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# LOW FROUDE NUMBER LIMIT OF THE ROTATING SHALLOW WATER AND EULER EQUATIONS

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ABSTRACT. We perform the mathematical derivation of the rotating lake equations (or anelastic system) from the classical solution of the rotating shallow water and Euler equations when the Froude number tends to zero.

## 1. INTRODUCTION

The isentropic Euler equations with additional rotating forcing read, in a two dimensional bounded domain $\Omega$:

$$\begin{cases} \partial_t h + \nabla \cdot (hu) = 0\,, \\ \partial_t (hu) + \nabla \cdot \big(hu \otimes u\big) + \dfrac{hu^\perp}{Ro} + h\dfrac{\nabla\big(h^{\gamma-1} - h_0^{\gamma-1}\big)}{Fr^2} = 0\,, \\ (hu)\cdot n|_{\partial\Omega} = 0\,, \quad h|_{t=0} = h(x,0)\,, \quad u|_{t=0} = u(x,0)\,, \end{cases} \tag{1.1}$$

where $\gamma \geq 2$. In particular, when $\gamma = 2$, (1.1) comprises the inviscid rotating shallow water equations which are frequently used for modeling both oceanographic and atmospheric fluid flow in the midlatitudes with relatively large length and time scales [7,15,19]. In this case the unknowns are $h = h(t,x)$, the height of water, and $u = u(t,x) = (u_1(t,x), u_2(t,x))$, the horizontal component of the fluid velocity. The orthogonal velocity is denoted by $u^\perp = (-u_2, u_1)$, and the strictly positive function $h_0 = h_0(x)$ describes the bottom topography. Note that the two parameters $Ro$ and $Fr$ are, respectively, the Rossby number measuring the inverse rotational forcing, and the Froude number measuring the inverse pressure forcing. They penalize the Coriolis force $hu^\perp$ and the pressure forcing $h\nabla\big(h^{\gamma-1} - h_0^{\gamma-1}\big)$ respectively. In real geophysical interest, e.g., the large-scale motions in the atmosphere, at least one of the parameters $Ro$ or $Fr$ is very small, which will lead asymptotically to reduced models. As noted by Majda in [15], although the rotating shallow water equations are mathematically similar to the compressible flow equations, whether or not compressibility effects are important depends on the scales associated with the

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fluid motion. In gas dynamics, the measure of the importance of the compressibility effects is given by the Mach number. For rotating shallow water equations, the Froude number plays the analogous role of the Mach number. In this paper, we will consider the low Froude number limit, i.e. $Fr \to 0$ of (1.1). For simplicity of notation, we may assume $Ro = 1$, $Fr = \varepsilon$, and rewrite (1.1) as

$$\begin{cases} \partial_t h^\varepsilon + \nabla \cdot (h^\varepsilon u^\varepsilon) = 0\,, \\ \partial_t (h^\varepsilon u^\varepsilon) + \nabla \cdot \big(h^\varepsilon u^\varepsilon \otimes u^\varepsilon\big) + h^\varepsilon (u^\varepsilon)^\perp + h^\varepsilon \dfrac{\nabla\big((h^\varepsilon)^{\gamma-1} - h_0^{\gamma-1}\big)}{\varepsilon^2} = 0\,, \\ (h^\varepsilon u^\varepsilon)\cdot n|_{\partial\Omega} = 0\,, \quad h^\varepsilon|_{t=0} = h_0^\varepsilon(x)\,, \quad u^\varepsilon|_{t=0} = u_0^\varepsilon(x)\,. \end{cases} \tag{1.2}$$

When $\varepsilon$ is a fixed number, under the assumption of initial conditions $h_0^\varepsilon \geq c > 0$, $(h_0^\varepsilon, u_0^\varepsilon) \in H^3(\Omega) \times (H^3(\Omega))^2$ and some appropriate compatibility conditions on $\partial\Omega$, Beirao da Veiga proved in [2] the local existence and uniqueness of the classical solution of (1.1). Moreover, the solution satisfies the energy equality

$$\frac{d}{dt}\int_\Omega e^\varepsilon(\cdot,t)dx = 0\,, \qquad e^\varepsilon = \frac{1}{2}h^\varepsilon|u^\varepsilon|^2 + \frac{1}{\varepsilon^2}\Theta(h^\varepsilon)\,, \tag{1.3}$$

