb{G}_n$ will also be denoted by $p = (y_1, y_2)$. The Carnot-Carathéodory distance on $\mathbb{G}_n$ is defined for points p and q as follows:

$$d_{\mathbb{G}}(p, q) = \inf_{\Gamma} \int \|\gamma'(t)\|_{\mathbb{G}}\, dt,$$

with Γ the set of curves γ such that $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma'(t) \in \mathrm{span}\{Y_1(\gamma(t)), Y_2(\gamma(t))\}$. By Chow's theorem, this is an honest metric.

We shall now discuss calculus on the Grushin-type planes. Given a smooth function f on $\mathbb{G}_n$, we define the horizontal gradient of f as

$$\nabla_0 f(p) = \big(Y_1 f(p), Y_2 f(p)\big).$$

Using these derivatives, we consider a key operator on $C^2_{\mathbb{G}}$ functions, namely the $\mathtt{p}$-Laplacian for $1 < \mathtt{p} < \infty$, given by

$$\begin{aligned} \Delta_{\mathtt{p}} f &= \mathrm{div}_{\mathbb{G}}(\|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-2}\nabla_0 f) = Y_1\big(\|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-2} Y_1 f\big) + Y_2\big(\|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-2} Y_2 f\big) \\ &= \frac{1}{2}(\mathtt{p}-2)\|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-4}\big(Y_1\|\nabla_0 f\|_{\mathbb{G}}^2 Y_1 f + Y_2\|\nabla_0 f\|_{\mathbb{G}}^2 Y_2 f\big) \\ &\quad + \|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-2}\big(Y_1 Y_1 f + Y_2 Y_2 f\big). \end{aligned} \tag{2.2}$$

3. Motivating results

3.1. **Grushin-type planes.** Bieske and Gong [5] proved the following theorem in the Grushin-type planes.

Theorem 3.1 ([5]). *Let $1 < p < \infty$ and define*

$$f(y_1, y_2) = c^2(y_1 - a)^{(2n+2)} + (n+1)^2(y_2 - b)^2.$$

For $p \neq n+2$, consider

$$\tau_p = \frac{n+2-p}{(2n+2)(1-p)}$$

so that in $\mathbb{G}_n \setminus \{(a,b)\}$ we have the well-defined function

$$\psi_p = \begin{cases} f(y_1, y_2)^{\tau_p} & p \neq n+2 \\ \log f(y_1, y_2) & p = n+2. \end{cases}$$

Then, $\Delta_p \psi_p = 0$ in $\mathbb{G}_n \setminus \{(a,b)\}$.

In the Grushin-type planes, Beals, Gaveau and Greiner [2] extend this equation as shown in the following theorem.

Theorem 3.2 ([2]). *Let $L \in \mathbb{R}$. Consider the following quantities:*

$$\alpha = \frac{-n}{(2n+2)}(1+L) \quad \text{and} \quad \beta = \frac{-n}{(2n+2)}(1-L).$$

We use these constants with the functions

$$\begin{aligned} g(y_1, y_2) &= c(y_1 - a)^{n+1} + i(n+1)(y_2 - b), \\ h(y_1, y_2) &= c(y_1 - a)^{n+1} - i(n+1)(y_2 - b) \end{aligned}$$

to define our main function $f(y_1, y_2)$, given by

$$f(y_1, y_2) = g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}.$$

Then, $\Delta_2 f + iL[Y_1, Y_2]f = 0$ in $\mathbb{G}_n \setminus \{(a,b)\}$.

3.2. **The Heisenberg group.** Capogna, Danielli, and Garofalo [6] proved the following theorem.

Theorem 3.3 ([6]). *Let $1 < p < \infty$. In $\mathbb{H}^n \setminus \{0\}$, let*

$$u(x_1, x_2, \ldots, x_{2n}, z) = \left(\left(\sum_{i=1}^{2n} x_i^2 \right)^2 + 16z^2 \right).$$

For $p \neq 2n+2$, let

$$\eta_p = \frac{2n+2-p}{4(1-p)},$$

and let

$$\zeta_p = \begin{cases} u(x_1, x_2, \ldots, x_{2n}, z)^{\eta_p} & p \neq 2n+2 \\ \log u(x_1, x_2, \ldots, x_{2n}, z) & p = 2n+2. \end{cases}$$

Then we have $\Delta_p \zeta_p = 0$ in $\mathbb{H}^n \setminus \{0\}$.

In the Heisenberg group, Beals, Gaveau, and Greiner [2] extend this equation as shown in the following theorem.

Theorem 3.4 ([2]). *Let $L \in \mathbb{R}$. Consider the constants*

$$\eta = \frac{L-1}{2} \quad \text{and} \quad \tau = \frac{-(L+1)}{2}$$

together with the functions

$$v(x_1, x_2, \ldots, x_{2n}, z) = \big(\sum_{j=1}^{2n} x_j^2\big) - 4iz \quad \text{and} \quad w(x_1, x_2, \ldots, x_{2n}, z) = \big(\sum_{j=1}^{2n} x_j^2\big) + 4iz$$

to define our main function, $u(x_1, x_2, \ldots, x_{2n}, z)$, given by

$$u(x_1, x_2, \ldots, x_{2n}, z) = v(x_1, x_2, \ldots, x_{2n}, z)^{\eta} w(x_1, x_2, \ldots, x_{2n}, z)^{\tau}.$$

Then, in $\mathbb{H}^n \setminus \{0\}$, we have $\Delta_2 u + iL \sum_{j=1}^{n} [X_j, X_{j+n}] u = 0$.

