PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 4, April 2014, Pages 1399–1412
S 0002-9939(2014)11860-0
Article electronically published on January 16, 2014

STOCHASTIC PERRON'S METHOD AND VERIFICATION WITHOUT SMOOTHNESS USING VISCOSITY COMPARISON: OBSTACLE PROBLEMS AND DYNKIN GAMES

ERHAN BAYRAKTAR AND MIHAI SÎRBU

(Communicated by Edward C. Waymire)

Abstract. We adapt the stochastic Perron's method to the case of double obstacle problems associated to Dynkin games. We construct, symmetrically, a viscosity sub-solution which dominates the upper value of the game and a viscosity super-solution lying below the lower value of the game. If the double obstacle problem satisfies the viscosity comparison property, then the game has a value which is equal to the unique and continuous viscosity solution. In addition, the optimal strategies of the two players are equal to the first hitting times of the two stopping regions, as expected. The (single) obstacle problem associated to optimal stopping can be viewed as a very particular case. This is the first instance of a non-linear problem where the stochastic Perron's method can be applied successfully.

1. Introduction

In [1], the authors introduce a stochastic version of Perron's method to construct viscosity (semi)-solutions for *linear* parabolic (or elliptic) equations and use viscosity comparison as a substitute for verification (Itô's lemma). The present note extends the stochastic Perron's method to the case of (double) obstacle problems associated to games of optimal stopping, the so-called Dynkin games introduced in [2]. This is the first instance of a non-linear problem that can be treated using stochastic Perron and represents a very important step towards treating general stochastic control problems and their corresponding Hamilton-Jacobi-Bellman equations. As a matter of fact, we conjecture that basically any partial differential equation which is related to a stochastic representation can be potentially treated using some modification of what we call the stochastic Perron's method. We intend to present some other important cases in future work.

Received by the editors January 20, 2012 and, in revised form, May 8, 2012.

2010 *Mathematics Subject Classification.* Primary 60G40, 60G46, 60H30; Secondary 35R35, 35K65, 35K10.

Key words and phrases. Perron's method, viscosity solutions, non-smooth verification, comparison principle.

The research of the first author was supported in part by the National Science Foundation under grants DMS 0955463 and DMS 1118673.

The research of the second author was supported in part by the National Science Foundation under grants DMS 0908441 and DMS 1211988.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

1.1. **Overview of existing literature on (games of) optimal stopping.** Optimal stopping and the more general problem of optimal stopping games are fundamental problems in stochastic optimization. Such problems have been well studied for more than fifty years to various degrees of generality, and very general results have been obtained. If the optimal stopping is associated to Markov diffusions, there are two classic approaches to solving the problem:

1. The analytic approach consists of writing the Hamilton-Jacobi-Bellman equation (which takes the form of an obstacle problem), finding a smooth solution and then going over *verification arguments*. The method works only if the solution to the HJB is smooth enough to apply Itô's formula along the diffusion. This is particularly delicate if the diffusion degenerates.

2. The probabilistic approach consists of a very fine analysis of the value function(s), heavily using the Markov property and conditioning, to show a similar conclusion to the analytic approach: it is optimal to stop as soon as the player(s) reach(es) the contact region between the value function and the obstacle. In the case of optimal stopping (only one player) the value function can be characterized as the least excessive (super-harmonic) function. Recently, a similar characterization of the value function was studied for the case of games in [3]. Usually, the probabilistic approach is further used to draw other important conclusions, resembling the analytic approach. More precisely, it can be shown that the value function is a viscosity solution of the HJB. If a comparison result holds, then the value function is *the unique viscosity solution*, and finite-difference numerical methods can be used to approximate it.

1.2. **Our contribution.** Compared to the existing large body of work on optimal stopping (games), we view our contribution as mostly conceptual. Here we provide a new approach that lies *in between the analytic and the probabilistic approaches* described above. More precisely, we propose a *probabilistic version of the analytic approach*. We believe our method is novel in that

1. Compared to the analytic approach, it does not require the existence of a smooth solution. This is because we do not apply Itô's formula to the solution of the PDE, but *only* to the smooth test functions.

2. Compared to the probabilistic approach, we do not perform *any* direct analysis on the value function(s). As a matter of fact, even the very Markov property needed for such analysis is not assumed. The Markov property is hidden behind the uniqueness of the viscosity solution. This is all a consequence of the (same) fact that we apply Itô's lemma to the smooth test functions (as described above) along solutions of SDE without any Markov assumption on the SDE. We believe our method displays a deeper connection between (stopped) diffusions and (viscosity solutions of) free boundary problems. The fine interplay between how much smoothness is needed for a solution of a PDE in order to apply Itô's formula along the SDE (which is needed in the classical analytic approach) is hidden behind the definitions of *stochastic* super- and sub-solutions, which traces back to the seminal work of Stroock and Varadhan [6].