where the potential energy

$$\Theta(h^\varepsilon) = \frac{1}{\gamma}\Big((h^\varepsilon)^\gamma + (\gamma-1)h_0^\gamma - \gamma h^\varepsilon h_0^{\gamma-1}\Big)$$

is a convex function with minimum occurring at $h^\varepsilon = h_0$ which satisfies $\Theta(h^\varepsilon) \geq 0$. Formally, letting $\varepsilon \to 0$, $h^\varepsilon$ will converge to $h_0$ from the energy equality (1.3) and the limiting velocity $u$ will solve the rotating lake equations (or anelastic system)

$$\begin{cases} \nabla \cdot (h_0 u) = 0\,, \\ \partial_t (h_0 u) + \nabla \cdot \big(h_0 u \otimes u\big) + h_0 u^\perp + h_0 \nabla \pi = 0\,, \\ (h_0 u)\cdot n|_{\partial\Omega} = 0\,, \quad u|_{t=0} = u_0(x)\,, \quad \nabla\cdot(h_0 u_0) = 0\,. \end{cases} \tag{1.4}$$

Thus, the rotating lake equations may be seen as the low Froude number limit of the usual inviscid rotating shallow water and Euler equations when the initial height converges to a nonconstant function $h_0(x)$ depending on the space variable (see [5] for the case without rotating forcing). Note that for nonvarying bottom $h_0 = 1$, (1.4) reverts to the rotating incompressible Euler equations.

Before the presentation of the main result of this paper, let us make the following assumptions on the initial conditions:

(A1) $h_0^\varepsilon \geq c > 0$, $(h_0^\varepsilon, u_0^\varepsilon) \in H^3(\Omega) \times (H^3(\Omega))^2$ and some appropriate compatibility conditions on $\partial\Omega$; this guarantees the local existence and uniqueness of the classical solution of the inviscid rotating shallow water equations (1.2).

(A2) $\varepsilon^{-2}\int_\Omega \Theta(h_0^\varepsilon)dx \to 0$ as $\varepsilon \to 0$; this means the initial potential energy converges to 0 as $\varepsilon$ goes to zero.

(A3) $\sqrt{h_0^\varepsilon}u_0^\varepsilon \to \sqrt{h_0}u_0$ in $L^2(\Omega)$ as $\varepsilon \to 0$; this means the initial kinetic energy is well prepared.

(A4) $h_0 \geq c > 0$, $u_0 \in (H^3(\Omega))^2$, $\nabla\cdot(h_0 u_0) = 0$ and some appropriate compatibility conditions on $\partial\Omega$; this guarantees the existence and uniqueness of the classical solution of the rotating lake equations (1.4) and the well prepared initial condition.

Under the assumption (A4), Levermore, Oliver and Titi proved in [9] the existence and uniqueness of the classical solution of the lake equations (1.4). The main result of this paper is stated as follows:

**Theorem 1.1.** *Let* $(h^\varepsilon, u^\varepsilon) \in H^3(\Omega) \times (H^3(\Omega))^2$ *be the solution of* (1.2) *and the initial conditions* $(h_0^\varepsilon, u_0^\varepsilon)$ *satisfy assumptions* (A1) – (A4). *Then there exists* $T_* > 0$ *such that*

$$\|(h^\varepsilon - h_0)(\cdot, t)\|_{L^\gamma(\Omega)} \to 0\,,$$
$$\|(h^\varepsilon u^\varepsilon - h_0 u)(\cdot, t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega)} \to 0$$

*as* $\varepsilon \to 0$, *where* $u \in (H^3(\Omega))^2$ *is a classical solution of the rotating lake equations* (1.4).

The question of the singular limits, e.g. incompressible, low Froude number limits, in fluid mechanics has received considerable attention. For a low Mach number or incompressible limit, some fundamental facts on this problem have been established by Klainerman and Majda in [8] (see also [14]). The basic result, which has been proven in various contexts, is that slightly compressible fluid flows are close to incompressible flows even though the equations for the latter are related to those for the former via a singular limit. This justifies the use of the incompressible flow equations for certain real fluids that are actually slightly compressible. For weak solutions, this problem was done by P.L. Lions and Masmoudi in [13], where Leray global weak solutions of the incompressible Navier-Stokes equation are recovered from the global weak solutions of the compressible Navier-Stokes equation.