Observation 3.5. In $\mathbb{G}_n \setminus \{(a,b)\}$, we have when $\mathtt{p} = 2$ that

$$f_2(y_1, y_2) = \big(c^2 (y_1 - a)^{2n+2} + (n+1)^2 (y_2 - b)^2\big)^{-\frac{n}{2n+2}}$$

solves

$$\Delta_2 f_2 = 0.$$

Also,

$$\widehat{f_L}(y_1, y_2) = g(y_1, y_2)^{-\frac{n}{2n+2}(1+L)} h(y_1, y_2)^{-\frac{n}{2n+2}(1-L)}$$

solves

$$\Delta_2 \widehat{f_L} + iL[Y_1, Y_2] \widehat{f_L} = 0.$$

Notice that the equations and solutions coincide when $L = 0$. That is,

$$\widehat{f_0} = f_2.$$

Similarly, in $\mathbb{H}^n \setminus \{0\}$,

$$u_2(x_1, x_2, \ldots, x_{2n}, z) = \Big(\big(\sum_{i=1}^{2n} x_i^2\big)^2 + 16z^2\Big)^{-\frac{2n}{4}}$$

solves

$$\Delta_2 u_2 = 0.$$

Also,

$$\widehat{u_L}(x_1, x_2, \ldots, x_{2n}, z) = v(x_1, x_2, \ldots, x_{2n}, z)^{\frac{L-1}{2}} w(x_1, x_2, \ldots, x_{2n}, z)^{-\frac{(L+1)}{2}}$$

solves

$$\Delta_2 \widehat{u_L} + iL \sum_{j=1}^{n} [X_j, X_{j+n}] \widehat{u_L} = 0.$$

Again, the equations and solutions coincide when $L = 0$. So again,

$$\widehat{u_0} = u_2.$$

We then ask the following:

Main Question. *Can we extend this relationship in both $\mathbb{G}_n \setminus \{(a,b)\}$ and in $\mathbb{H}^n \setminus \{0\}$ for all $\mathtt{p}$, $1 < \mathtt{p} < \infty$? That is, can we find an operator $\Phi(\mathtt{p}, L)$ so that $\phi_{\mathtt{p},L}$ is a solution to $\Phi(\mathtt{p}, L)\phi_{\mathtt{p},L} = 0$, for all $\mathtt{p}$, $1 < \mathtt{p} < \infty$, and for all $L \in \mathbb{R}$? In addition, we should have*

$$\Phi(\mathtt{p}, 0) = \Delta_{\mathtt{p}} \tag{3.1}$$

and

$$\begin{cases} \Phi(2,L) = \Delta_2 + iL[Y_1, Y_2] & \textit{in the Grushin case,} \\ \Phi(2,L) = \Delta_2 + iL\sum_{j=1}^{n}[X_j, X_{j+n}] & \textit{in the Heisenberg case.} \end{cases} \tag{3.2}$$

Additionally, we would like to have that $\phi_{2,L}$ is the solution from [2] *and $\phi_{\mathtt{p},0}$ is the solution from* [5] *in the Grushin case or* [6] *in the Heisenberg case.*

In order to answer this question, we first look at a good candidate for what the solution should be in each environment.

3.3. **The core Grushin function.** For the Grushin-type planes, we consider the following for $\mathtt{p} \neq n+2$:

$$\alpha = \frac{n+2-\mathtt{p}}{(1-\mathtt{p})(2n+2)}(1+L) \quad \text{and} \quad \beta = \frac{n+2-\mathtt{p}}{(1-\mathtt{p})(2n+2)}(1-L),$$

where $L \in \mathbb{R}$. We use these constants with the functions

$$\begin{aligned} g(y_1,y_2) &= c(y_1-a)^{n+1} + i(n+1)(y_2-b), \\ h(y_1,y_2) &= c(y_1-a)^{n+1} - i(n+1)(y_2-b) \end{aligned}$$

to define our main function $f_{\mathtt{p},L}(y_1,y_2)$, given by

$$f_{\mathtt{p},L}(y_1,y_2) = \begin{cases} g(y_1,y_2)^{\alpha}h(y_1,y_2)^{\beta} & \mathtt{p} \neq n+2 \\ \log\left(g^{1+L}h^{1-L}\right) & \mathtt{p} = n+2. \end{cases}$$

From Theorems 3.1 and 3.2, we have that $f_{\mathtt{p},L}(y_1,y_2)$ solves

$$\Delta_{\mathtt{p}} f_{\mathtt{p},L} + iL[Y_1,Y_2]f_{\mathtt{p},L} = 0$$

in $\mathbb{G}_n \setminus \{(a,b)\}$ when $\mathtt{p}$ is arbitrary and $L=0$ or when $\mathtt{p}=2$ and for all L.

