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ON THE CATEGORY OF COFINITE MODULES WHICH IS ABELIAN

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Dedicated to Professor Robin Hartshorne

Abstract. Let R denote a commutative Noetherian (not necessarily local) ring and I an ideal of R of dimension one. The main purpose of this paper is to generalize, and to provide a short proof of, K. I. Kawasaki's theorem that the category $\mathscr{M}(R,I)_{cof}$ of I-cofinite modules over a commutative Noetherian local ring R forms an Abelian subcategory of the category of all R-modules. Consequently, this assertion answers affirmatively the question raised by R. Hartshorne in his article *Affine duality and cofiniteness* [Invent. Math. **9** (1970), 145-164] for an ideal of dimension one in a commutative Noetherian ring R.

1. Introduction

Let R denote a commutative Noetherian ring, and let I be an ideal of R. In [4], Hartshorne defined an R-module L to be *I-cofinite* if $\operatorname{Supp}(L) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, L)$ is finitely generated module for all i. He posed the following question:

Does the category $\mathscr{M}(R,I)_{cof}$ of I-cofinite modules form an Abelian subcategory of the category of all R-modules? That is, if $f : M \longrightarrow N$ is an R-module homomorphism of I-cofinite modules, are $\ker f$ *and* $\operatorname{coker} f$ *I-cofinite?*

Hartshorne gave the following counterexample (see [3]): Let k be a field and let $R = k[[x,y,z,u]]/(xy - zu)$. Set $M = R/(xy - uv)R$ and $I = (x,u)R$. Then applying functor $H^0_I(-)$ to the exact sequence

$$0 \longrightarrow R \overset{xy-uv}{\longrightarrow} R \longrightarrow M \longrightarrow 0,$$

we obtain the exact sequence

$$\cdots \longrightarrow H^2_I(R) \overset{f}{\longrightarrow} H^2_I(R) \longrightarrow H^2_I(M) \longrightarrow 0.$$

Since $H^i_I(R) = 0$ for all $i \neq 2$, one can show that

$$\operatorname{Ext}^i_R(R/I, H^2_I(R)) \cong \operatorname{Ext}^{i+2}_R(R/I, R),$$

for all i. Thus, $H^2_I(R)$ is I-cofinite. However, $\operatorname{coker} f = H^2_I(M)$ is not I-cofinite. On the positive side, Hartshorne proved that if I is a prime ideal of dimension one in a complete regular local ring R, then the answer to his question is yes. On the other hand, in [3], Delfino and Marley extended this result to arbitrary complete

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local rings. Recently, Kawasaki [6] generalized Delfino and Marley's result for an arbitrary ideal I of dimension one in a local ring R. Kawasaki's proof relies on a *spectral sequence*, and several pages of his paper are devoted to a proof of that theorem. See also [7] and [8].

The main purpose of this paper is to generalize and to present a much shorter proof of Kawasaki's theorem, using somewhat more elementary methods than those used by Kawasaki. More precisely, we shall show that:

Theorem 1.1. *Let R be a Noetherian ring and I an ideal of R of dimension one. Let $\mathscr{M}(R,I)_{cof}$ denote the category of I-cofinite modules. Then $\mathscr{M}(R,I)_{cof}$ forms an Abelian subcategory of the category of all R-modules.*

One of our tools for proving Theorem 1.1 is the following, which is a generalization of a result of Melkersson (cf. [11, Proposition 4.3]).

Proposition 1.2. *Let I denote an ideal of a Noetherian ring R and let M be an R-module such that $\dim M \leq 1$ and $\mathrm{Supp}(M) \subseteq V(I)$. Then M is I-cofinite if and only if the R-modules $\mathrm{Hom}_R(R/I, M)$ and $\mathrm{Ext}^1_R(R/I, M)$ are finitely generated.*

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. For an Artinian R-module A, we denote by $\mathrm{Att}_R A$ the set of attached prime ideals of A. For each R-module L, we denote by $\mathrm{Assh}_R L$ the set $\{\mathfrak{p} \in \mathrm{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$. We shall use $\mathrm{Max}\, R$ to denote the set of all maximal ideals of R. Also, for any ideal $\mathfrak{a}$ of R, we denote $\{\mathfrak{p} \in \mathrm{Spec}\, R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. Finally, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b} \text{ for some } n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [2] and [9].