The modulated energy method is a popular way to study the hydrodynamic limits. It was introduced by Brenier [3] to prove the convergence of the Vlasov-Poisson system to the incompressible Euler equation. It has also been applied to study various singular limits of the other equations, for example the Schrödinger-Poisson equation [20], the Gross-Pitaevskii equation [12], the Klein-Gordon equation [11] and the quantum hydrodynamic model [10]. In fact, we will employ this method to study the low Froude number limit of the rotating shallow water and Euler equations. We limit ourselves in this paper to the case when the initial data is well prepared (see assumption (A4)). For a general not *well-prepared* initial condition, as mentioned in [10, 16], we must consider the oscillation part generated by the non-divergence free part of the initial momentum. Indeed, this is a challenging problem and will be our next research project.

The fact that density variations in real fluids are related to both pressure and entropy variations, even though the low Mach or Froude number limit and the limiting density may not necessarily be a constant, is the main focus of recent research about the singular limit problems. The only known results concerning the nonconstant limiting density we will refer to are [4, 5], where the viscous shallow water equation is discussed for the periodic domain. We also refer to [18] for the nonisentropic Euler equation and [1] for the full Navier-Stokes equation. Moreover, we mention the works about the anelastic system by Feireisl, Novotný and Petzeltová [6] and Masmoudi [17], where they extended the classical Leray global weak solutions of the incompressible Navier-Stokes equation to the anelastic system.

In this paper, we use the modulated energy functional to control the propagation of the height $h^\varepsilon$ and velocity $u^\varepsilon$. We have to check the evolution of the modulated energy and calculate the kinetic part $R_1$, the potential part $R_2$, and the rotating part $R_3$ carefully, as shown in (2.7). Fortunately, we can treat the kinetic part $R_1$

similarly to [3, 10, 11, 20] and control $R_2$ and $R_3$ successfully. Note that it is easy to control $R_2$ if the bottom topography $h_0 = 1$. Besides the introduction, section 2 is devoted to the rigorous proof of the main theorem.

## 2. Proof of the theorem

The assumptions of initial conditions (A2)–(A3) give the uniform bound of initial energy. By the energy estimate (1.3), we have the uniform bound of total energy

$$\int_\Omega e^\varepsilon(\cdot,t)dx = \int_\Omega e^\varepsilon(\cdot,0)dx \le C\,.$$

Especially,

$$\int_\Omega \Theta(h^\varepsilon)dx = O(\varepsilon^2)\,.$$

We will have

$$\|(h^\varepsilon - h_0)(\cdot,t)\|_{L^\gamma(\Omega)} = O\big(\varepsilon^{\frac{2}{\gamma}}\big)\,, \qquad t\in[0,T], \tag{2.1}$$

by the following elementary convexity inequality:

$$\frac{1}{\gamma}\big|h^\varepsilon - h_0\big|^\gamma \le \Theta(h^\varepsilon) \quad \text{for} \quad \gamma \ge 2\,.$$

Now, we define the modulated energy as follows:

$$H^\varepsilon(t) = \frac{1}{2}\int_\Omega h^\varepsilon|u^\varepsilon - u|^2dx + \frac{1}{\varepsilon^2}\int_\Omega \Theta(h^\varepsilon)dx\,. \tag{2.2}$$

The modulated energy $H^\varepsilon(t)$ can be further rewritten as

$$H^\varepsilon(t) = \int_\Omega e^\varepsilon dx - \int_\Omega h^\varepsilon u\cdot u^\varepsilon dx + \frac{1}{2}\int_\Omega h^\varepsilon|u|^2dx\,. \tag{2.3}$$

Differentiating the modulated energy (2.3) with respect to $t$ and using energy equation (1.3), we obtain

$$\frac{d}{dt}H^\varepsilon(t) = -\frac{d}{dt}\int_\Omega h^\varepsilon u\cdot u^\varepsilon dx + \frac{d}{dt}\int_\Omega \frac{1}{2}h^\varepsilon|u|^2dx \equiv I_1 + I_2\,. \tag{2.4}$$

By momentum equation $(1.2)_2$, integration by parts and the boundary condition of $h^\varepsilon u^\varepsilon$, we obtain

$$\begin{aligned} I_1 = &-\int_\Omega h^\varepsilon\partial_t u\cdot u^\varepsilon dx - \int_\Omega \big(h^\varepsilon u^\varepsilon\otimes u^\varepsilon\big):\nabla u dx + \int_\Omega h^\varepsilon u\cdot(u^\varepsilon)^\perp dx \\ &+\frac{1}{\varepsilon^2}\int_\Omega h^\varepsilon u\cdot\nabla\Big((h^\varepsilon)^{\gamma-1} - h_0^{\gamma-1}\Big)dx\,. \end{aligned} \tag{2.5}$$