3.4. **The core Heisenberg function.** In the Heisenberg group, for $\mathtt{p} \neq 2n+2$, we consider the following quantities:

$$\eta = \frac{2n+2-\mathtt{p}}{4(1-\mathtt{p})}(1-L) \quad \text{and} \quad \tau = \frac{2n+2-\mathtt{p}}{4(1-\mathtt{p})}(1+L),$$

where $L \in \mathbb{R}$. We use these constants with the functions

$$\begin{aligned} v(x_1,x_2,\ldots,x_{2n},z) &= \Big(\sum_{j=1}^{2n} x_1^2\Big) - 4iz, \\ w(x_1,x_2,\ldots,x_{2n},z) &= \Big(\sum_{j=1}^{2n} x_1^2\Big) + 4iz \end{aligned}$$

to define our main function $u_{\mathtt{p},L}(x_1,x_2,\ldots,x_{2n},z)$, given by

$$u_{\mathtt{p},L}(x_1,x_2,\ldots,x_{2n},z) = \begin{cases} v(x_1,x_2,\ldots,x_{2n},z)^{\eta}w(x_1,x_2,\ldots,x_{2n},z)^{\tau} & \mathtt{p} \neq 2n+2 \\ \log\left(v^{1-L}w^{1+L}\right) & \mathtt{p} = 2n+2. \end{cases}$$

From Theorems 3.3 and 3.4, we have that $u_{\mathtt{p},L}(x_1,x_2,\ldots,x_{2n},z)$ solves

$$\Delta_{\mathtt{p}} u_{\mathtt{p},L} + iL\sum_{j=1}^{2n}[X_j, X_{j+n}]u_{\mathtt{p},L} = 0$$

in $\mathbb{H}^n \setminus \{0\}$ when $\mathtt{p}$ is arbitrary and $L=0$ or when $\mathtt{p}=2$ and for all L.

4. A negative result

A "natural" generalization of the equation $\Delta_2\phi+iL[Z_1,Z_2]\phi$ is $\Delta_{\mathtt{p}}\phi+iL[Z_1,Z_2]\phi$ where $Z_i=Y_i$ in the Grushin-type planes and $Z_i=X_i$ in the Heisenberg group $\mathbb{H}^1$. We now consider this equation in each of our environments. We will suppress the subscripts on the function f and on $\|\cdot\|$ for the upcoming formal computations.

Theorem 4.1. *Let $f_{p,L},\alpha,\beta$ be as in the previous section. Let $p\neq n+2$ and let $L\in\mathbb{R}$ with $L\neq\pm1$. Then in $\mathbb{G}\setminus\{(a,b)\}$*

$$\begin{aligned}&\Delta_p f_{p,L}+iL(p-1)\|\nabla_0 f_{p,L}\|^{p-2}[Y_1,Y_2]f_{p,L}\\&-\|\nabla_0 f_{p,L}\|^{p-2}\frac{L^2}{L^2-1}(-4)\left(\frac{(p-2)(1+np)}{2+n-p}\right)(Y_2g^\alpha)(Y_2h^\beta)=0.\end{aligned}$$

In particular, $\Delta_p f_{p,L}+iL[Y_1,Y_2]f_{p,L}$ need not be zero.

Proof. We compute for $\mathtt{p}\neq n+2$,

$$\begin{aligned}Y_1f&=c(n+1)(y_1-a)^ng^{\alpha-1}h^{\beta-1}(\alpha h+\beta g),\\Y_2f&=ic(n+1)(y_1-a)^ng^{\alpha-1}h^{\beta-1}(\alpha h-\beta g),\\\|\nabla_0 f\|^2&=4c^2(n+1)^2(y_1-a)^{2n}g^{\alpha+\beta-1}h^{\alpha+\beta-1}(\alpha^2+\beta^2),\\Y_1\|\nabla_0 f\|^2&=2^2c^2(n+1)^2(\alpha^2+\beta^2)(y_1-a)^{2n-1}g^{\alpha+\beta-2}h^{\alpha+\beta-2}\\&\quad\times\big(ngh+c^2(n+1)(\alpha+\beta-1)(y_1-a)^{2n+2}\big),\\\text{and}\quad Y_2\|\nabla_0 f\|^2&=2^2c^3(n+1)^4(\alpha^2+\beta^2)(y_1-a)^{3n}(y_2-b)\\&\quad\times(\alpha+\beta-1)g^{\alpha+\beta-2}h^{\alpha+\beta-2}.\end{aligned}$$

We let Λ be defined as

$$\begin{aligned}\Lambda&=\Big((\mathtt{p}-2)\frac{1}{2}\Big(Y_1\|\nabla_0 f\|^2(Y_1f)+Y_2\|\nabla_0 f\|^2(Y_2f)\Big)\\&\quad+\|\nabla_0 f\|^2\big(Y_1Y_1f+Y_2Y_2f\big)\Big)+iL(\mathtt{p}-1)\|\nabla_0 f\|^2[Y_1,Y_2]f\\&\quad-\|\nabla_0 f\|^2\frac{L^2}{L^2-1}(-4)\left(\frac{(\mathtt{p}-2)(1+n\mathtt{p})}{2+n-\mathtt{p}}\right)(X_2g^\alpha)(X_2h^\beta).\end{aligned}$$

We then compute

$$\begin{aligned}\Lambda&=2c^3(n+1)^3(\alpha^2+\beta^2)(y_1-a)^{3n-1}g^{2\alpha+\beta-3}h^{\alpha+2\beta-3}c(y_1-a)^{n+1}\\&\quad\times\left(\left(\frac{n(-1+L^2)(2+n-\mathtt{p})}{(n+1)}\right)+\left(-\frac{n(-1+L^2)(2+n-\mathtt{p})}{(n+1)}\right)\right)=0.\end{aligned}$$