Recall that a module is called a *minimax module* when it has a finitely generated submodule, such that the quotient by it is an Artinian module [12].

2. The results

Let us first recall the important concept of the arithmetic rank of an ideal. The *arithmetic rank* of an ideal $\mathfrak{b}$ in a commutative Noetherian ring T, denoted by $\mathrm{ara}(\mathfrak{b})$, is the least number of elements of T required to generate an ideal which has the same radical as $\mathfrak{b}$, i.e.,

$$\mathrm{ara}(\mathfrak{b}) = \min\{n \in \mathbb{N}_0 : \exists b_1, \ldots, b_n \in T \text{ with } \mathrm{Rad}(b_1, \ldots, b_n) = \mathrm{Rad}(\mathfrak{b})\}.$$

Let M be a T-module. The arithmetic rank of an ideal $\mathfrak{b}$ of T with respect to M, denoted by $\mathrm{ara}_M(\mathfrak{b})$, is defined by the arithmetic rank of the ideal $\mathfrak{b} + \mathrm{Ann}_T M / \mathrm{Ann}_T M$ in the ring $T/\mathrm{Ann}_T M$.

The main point of this note is to generalize and to provide a short proof the main result of Kawasaki [6, Theorem 1] concerning a question raised by R. Hartshorne. The following proposition plays a key role in the proof of that theorem. Before we state Proposition 2.6, we recall some lemmas that we will use in the proof of this proposition.

Lemma 2.1. *Let R be a Noetherian ring and I an ideal of R. Then, for any R-module T, the following conditions are equivalent:*

(i) $\mathrm{Ext}^n_R(R/I, T)$ *is finitely generated for all* $n \geq 0$.

(ii) $\mathrm{Ext}^n_R(N, T)$ *is finitely generated for all* $n \geq 0$ *and for each finitely generated R-module N for which* $\mathrm{Supp}\, N \subseteq V(I)$.

Proof. See [5, Lemma 1]. □

Lemma 2.2. *Let R be a Noetherian ring and I an ideal of R. Let $x \in I$ and M be an R-module such that* $\operatorname{Supp} M \subseteq V(I)$. *If $(0:_M x)$ and M/xM are I-cofinite, then M is also I-cofinite.*

Proof. See [11, Corollary 3.4]. □

Lemma 2.3. *Let R be a Noetherian ring and I an ideal of R. Let M be a minimax R-module such that* $\operatorname{Supp} M \subseteq V(I)$. *Then M is I-cofinite if and only if $(0:_M I)$ is finitely generated.*

Proof. See [10, Proposition 4.3]. □

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a local (Noetherian) ring and let A be an Artinian R-module. Let I be an ideal of R such that the R-module* $\operatorname{Hom}_R(R/I, A)$ *is finitely generated. Then* $V(I) \cap \operatorname{Att}_R A \subseteq V(\mathfrak{m})$.

Proof. See [1, Lemma 2.5]. □

Lemma 2.5. *Let $(R, \mathfrak{m})$ and A be as in Lemma* 2.4. *Suppose that x is an element in $\mathfrak{m}$ such that* $V(Rx) \cap \operatorname{Att}_R A \subseteq \{\mathfrak{m}\}$. *Then the R-module A/xA has finite length.*

Proof. See [1, Lemma 2.4]. □

Proposition 2.6. *Let I be an ideal of a Noetherian ring R and M be an R-module such that* $\dim M \leq 1$ *and* $\operatorname{Supp} M \subseteq V(I)$. *Then the following statements are equivalent:*
(i) *M is I-cofinite,*
(ii) *the R-modules* $\operatorname{Hom}_R(R/I, M)$ *and* $\operatorname{Ext}^1_R(R/I, M)$ *are finitely generated.*

Proof. The conclusion (i) $\Longrightarrow$ (ii) is obviously true. In order to prove that (ii) $\Longrightarrow$ (i), since by assumption (ii) $\operatorname{Hom}_R(R/I, M)$ is finitely generated, using Lemma 2.3 and [10, Theorem 1.3], we may assume $\dim M = 1$.