Next, employing the continuity equation $(1.2)_1$, integration by parts and using the boundary condition of $h^\varepsilon u^\varepsilon$, we have

$$I_2 = \int_\Omega h^\varepsilon u\cdot\partial_t u dx + \int_\Omega \frac{1}{2}\nabla|u|^2\cdot(h^\varepsilon u^\varepsilon)dx\,. \tag{2.6}$$

Consequently, by (2.4)–(2.6) we have

$$\frac{d}{dt}H^\varepsilon(t) = \int_\Omega \frac{1}{2}\nabla|u|^2\cdot(h^\varepsilon u^\varepsilon)dx + \int_\Omega \partial_t u\cdot(h^\varepsilon u - h^\varepsilon u^\varepsilon)dx + \sum_{i=1}^3 R_i\,,$$

where

$$R_1 = -\int_\Omega \big(h^\varepsilon u^\varepsilon \otimes u^\varepsilon\big) : \nabla u dx\,,$$

$$R_2 = \frac{1}{\varepsilon^2}\int_\Omega h^\varepsilon u \cdot \nabla\Big((h^\varepsilon)^{\gamma-1} - h_0^{\gamma-1}\Big)dx\,, \tag{2.7}$$

$$R_3 = \int_\Omega h^\varepsilon u \cdot (u^\varepsilon)^\perp dx\,.$$

To deal with the kinetic part $R_1$, we rewrite $R_1$ as

$$R_1 = -\int_\Omega \big(h^\varepsilon(u^\varepsilon - u)\otimes(u^\varepsilon - u)\big) : \nabla u dx - \int_\Omega \big(h^\varepsilon u \otimes u^\varepsilon\big) : \nabla u dx$$

$$+\int_\Omega \big(h^\varepsilon u \otimes u\big) : \nabla u dx - \int_\Omega \big(h^\varepsilon u^\varepsilon \otimes u\big) : \nabla u dx\,.$$

One can calculate, using an integration by parts and the boundary conditions of $h^\varepsilon u^\varepsilon$ and $u$,

$$-\int_\Omega \big(h^\varepsilon u \otimes u^\varepsilon\big) : \nabla u dx = \int_\Omega \frac{1}{2}|u|^2 \nabla\cdot(h^\varepsilon u^\varepsilon)dx$$

and

$$\int_\Omega \big(h^\varepsilon u \otimes u\big) : \nabla u dx - \int_\Omega \big(h^\varepsilon u^\varepsilon \otimes u\big) : \nabla u dx$$

$$= \int_\Omega \Big[\big(u\cdot\nabla\big)u\Big]\cdot(h^\varepsilon u - h^\varepsilon u^\varepsilon)dx.$$

This means that

$$R_1 = -\int_\Omega \big(h^\varepsilon(u^\varepsilon - u)\otimes(u^\varepsilon - u)\big) : \nabla u dx$$

$$+\int_\Omega \frac{1}{2}|u|^2\nabla\cdot(h^\varepsilon u^\varepsilon)dx + \int_\Omega \Big[\big(u\cdot\nabla\big)u\Big]\cdot(h^\varepsilon u - h^\varepsilon u^\varepsilon)dx\,.$$

To deal with the potential part $R_2$, we need

$$h^\varepsilon u\cdot\nabla(h^\varepsilon)^{\gamma-1} = \tfrac{\gamma-1}{\gamma} u\cdot\nabla(h^\varepsilon)^\gamma\,, \tag{2.8}$$

and using the divergence free part of $h_0 u$ we obtain

$$-h^\varepsilon u\cdot\nabla h_0^{\gamma-1} = (\gamma-1)h^\varepsilon h_0^{\gamma-1}\nabla\cdot u\,. \tag{2.9}$$

Combining (2.8) and (2.9) we have

$$h^\varepsilon u\cdot\nabla\Big((h^\varepsilon)^{\gamma-1} - h_0^{\gamma-1}\Big) = \tfrac{\gamma-1}{\gamma}\Big[u\cdot\nabla(h^\varepsilon)^\gamma + \gamma h^\varepsilon h_0^{\gamma-1}\nabla\cdot u\Big]\,. \tag{2.10}$$