□

4.1. **The Heisenberg group.** Similarly to the Grushin case (Theorem 4.1), in the Heisenberg group $\mathbb{H}^1$, Theorems 3.3 and 3.4 lead us to hypothesize that $u_{\mathtt{p},L}(x_1,x_2,x_3)$ should solve

$$\Delta_{\mathtt{p}}u_{\mathtt{p},L}+iL[X_1,X_2]u_{\mathtt{p},L}=0$$

for $\mathtt{p}$ arbitrary and $L\in\mathbb{R}$

Unfortunately, we discover that this is not the case. Again, we will suppress the subscripts on the function u and on $\|\cdot\|$ throughout our calculations.

Theorem 4.2. *Let $p\neq4$. Then in $\mathbb{H}^1\setminus\{0\}$, $\Delta_p u+iL[X_1,X_2]u$ need not be zero.*

Proof. Similarly to the Grushin case, we compute:

$$\begin{aligned}
X_1 u &= 2v^{\eta-1}w^{\tau-1}\big((\eta w+\tau v)x_1+(\eta w-\tau v)ix_2\big),\\
X_2 u &= 2v^{\eta-1}w^{\tau-1}\big((\eta w+\tau v)x_2-(\eta w-\tau v)ix_1\big),\\
\|\nabla_0 u\|^2 &= 2^3(\eta^2+\tau^2)v^{\eta+\tau-1}w^{\eta+\tau-1}(x_1^2+x_2^2),\\
X_1X_1u &= 2v^{\eta-2}w^{\tau-2}\Big(2\big((\eta w+\tau v)x_1^2+vw\big(2(x_1^2+x_2^2)(\tau+\eta)+vw(\eta w+\tau v)\big)\\
&\quad +(\eta w-\tau v)ix_1x_2\big)\big((\eta-1)w+(\tau-1)v\big)\\
&\quad -2i\big((\eta w+\tau v)x_1x_2+(\eta w-\tau v)ix_2^2\big)\big(-(\eta-1)w+(\tau-1)v\big)\Big),\\
X_2X_2u &= 2v^{\eta-2}w^{\tau-2}\Big(2\big((\eta w+\tau v)x_2^2+vw\big(2(x_1^2+x_2^2)(\eta+\tau)+(\eta w+\tau v)\big)\\
&\quad +(-\eta w+\tau v)ix_1x_2\big)\big((\eta-1)w+(\tau-1)v\big)\\
&\quad +2i\big((\eta w+\tau v)x_1x_2+(-\eta w+\tau v)ix_1^2\big)\big(-(\eta-1)w+(\tau-1)v\big)\Big),\\
iLZu &= -4Lv^{\eta-1}w^{\tau-1}(-\eta w+\tau v).
\end{aligned}$$

We can now compute

$$\begin{aligned}
&(\mathtt{p}-2)\frac{1}{2}\Big(X_1(\|\nabla_0 u\|^2)X_1u+X_2(\|\nabla_0 u\|^2)X_2u\Big)+\|\nabla_0 u\|^2\big(X_1X_1u+X_2X_2u\big)\\
&\quad=-\left(\frac{1}{(-1+\mathtt{p})^4}\right)L(1+L^2)(-4+\mathtt{p})^3(x_1^2+x_2^2)g^{\frac{4+L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}}h^{\frac{4+4L+5\mathtt{p}-L\mathtt{p}}{4-4\mathtt{p}}}\\
&\quad\times\big(L(-4+\mathtt{p})(x_1^2+x_2^2)+4i(-1+\mathtt{p})\mathtt{p}z\big).
\end{aligned}$$

Using this, we calculate

$$\Delta_{\mathtt{p}}u+iLZu=-8Lv^{\frac{1}{2}(-3+L)}\big(L(x_1^2+x_2^2)-4iz\big)w^{\frac{1}{2}(-3-L)}.$$

□

5. A generalization

5.1. **Grushin-type planes.** In the previous section, we see that our first instinct does not work as well as we had wished. We will now put the equation given in Theorem 4.1 into divergence form. For the Grushin-type planes, we have

$$\Delta_2\phi+iL[Y_1,Y_2]\phi=\operatorname{div}_G\begin{pmatrix} Y_1\phi+iLY_2\phi\\ Y_2\phi-iLY_1\phi\end{pmatrix}. \tag{5.1}$$

Inspired by the definition of $\Delta_{\mathtt{p}}$ in equation (2.2) we consider

$$\overline{\Delta_{\mathtt{p}}}\phi=\operatorname{div}_G\left(\left\|\begin{matrix} Y_1\phi+iLY_2\phi\\ Y_2\phi-iLY_1\phi\end{matrix}\right\|^{\mathtt{p}-2}\begin{pmatrix} Y_1\phi+iLY_2\phi\\ Y_2\phi-iLY_1\phi\end{pmatrix}\right). \tag{5.2}$$

Using equation (5.2), we have the following theorem. We will again suppress the subscripts on the function f and on $\|\cdot\|$.