We now prove by induction on $t := \operatorname{ara}_M(I) = \operatorname{ara}(I + \operatorname{Ann}_R M/\operatorname{Ann}_R M)$ that M is I-cofinite. If $t = 0$, then it follows from definition that $I^n \subseteq \operatorname{Ann}_R(M)$ for some positive integer n, and so $M = (0:_M I^n)$. Therefore the assertion follows from Lemma 2.1. So assume that $t > 0$, and the result has been proved for all $i \leq t-1$. Let

$$\mathcal{T} := \{\mathfrak{p} \in \operatorname{Supp} M \mid \dim R/\mathfrak{p} = 1\}.$$

It is easy to see that $\mathcal{T} = \operatorname{Assh}_R M$. As $\operatorname{Ass}_R \operatorname{Hom}_R(R/I, M) = V(I) \cap \operatorname{Ass}_R M = \operatorname{Ass}_R M$, it follows that the set $\operatorname{Ass}_R M$ is finite. Hence $\mathcal{T}$ is finite. Moreover, since for each $\mathfrak{p} \in \mathcal{T}$ the $R_\mathfrak{p}$-module $\operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, M_\mathfrak{p})$ is finitely generated, by [9, Ex. 7.7], and $M_\mathfrak{p}$ is an $IR_\mathfrak{p}$-torsion $R_\mathfrak{p}$-module, with $\operatorname{Supp} M_\mathfrak{p} \subseteq V(\mathfrak{p}R_\mathfrak{p})$, it follows that the $R_\mathfrak{p}$-module $\operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, M_\mathfrak{p})$ is Artinian. Consequently, according to Melkersson's results [10, Theorem 1.3] and Lemma 2.3, $M_\mathfrak{p}$ is an Artinian and $IR_\mathfrak{p}$-cofinite $R_\mathfrak{p}$-module. Let

$$\mathcal{T} := \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}.$$

By Lemma 2.4, we have

$$V(IR_{\mathfrak{p}_j}) \cap \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) \subseteq V(\mathfrak{p}_j R_{\mathfrak{p}_j}),$$

for all $j = 1, 2, ..., n$. Next, let

$$\mathscr{U} := \bigcup_{j=1}^{n} \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q}R_{\mathfrak{p}_j} \in \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j})\}.$$

Then it is easy to see that $\mathscr{U} \cap V(I) \subseteq \mathcal{T}$.

On the other hand, since $t = \operatorname{ara}_M(I) \geq 1$, there exist elements $y_1, ..., y_t \in I$ such that

$$\operatorname{Rad}(I + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)) = \operatorname{Rad}((y_1, ..., y_t) + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)).$$

Now, as $I \not\subseteq \bigcup_{\mathfrak{q} \in \mathscr{U} \setminus V(I)} \mathfrak{q}$, it follows that $(y_1, ..., y_t) + \operatorname{Ann}_R(M) \not\subseteq \bigcup_{\mathfrak{q} \in \mathscr{U} \setminus V(I)} \mathfrak{q}$.