Moreover, using integration by parts, the boundary condition of $u$, and divergence free of $h_0 u$, we have

$$\begin{aligned}\int_\Omega (h_0)^\gamma \nabla\cdot u dx &= -\int_\Omega u\cdot\nabla(h_0)^\gamma dx\\ &= -\frac{\gamma}{\gamma-1}\int_\Omega h_0 u\cdot\nabla(h_0)^{\gamma-1}dx = 0\,.\end{aligned} \tag{2.11}$$

Consequently, by (2.10) and (2.11)

$$R_2 = -\frac{1}{\varepsilon^2}\int_\Omega \frac{\gamma-1}{\gamma}\nabla\cdot u\Big[(h^\varepsilon)^\gamma - \gamma h^\varepsilon h_0^{\gamma-1} + (\gamma-1)h_0^\gamma\Big]dx\,.$$

For the rotating part $R_3$, one can prove that

$$\begin{aligned} R_3 &= \int_\Omega h^\varepsilon u\cdot(u^\varepsilon)^\perp dx \\ &= \int_\Omega -u^\perp\cdot(h^\varepsilon u^\varepsilon)dx = \int_\Omega u^\perp\cdot(h^\varepsilon u - h^\varepsilon u^\varepsilon)dx\,, \end{aligned}$$

where the anti-symmetric property $u\cdot(u^\varepsilon)^\perp = -u^\perp\cdot u^\varepsilon$ and the orthogonal property $u^\perp\cdot u = 0$ have been used. Combining the above equalities, we have

$$\begin{aligned} \frac{d}{dt}H^\varepsilon(t) = &-\int_\Omega \big(h^\varepsilon(u^\varepsilon-u)\otimes(u^\varepsilon-u)\big):\nabla u dx \\ &-\frac{1}{\varepsilon^2}\int_\Omega \frac{\gamma-1}{\gamma}\Big[(h^\varepsilon)^\gamma - \gamma h^\varepsilon h_0^{\gamma-1} + (\gamma-1)h_0^\gamma\Big]\nabla\cdot u dx \\ &+\int_\Omega \Big[\partial_t u + \big(u\cdot\nabla\big)u + u^\perp\Big]\cdot(h^\varepsilon u - h^\varepsilon u^\varepsilon)dx\,. \end{aligned} \tag{2.12}$$

We can estimate the first two integrals of the right side of (2.12); it can be bounded by $\|\nabla u\|_{L^\infty(\Omega)}H^\varepsilon(t)$. Moreover, (2.12) can be transformed into

$$\frac{d}{dt}H^\varepsilon(t) \le C_1 H^\varepsilon(t) - \int_\Omega \nabla\pi\cdot(h^\varepsilon u - h^\varepsilon u^\varepsilon)dx\,, \tag{2.13}$$

where $\pi$ is the pressure of the lake equations (anelastic system) (1.4). Now we will estimate the second term of the right side of (2.13). By (2.1), the divergence free part of $h_0 u$ and the Hölder inequality, we arrive at the inequality

$$\begin{aligned} \int_\Omega h^\varepsilon u\cdot\nabla\pi dx &= \int_\Omega (h^\varepsilon - h_0)u\cdot\nabla\pi dx \\ &\le \varepsilon^{\frac{2}{\gamma}}\|u\|_{L^\infty(\Omega)}\|\nabla\pi\|_{L^{\frac{\gamma}{\gamma-1}}(\Omega)}\,. \end{aligned}$$

To go further, we need the relation (which follows from the continuity equation $(1.2)_1$, integration by parts and the boundary condition of $h^\varepsilon u^\varepsilon$)

$$\begin{aligned} \int_\Omega (h^\varepsilon u^\varepsilon)\cdot\nabla\pi dx &= \int_\Omega \pi\partial_t(h^\varepsilon - h_0)dx \\ &= \frac{d}{dt}\int_\Omega \pi(h^\varepsilon - h_0)dx - \int_\Omega \partial_t\pi(h^\varepsilon - h_0)dx\,. \end{aligned} \tag{2.14}$$

The last integral of (2.14) can be estimated by the Hölder inequality,

$$\int_\Omega \partial_t\pi(h^\varepsilon - h_0)dx \le \varepsilon^{\frac{2}{\gamma}}\|\partial_t\pi\|_{L^{\frac{\gamma}{\gamma-1}}(\Omega)}\,.$$