Theorem 5.1. *On $\mathbb{G}_n\setminus\{(a,b)\}$, we have*

$$\overline{\Delta_p}f=\operatorname{div}_G\left(\left\|\begin{matrix} Y_1f+iLY_2f\\ Y_2f-iLY_1f\end{matrix}\right\|^{p-2}\begin{pmatrix} Y_1f+iLY_2f\\ Y_2f-iLY_1f\end{pmatrix}\right)=0.$$

Proof. First, we let

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{pmatrix},$$

and from the definition of $\overline{\Delta_{\mathtt{p}}} f$ we only need to show:

$$\Lambda \equiv \frac{1}{2}(\mathtt{p}-2)\big(Y_1\|\xi\|^2\xi_1 + Y_2\|\xi\|^2\xi_2\big) + \|\xi\|^2\big(Y_1\xi_1 + Y_2\xi_2\big) = 0.$$

We compute for $\mathtt{p} \neq n+2$,

$$\begin{aligned}
Y_1 f &= c(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h + \beta g),\\
Y_2 f &= ic(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h - \beta g),\\
Y_1 f + iLY_2 f &= c(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}\big(\alpha h(1-L) + \beta g(1+L)\big),\\
Y_2 f - iLY_1 f &= ic(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}\big(\alpha h(1-L) - \beta g(1+L)\big),\\
\|\xi\|^2 &= 2c^2(n+1)^2(y_1-a)^{2n}g^{\alpha+\beta-1}h^{\alpha+\beta-1}\big(\alpha^2(1-L)^2 + \beta^2(1+L)^2\big).
\end{aligned}$$

We then calculate:

$$\begin{aligned}
Y_1\xi_1 + Y_2\xi_2 &= \left(\frac{1}{(-1+\mathtt{p})^2 gh}\right)\Bigg(c^2(-1+L^2)(1+n)(2+n-\mathtt{p})(-2+\mathtt{p})\\
&\qquad \times (y_1-a)^{2n} h^{\frac{(-1+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}} g^{-\frac{(1+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}}\Bigg),\\
Y_1\left(\|\xi\|^2\right) &= -\left(\frac{1}{(-1+\mathtt{p})^3 gh}\right)\Bigg(2c^2(-1+L^2)^2\\
&\quad \times (1+n)(2+n-\mathtt{p})^2(y_1-a)^{-1+2n}h^{\frac{1}{1+n}+\frac{1}{1-\mathtt{p}}}\\
&\quad \times \big(c^2(y_1-a)^{2+2n} + n(1+n)(-1+\mathtt{p})(y-b)^2\big)g^{\frac{1}{1+n}+\frac{1}{1-\mathtt{p}}}\Bigg),\\
\text{and}\quad Y_2\left(\|\xi\|^2\right) &= \left(\frac{1}{(-1+\mathtt{p})^3 gh}\right)\Bigg(2c^3(-1+L^2)^2\\
&\quad \times (1+n)(2+n-\mathtt{p})^2(1+n\mathtt{p})(y_1-a)^{3n}h^{\frac{1}{1+n}+\frac{1}{1-\mathtt{p}}}\\
&\qquad \times (b-y_2)g^{\frac{1}{1+n}+\frac{1}{1-\mathtt{p}}}\Bigg).
\end{aligned}$$

Using the above quantities, we compute

$$\begin{aligned}
&Y_1\left(\|\xi\|^2\right)\xi_1 + Y_2\left(\|\xi\|^2\right)\xi_2 = -\left(\frac{1}{(-1+\mathtt{p})^4(gh)^2}\right)\\
&\quad \times\left(2c^4(-1+L^2)^3(1+n)(2+n-\mathtt{p})^3(y_1-a)^{4n}h^{\frac{(-3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}}g^{-\frac{(3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}}\right)\\
&\text{and}\quad \|\xi\|^2\big(Y_1\xi_1 + Y_2\xi_2\big) = \left(\frac{1}{(-1+\mathtt{p})^4(gh)^2}\right)\\
&\quad \times\Bigg(c^4(-1+L^2)^3(1+n)(2+n-\mathtt{p})^3(-2+\mathtt{p})\\
&\qquad \times(y_1-a)^{4n}h^{\frac{(-3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}}g^{-\frac{(3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}}\Bigg).
\end{aligned}$$

We can then calculate

$$\Lambda = \left(\frac{1}{(-1+\mathtt{p})^4(gh)^2}\right) h^{\frac{(-3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}} g^{-\frac{(3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}} c^4(-1+L^2)^3(1+n)(2+n-\mathtt{p})^3$$
$$\times \left(-\frac{1}{2}(\mathtt{p}-2)\Big(2(y_1-a)^{4n}\Big) + \Big((-2+\mathtt{p})(y_1-a)^{4n}\Big)\right) = 0.$$

So we have $\overline{\Delta_{\mathtt{p}}} f = 0$ when $\mathtt{p} \neq n+2$. The case $\mathtt{p} = n+2$ is similar and omitted. □

We then conclude the following corollary.