On the other hand, for each $\mathfrak{q} \in \mathscr{U}$ we have

$$\mathfrak{q}R_{\mathfrak{p}_j} \in \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}),$$

for some integer $1 \leq j \leq n$. Hence

$$\operatorname{Ann}_R(M)R_{\mathfrak{p}_j} \subseteq \operatorname{Ann}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) \subseteq \mathfrak{q}R_{\mathfrak{p}_j}.$$

Since $\mathfrak{q}$ is prime we obtain that $\operatorname{Ann}_R(M) \subseteq \mathfrak{q}$. Consequently, it follows from

$$\operatorname{Ann}_R(M) \subseteq \bigcap_{\mathfrak{q} \in \mathscr{U} \setminus V(I)} \mathfrak{q}$$

that $(y_1, \ldots, y_t) \not\subseteq \bigcup_{\mathfrak{q} \in \mathscr{U} \setminus V(I)} \mathfrak{q}$. By [9, Ex. 16.8] there is $a \in (y_2, \ldots, y_t)$ such that $y_1 + a \notin \bigcup_{\mathfrak{q} \in \mathscr{U} \setminus V(I)} \mathfrak{q}$. Let $x := y_1 + a$. Then $x \in I$ and

$$\operatorname{Rad}(I + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)) = \operatorname{Rad}((x, y_2, ..., y_t) + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)).$$

Next, let $N := (0 :_M x)$. Then, it is easy to see that

$$\operatorname{ara}_N(I) = \operatorname{ara}(I + \operatorname{Ann}_R(N)/\operatorname{Ann}_R(N)) \leq t - 1$$

(note that $x \in \operatorname{Ann}_R(N)$), and hence

$$\operatorname{Rad}(I + \operatorname{Ann}_R(N)/\operatorname{Ann}_R(N)) = \operatorname{Rad}((y_2, ..., y_t) + \operatorname{Ann}_R(N)/\operatorname{Ann}_R(N)).$$

Now, the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow xM \longrightarrow 0$$

induces an exact sequence

$$(*) \qquad \begin{aligned} 0 \longrightarrow \operatorname{Hom}_R(R/I, N) &\longrightarrow \operatorname{Hom}_R(R/I, M) \longrightarrow \operatorname{Hom}_R(R/I, xM) \\ &\longrightarrow \operatorname{Ext}^1_R(R/I, N) \longrightarrow \operatorname{Ext}^1_R(R/I, M), \end{aligned}$$

which implies that the R-modules $\operatorname{Hom}_R(R/I, N)$ and $\operatorname{Ext}^1_R(R/I, N)$ are finitely generated. Consequently, by the inductive hypothesis, the R-module N is I-cofinite.

Moreover, the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow xM \longrightarrow 0$$

induces an exact sequence

$$\operatorname{Ext}^1_R(R/I, M) \longrightarrow \operatorname{Ext}^1_R(R/I, xM) \longrightarrow \operatorname{Ext}^2_R(R/I, N),$$

which implies that the R-module $\operatorname{Ext}^1_R(R/I, xM)$ is finitely generated.

Also, from the exact sequence

$$0 \longrightarrow xM \longrightarrow M \longrightarrow M/xM \longrightarrow 0$$

we get the exact sequence

$$\mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{Hom}_R(R/I, M/xM) \longrightarrow \mathrm{Ext}^1_R(R/I, xM),$$

which implies that the R-module $\mathrm{Hom}_R(R/I, M/xM)$ is finitely generated.

Now, from Lemma 2.5, it is easy to see that $(M/xM)_{\mathfrak{p}_j}$ has finite length for all $j = 1, \ldots, n$. Therefore there exists a finitely generated submodule L_j of M/xM such that

$$(M/xM)_{\mathfrak{p}_j} = (L_j)_{\mathfrak{p}_j}.$$

Let $L := L_1 + \cdots + L_n$. Then L is a finitely generated submodule of M/xM such that

$$\mathrm{Supp}(M/xM)/L \subseteq \mathrm{Supp}(M) \setminus \{\mathfrak{p}_1, ..., \mathfrak{p}_n\} \subseteq \mathrm{Max}\, R.$$