We have to introduce one more correction term of the modulated energy defined by

$$W^\varepsilon(t) = \int_\Omega -(h^\varepsilon - h_0)\pi dx\,.$$

The correction term $W^\varepsilon(t)$ can serve as the acoustic part (density fluctuation) of the modulated energy $H^\varepsilon(t)$. This term describes the propagation of the density fluctuation in order to obtain the incompressible limit (see [11] for the Klein-Gordon equation). This is similar to the low Mach number limit of the compressible fluid [4, 13, 14]. Hence for $t \in [0, T_*)$ we have

$$\frac{d}{dt}\Big(H^\varepsilon(t) + W^\varepsilon(t)\Big) \le C_1 H^\varepsilon(t) + O(\varepsilon^{\frac{\gamma}{2}})\,.$$

Integrating this inequality yields

$$H^\varepsilon(t) \le H^\varepsilon(0) - W^\varepsilon(t) + C_1 \int_0^t H^\varepsilon(\tau)d\tau + O(\varepsilon^{\frac{\gamma}{2}})\,.$$

One can show that $W^\varepsilon(t) = O(\varepsilon^{\frac{\gamma}{2}})$, and hence

$$H^\varepsilon(t) \le C_1 \int_0^t H^\varepsilon(\tau)d\tau + H^\varepsilon(0) + O(\varepsilon^{\frac{\gamma}{2}})\,.$$

In order to obtain the convergence result, we need to estimate the initial modulated energy functional $H^\varepsilon(0)$. It is easy to see that

$$\begin{aligned} &\|\sqrt{h_0^\varepsilon}u_0^\varepsilon - \sqrt{h_0^\varepsilon}u_0\|_{L^2(\Omega)} \\ &\le \|\sqrt{h_0^\varepsilon}u_0^\varepsilon - \sqrt{h_0}u_0\|_{L^2(\Omega)} + \|(\sqrt{h_0} - \sqrt{h_0^\varepsilon})u_0\|_{L^2(\Omega)} \end{aligned} \tag{2.15}$$

and the first term of the right hand side of (2.15) converges to 0 by assumption (A3). For the second term, using the finite measure of $\Omega$, assumption (A1) and an elementary inequality

$$|\sqrt{x} - \sqrt{a}|^2 \le a^{-1}|x-a|^2, \quad x \ge 0, \quad a \ge c > 0\,,$$

we have

$$\begin{aligned} \|(\sqrt{h_0} - \sqrt{h_0^\varepsilon})u_0\|_{L^2(\Omega)} &\le \|u_0\|_{L^\infty(\Omega)}\|\sqrt{h_0} - \sqrt{h_0^\varepsilon}\|_{L^2(\Omega)} \\ &\le \frac{1}{\sqrt{c}}\|u_0\|_{L^\infty(\Omega)}\|h_0 - h_0^\varepsilon\|_{L^2(\Omega)} \\ &\le C(\Omega)\|u_0\|_{L^\infty(\Omega)}\|h_0 - h_0^\varepsilon\|_{L^\gamma(\Omega)}\,, \end{aligned}$$

which converges to 0 by assumption (A2), and hence $H^\varepsilon(0) \to 0$ as $\varepsilon \to 0$. Applying the Gronwall inequality, we can show that $H^\varepsilon(t) \to 0$ for $t \in [0, T_*)$. It is easy to rewrite the modulated energy (2.2) as

$$H^\varepsilon(t) = \frac{1}{2}\int_\Omega \Big|\frac{1}{\sqrt{h^\varepsilon}}(h^\varepsilon u^\varepsilon - h^\varepsilon u)\Big|^2 dx + \frac{1}{\varepsilon^2}\int_\Omega \Theta(h^\varepsilon)dx.$$

Then we have

$$\int_\Omega \Big|\frac{1}{\sqrt{h^\varepsilon}}(h^\varepsilon u^\varepsilon - h^\varepsilon u)\Big|^2 dx \to 0 \tag{2.16}$$

as $\varepsilon \to 0$. Therefore we can deduce from (2.16) and the Hölder inequality that

$$\begin{aligned} \|h^\varepsilon u^\varepsilon - h_0 u\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega)} &\le \Big\|\sqrt{h^\varepsilon}\Big\|_{L^{2\gamma}(\Omega)}\Big\|\frac{1}{\sqrt{h^\varepsilon}}(h^\varepsilon u^\varepsilon - h^\varepsilon u)\Big\|_{L^2(\Omega)} \\ &\quad + \|h^\varepsilon - h_0\|_{L^\gamma(\Omega)}\|u\|_{L^2(\Omega)}\,, \end{aligned}$$

which converges to zero as $\varepsilon \to 0$. This completes the proof of the theorem.

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