Corollary 5.2. *Let $\mathtt{p} > n+2$. The function $f_{\mathtt{p},L}$, as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \overline{\Delta_{\mathtt{p}}} f(p) = 0 & p \in \mathbb{G}_n \setminus \{(a,b)\} \\ 0 & p = (a,b). \end{cases}$$

5.2. **The Heisenberg group.** We begin by considering the $2n \times 1$ vector Υ with components

$$\Upsilon_k = \begin{cases} X_k u + iLX_{n+k}u & \text{when } k = 1,2,\dots,n \\ X_k u - iLX_{k-n}u & \text{when } k = n+1, n+2, \dots, 2n. \end{cases}$$

Motivated by the Grushin case, we consider the equation

$$\overline{\Delta_{\mathtt{p}}} u = \operatorname{div}(\|\Upsilon\|^{\mathtt{p}-2}\Upsilon) = 0$$

in $\mathbb{H}^n \setminus \{0\}$. We then have the following theorem. We will again suppress the subscripts on the function u and on $\|\cdot\|$.

Theorem 5.3. *In $\mathbb{H}^n \setminus \{0\}$, we have*

$$\overline{\Delta_{\mathtt{p}}} u = \operatorname{div}(\|\Upsilon\|^{\mathtt{p}-2}\Upsilon) = 0.$$

Proof. From equation (2.1), we have to show only that

$$\frac{1}{2}(\mathtt{p}-2)\sum_{j=1}^{2n} X_j \|\Upsilon\|^2 \Upsilon_j + \|\Upsilon\|^2 \sum_{j=1}^{2n} X_j \Upsilon_j = 0. \tag{5.3}$$

We consider the case $\mathtt{p} \neq 2n+2$. The case when $\mathtt{p} = 2n+2$ is similar and omitted. For $k = 1,2,\dots,n$, we have

$$X_k u = 2v^{\eta-1}w^{\tau-1}\big((\eta w + \tau v)x_k - (-\eta w + \tau v) i x_{n+k}\big),$$

and for $l = n+1, n+2, \dots, 2n$, we have

$$X_l u = 2v^{\eta-1}w^{\tau-1}\big((\eta w + \tau v)x_l + (-\eta w + \tau v) i x_{l-n}\big).$$

We then have for $k = 1,2,\dots,n$,

$$X_k u + iLX_{n+k}u = 2v^{\eta-1}w^{\tau-1}\Big((\eta w + \tau v)(x_k + iLx_{n+k}) + (\eta w - \tau v)(Lx_k + ix_{n+k})\Big),$$

and for $l = n+1, n+2, \dots, 2n$, we have

$$X_l u - iLX_{l-n}u = 2v^{\eta-1}w^{\tau-1}\Big((\eta w + \tau v)(x_l - iLx_{l-n}) + (\eta w - \tau v)(Lx_l - ix_{l-n})\Big).$$

A routine calculation produces

$$\|\Upsilon\|^2 = \frac{(\mathtt{p}-(2n+2))^2}{(\mathtt{p}-1)^2}(L^2-1)^2\Big(\sum_{j=1}^{2n} x_j^2\Big) v^{\frac{2n+\mathtt{p}}{2-2\mathtt{p}}} w^{\frac{2n+\mathtt{p}}{2-2\mathtt{p}}}, \tag{5.4}$$

which yields

$$\sum_{j=1}^{2n} X_j(\|\Upsilon\|^2)\Upsilon_j = (4n+2)\frac{(\mathtt{p}-(2n+2))^3}{(\mathtt{p}-1)^4}(L^2-1)^3\Big(\sum_{j=1}^{2n} x_j^2\Big)^2 v^{\mathcal{M}} w^{\mathfrak{M}}, \tag{5.5}$$

where

$$\mathcal{M} = \frac{L(\mathtt{p}-(2n+2))+(6n-2)+5\mathtt{p}}{4(1-\mathtt{p})}$$

and

$$\mathfrak{M} = \frac{-L(\mathtt{p}-(2n+2))+5\mathtt{p}+(6n-2))}{4(1-\mathtt{p})}.$$

In addition, we have

$$\sum_{j=1}^{2n} X_j\Upsilon_j = -(2n+1)\frac{(\mathtt{p}-2)(p-(2n+2))}{(\mathtt{p}-1)^2}(L^2-1)\Big(\sum_{j=1}^{2n} x_j^2\Big)v^{\mathcal{C}} w^{\mathfrak{C}}, \tag{5.6}$$

where

$$\mathcal{C} = \frac{L(\mathtt{p}-(2n+2))+(2n-2)+3\mathtt{p}}{4(1-\mathtt{p})} \quad \text{and} \quad \mathfrak{C} = \frac{-L(\mathtt{p}-(2n+2))+3\mathtt{p}+(2n-2))}{4(1-\mathtt{p})}.$$

Combining equations (5.4) and (5.6) we obtain

$$\|\Upsilon\|^2\sum_{j=1}^{2n} X_j\Upsilon_j = -(2n+1)(p-2)\frac{(\mathtt{p}-(2n+2))^3}{(\mathtt{p}-1)^4}(L^2-1)^3\Big(\sum_{j=1}^{2n} x_j^2\Big)^2 v^{\mathcal{M}} w^{\mathfrak{M}} \tag{5.7}$$

with $\mathcal{M}$ and $\mathfrak{M}$ defined as above. Equation (5.3) then holds. □

The following corollary naturally follows:

Corollary 5.4. *Let $\mathtt{p} > 2n+2$. The function $u_{\mathtt{p},L}$, as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \overline{\Delta_p}u(p) = 0 & p \in \mathbb{H}^n \setminus \{0\} \\ 0 & p = 0. \end{cases}$$