The sequence

$$0 \longrightarrow L \longrightarrow M/xM \longrightarrow (M/xM)/L \longrightarrow 0$$

provides the exact sequence

$$\mathrm{Hom}_R(R/I, M/xM) \longrightarrow \mathrm{Hom}_R(R/I, (M/xM)/L) \longrightarrow \mathrm{Ext}^1_R(R/I, L),$$

which implies that the R-module $\mathrm{Hom}_R(R/I, (M/xM)/L)$ is finitely generated. We now show that M/xM is a minimax R-module. To do this, since

$$\mathrm{Supp}(M/xM)/L \subseteq \mathrm{Max}\, R$$

and $(M/xM)/L$ is I-torsion, it follows from [10, Theorem 1.3] that the R-module $(M/xM)/L$ is Artinian. Hence M/xM is a minimax R-module. Now, as

$$\mathrm{Hom}_R(R/I, M/xM)$$

is a finitely generated R-module, it follows from Melkersson's theorem (see Lemma 2.3) that M/xM is I-cofinite. Also, since the R-modules $N = (0 :_M x)$ and M/xM are I-cofinite, it follows from Lemma 2.2 that M is I-cofinite. This completes the inductive step. $\square$

We are now in a position to use the previous result to produce a proof of the main theorem, which is a generalization of the main result of [6, Theorem 1].

Theorem 2.7. *Let I be an ideal of a Noetherian ring R. Let $\mathscr{C}^1(R, I)_{cof}$ denote the category of I-cofinite R-modules M with $\dim M \leq 1$. Then $\mathscr{C}^1(R, I)_{cof}$ is an Abelian category.*

Proof. Let $M, N \in \mathscr{C}^1(R, I)_{cof}$ and let $f : M \longrightarrow N$ be an R-homomorphism. It is enough to show that the R-modules $\ker f$ and $\mathrm{coker} f$ are I-cofinite.

To this end, the exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow \mathrm{im} f \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/I, \ker f) \longrightarrow \mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{im} f)$$

$$\longrightarrow \mathrm{Ext}^1_R(R/I, \ker f) \longrightarrow \mathrm{Ext}^1_R(R/I, M)$$

that implies the R-modules $\mathrm{Hom}_R(R/I, \ker f)$ and $\mathrm{Ext}^1_R(R/I, \ker f)$ are finitely generated. Therefore it follows from Proposition 2.6 that $\ker f$ is I-cofinite. Now, the assertion follows from the exact sequences

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow \mathrm{im} f \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{im} f \longrightarrow N \longrightarrow \mathrm{coker} f \longrightarrow 0.$$

□

As an immediate consequence of Theorem 2.7, we derive the following extension of Delfino-Marley's result in [3] and Kawasaki's result in [6] for an arbitrary Noetherian ring.

Corollary 2.8. *Let I be an ideal of a commutative Noetherian ring R of dimension one. Then the category $\mathscr{M}(R,I)_{cof}$ of I-cofinite modules forms an Abelian subcategory of the category of all R-modules.*

Proof. As $\mathrm{Supp}\, M \subseteq \mathrm{Supp}\, R/I$ for all $M \in \mathscr{M}(R,I)_{cof}$, and $\dim R/I = 1$, it follows that

$$\dim M \leq 1.$$

Now the assertion follows from Theorem 2.7. □

Corollary 2.9. *Let I be an ideal of a commutative Noetherian ring R of dimension one. Let $\mathscr{M}(R,I)_{cof}$ denote the category of I-cofinite modules over R. Let*

$$X^\bullet : \cdots \longrightarrow X^i \xrightarrow{f^i} X^{i+1} \xrightarrow{f^{i+1}} X^{i+2} \longrightarrow \cdots$$

be a complex such that $X^i \in \mathscr{M}(R,I)_{cof}$ for all $i \in \mathbb{Z}$. Then the i^{th} homology module $H^i(X^\bullet)$ is in $\mathscr{M}(R,I)_{cof}$.

Proof. The assertion follows from Corollary 2.8. □

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