6. The limit as $\mathtt{p} \to \infty$

6.1. **Grushin-type planes.** We recall that on $\mathbb{G}_n \setminus \{(a,b)\}$, we have

$$\begin{aligned}\overline{\Delta_{\mathtt{p}}}f &= \mathrm{div}_G(\|\xi\|^{\mathtt{p}-2}\xi) \\ &= \|\xi\|^{\mathtt{p}-4}\left(\frac{1}{2}(\mathtt{p}-2)\big(Y_1\|\xi\|^2\xi_1 + Y_2\|\xi\|^2\xi_2\big) + \|\xi\|^2\big(Y_1\xi_1 + Y_2\xi_2\big)\right)\end{aligned}$$

with ξ defined by

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{pmatrix}.$$

Formally letting $\mathtt{p} \to \infty$, we obtain

$$\overline{\Delta_\infty}f = (Y_1\|\xi\|^2)\xi_1 + (Y_2\|\xi\|^2)\xi_2.$$

Formally letting $\mathtt{p} \to \infty$ also produces the function

$$f_{\infty,L}(y_1,y_2) = g(y_1,y_2)^{\frac{1+L}{2n+2}}h(y_1,y_2)^{\frac{1-L}{2n+2}},$$

where we recall that the functions g and h are given by

$$\begin{aligned} g(y_1, y_2) &= c(y_1 - a)^{n+1} + i(n+1)(y_2 - b), \\ h(y_1, y_2) &= c(y_1 - a)^{n+1} - i(n+1)(y_2 - b). \end{aligned}$$

We have the following theorem.

Theorem 6.1. *The function $f_{\infty,L}$, as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \overline{\Delta_\infty} f_{\infty,L}(p) = 0, & p \in \mathbb{G}_n \setminus \{(a,b)\}, \\ 0, & p = (a,b). \end{cases}$$

Proof. We may prove this theorem by letting $\mathtt{p} \to \infty$ in equation (5.3) and invoking continuity (cf. Corollary 5.2). For completeness, though, we compute formally.

We first make the following definitions:

$$A \equiv \frac{1+L}{2n+2} \quad \text{and} \quad B \equiv \frac{1-L}{2n+2}$$

so that we may compute

$$\begin{aligned} Y_1 f &= c(n+1)(y_1 - a)^n g^{A-1} h^{B-1}(Ah + Bg), \\ Y_2 f &= ic(n+1)(y_1 - a)^n g^{A-1} h^{B-1}(Ah - Bg), \\ Y_1 f + iLY_2 f &= c^2(1 - L^2)(y_1 - a)^{2n+1} g^{A-1} h^{B-1}, \\ Y_2 f - iLY_1 f &= c(1 - L^2)(n+1)(y_1 - a)^n (y_2 - b) g^{A-1} h^{B-1}, \\ \text{and} \quad \|\xi\|^2 &= c^2(y_1 - a)^{2n} g^{A+B-1} h^{A+B-1}(1 - L^2)^2. \end{aligned}$$

We then have

$$\begin{aligned} Y_1 \|\xi\|^2 &= 2c^2(1 - L^2)^2 n(n+1)^2 (y_1 - a)^{2n-1} (y_2 - b)^2 (gh)^{\frac{-1-2n}{n+1}} \\ \text{and} \quad Y_2 \|\xi\|^2 &= -2c^3(1 - L^2)^2 n(n+1)(y_1 - a)^{3n} (y_2 - b)(gh)^{\frac{-1-2n}{n+1}} \end{aligned}$$

so that

$$\begin{aligned} Y_1 \|\xi\|^2 \xi_1 &= 2c^4(1 - L^2)^3 n(n+1)^2 (y_1 - a)^{4n} (y_2 - b)^2 (gh)^{\frac{-1-2n}{n+1}} g^{A-1} h^{B-1} \\ \text{and} \quad Y_2 \|\xi\|^2 \xi_2 &= -2c^4(1 - L^2)^3 n(n+1)^2 (y_1 - a)^{4n} (y_2 - b)^2 (gh)^{\frac{-1-2n}{n+1}} g^{A-1} h^{B-1}. \end{aligned}$$

The theorem follows. □

We notice that when $L = 0$, this result is already well-known as Corollary 3.2 in [5]. In particular, combined with [5], we have shown that the following diagram commutes in $\mathbb{G}_n \setminus \{(a,b)\}$:

$$\begin{array}{ccc} \overline{\Delta_\mathtt{p}} f_{\mathtt{p},L} = 0 & \xrightarrow[\mathtt{p}\to\infty]{} & \overline{\Delta_\infty} f_{\infty,L} = 0 \\ \Big\downarrow L \to 0 & & \Big\downarrow L \to 0 \\ \Delta_\mathtt{p} f_{\mathtt{p},0} = 0 & \xrightarrow[\mathtt{p}\to\infty]{} & \Delta_\infty f_{\infty,0} = 0 \end{array}$$

6.2. **Heisenberg group.** We recall that in $\mathbb{H}^n$, the vector Υ has components

$$\Upsilon_k = \begin{cases} X_k u + iLX_{n+k}u & \text{when } k = 1, 2, \ldots, n, \\ X_k u - iLX_{k-n}u & \text{when } k = n+1, n+2, \ldots, 2n \end{cases}$$

so that we have

$$\overline{\Delta_{\mathtt{p}}} u = \operatorname{div}(\|\Upsilon\|^{\mathtt{p}-2}\Upsilon) = 0$$

in $\mathbb{H}^n \setminus \{0\}$. As in the Grushin case, we formally let $\mathtt{p} \to \infty$ and obtain

$$\overline{\Delta_\infty} u = \sum_{j=1}^{2n} (X_j \|\Upsilon\|^2)\Upsilon_j.$$

Formally letting $\mathtt{p} \to \infty$ also produces the function

$$u_{\infty,L}(x_1, x_2, \ldots, x_{2n}, z) = v(x_1, x_2, \ldots, x_{2n}, z)^{\frac{1-L}{4}} w(x_1, x_2, \ldots, x_{2n}, z)^{\frac{1+L}{4}},$$

where we recall that the functions v and w are given by

$$\begin{aligned} v(x_1, x_2, \ldots, x_{2n}, z) &= \Big(\sum_{j=1}^{2n} x_1^2\Big) - 4iz, \\ w(x_1, x_2, \ldots, x_{2n}, z) &= \Big(\sum_{j=1}^{2n} x_1^2\Big) + 4iz. \end{aligned}$$

We have the following theorem.

Theorem 6.2. *The function $u_{\infty,L}$, as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \overline{\Delta_\infty} u_{\infty,L}(p) = 0, & p \in \mathbb{H}^n \setminus \{(0)\}, \\ 0, & p = 0. \end{cases}$$

Proof. We may prove this theorem by letting $\mathtt{p} \to \infty$ in equation (5.5) and invoking continuity (cf. Corollary 5.4). For completeness, though, we compute formally.

We first make the following definitions:

$$A \equiv \frac{1+L}{4} \quad \text{and} \quad B \equiv \frac{1-L}{4},$$

so that we may compute. We then have for $k = 1, 2, \ldots, n$,

$$\begin{aligned} \Upsilon_k &= X_k u + iLX_{n+k}u \\ &= 2v^{B-1}w^{A-1}\Big((Bw + Av)(x_k + iLx_{n+k}) + (Bw - Av)(Lx_k + ix_{n+k})\Big), \end{aligned}$$

and for $l = n+1, n+2, \ldots, 2n$,

$$\begin{aligned} \Upsilon_l &= X_l u - iLX_{l-n}u \\ &= 2v^{B-1}w^{A-1}\Big((Bw + Av)(x_l - iLx_{l-n}) + (Bw - Av)(Lx_l - ix_{l-n})\Big), \end{aligned}$$

and so

$$\|\Upsilon\|^2 = (L^2 - 1)^2\Big(\sum_{j=1}^{2n} x_j^2\Big)(vw)^{-\frac{1}{2}}.$$

For $k = 1, 2, \ldots, n$, we have

$$X_k \|\Upsilon\|^2 \quad = \quad 8z(L^2-1)^2(vw)^{-\frac{3}{2}}\Big(4zx_k + x_{n+k}\sum_{j=1}^{2n} x_j^2\Big),$$

and for $l = n+1, n+2, \ldots, 2n$, we have

$$X_l \|\Upsilon\|^2 \quad = \quad 8z(L^2-1)^2(vw)^{-\frac{3}{2}}\Big(4zx_l - x_{l-n}\sum_{j=1}^{2n} x_j^2\Big).$$

For $k = 1, 2, \ldots, n$ this yields

$$\begin{aligned} X_k \|\Upsilon\|^2 \Upsilon_k = 8z(L^2-1)^3(vw)^{-\frac{3}{2}}&\Big(4zx_k + x_{n+k}\sum_{j=1}^{2n} x_j^2\Big)v^{B-1}w^{A-1} \\ &\times \Big(4zx_{n+k} - x_k\sum_{j=1}^{2n} x_j^2\Big), \end{aligned} \tag{6.1}$$

and for $l = n+1, n+2, \ldots, 2n$, we have

$$\begin{aligned} X_l \|\Upsilon\|^2 \Upsilon_l = 8z(L^2-1)^3(vw)^{-\frac{3}{2}}&\Big(4zx_l - x_{l-n}\sum_{j=1}^{2n} x_j^2\Big)v^{B-1}w^{A-1} \\ &\times \Big(-4zx_{l-n} - x_l\sum_{j=1}^{2n} x_j^2\Big). \end{aligned} \tag{6.2}$$

Combining equations (6.1) and (6.2) along with an index reordering produces

$$\sum_{j=1}^{2n} X_j(\|\Upsilon\|^2)\Upsilon_j = 8z(L^2-1)^3(vw)^{-\frac{3}{2}}v^{B-1}w^{A-1} \times 0 = 0.$$

□

We notice that when $L = 0$, this result was a part of the Ph.D. thesis of the first author [3]. In particular, combined with [3, 4], we have shown that the following diagram commutes in $\mathbb{H}^n \setminus \{0\}$:

$$\begin{array}{ccc} \overline{\Delta_{\mathrm{p}}} u_{\mathrm{p},L} = 0 & \xrightarrow[\mathrm{p}\to\infty]{} & \overline{\Delta_\infty} u_{\infty,L} = 0 \\ \Big\downarrow L\to 0 & & \Big\downarrow L\to 0 \\ \Delta_{\mathrm{p}} u_{\mathrm{p},0} = 0 & \xrightarrow[\mathrm{p}\to\infty]{} & \Delta_\infty u_{\infty,0} = 0 \end{array}$$

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