We could summarize the main message of Theorem 2.2 as: if a viscosity comparison result for the HJB holds, then there is no need either to find a smooth solution of the HJB or to analyze the value function(s) to solve the optimization problem. Formally, all classic results hold as expected; i.e., *the unique continuous (but possibly non-smooth) viscosity solution is equal to the value of the game, and*

it is optimal for the players to stop as soon as they reach their corresponding contact/stopping regions. This amounts to a verification without smoothness, in the spirit of the analytic approach to optimal stopping. This resolution of the problem seems shorter (and more elementary) than the probabilistic approach described above. In addition, Theorem 2.2 tells us that the value function is equal to the infimum over stochastic super-solutions or the supremum over stochastic sub-solutions, resembling the probabilistic results in [3].

Compared to the previous work on stochastic Perron's method [1], the contribution of the present note lies in *the precise and proper identification of stochastic sub- and super- solutions for the obstacle problem* (see Definitions 2.1 and 2.2). The technical contribution consists of proving that, having identified such a definition of stochastic solutions, Perron's method actually does produce viscosity super- and sub-solutions. The proofs turn out to be very different from [1]. The short paragraph before Lemma 2.3 singles out the difficulty of trying to follow the results in [1]. As the reader can see below, Definitions 2.1 and 2.2 are tailor-made to fit the proofs, thus avoiding considerations related to Markov processes.

2. The set-up and main results

In order to keep the presentation short and simple, we use a very similar framework (and notation) to the one in [1]. More precisely, the state space is the whole $\mathbb{R}^d$, there is a finite time horizon T, there is no running-cost and no discounting. The obstacle and the terminal pay-off are bounded in order to avoid any issue related to integrability. However, our method works for more general obstacle problems, including all features assumed as described above. In particular, the method can be applied to elliptic obstacle problems rather than parabolic.

Now, fix a time interval $T > 0$ and for each $0 \leq s < T$ and $x \in \mathbb{R}^d$ consider the stochastic differential equation

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, & s \leq t \leq T, \\ X_s = x. \end{cases} \tag{1}$$

We assume that the coefficients $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{M}_{d,d'}(\mathbb{R})$ are continuous. We also assume that for each (s, x), equation (1) has at least a weak non-exploding solution,

$$\Big((X_t^{s,x})_{s\leq t\leq T}, (W_t^{s,x})_{s\leq t\leq T}, \Omega^{s,x}, \mathcal{F}^{s,x}, \mathbb{P}^{s,x}, (\mathcal{F}_t^{s,x})_{s\leq t\leq T}\Big),$$

where the $W^{s,x}$ is a d'-dimensional Brownian motion on the stochastic basis

$$(\Omega^{s,x}, \mathcal{F}^{s,x}, \mathbb{P}^{s,x}, (\mathcal{F}_t^{s,x})_{s\leq t\leq T})$$

and the filtration $(\mathcal{F}_t^{s,x})_{s\leq t\leq T}$ satisfies the usual conditions. We denote by $\mathcal{X}^{s,x}$ the non-empty set of such weak solutions. It is well known, for example from [5], that a sufficient condition for the existence of non-exploding solutions, in addition to continuity of the coefficients, is the condition of linear growth:

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|), \quad (t, x) \in [0, T] \times \mathbb{R}^d.$$

We emphasize that we do *not* assume uniqueness in law of the weak solution. In order to insure that $\mathcal{X}^{s,x}$ is a set in the sense of axiomatic set theory, we restrict ourselves to weak solutions where the probability space Ω is an element of a fixed universal set $\mathcal{S}$ of possible probability spaces. For each $(s, x) \in [0, T] \times \mathbb{R}^d$ we *choose*

a fixed solution $X^{s,x}$ as above, using the axiom of choice. We do not assume that the selection is Markov.

Let $g : \mathbb{R}^d \to \mathbb{R}$ be a *bounded* and measurable function (terminal pay-off). Also let $l, u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be two *bounded* and measurable functions satisfying $l \leq u$. The functions l, u are the lower and the upper obstacles. Assume, in addition, that

$$l(T,\cdot) \leq g \leq u(T,\cdot).$$

For each weak solution $X^{s,x}$ we denote by $\mathcal{T}^{s,x}$ the set of stopping times τ (with respect to the filtration $(\mathcal{F}_t^{s,x})_{s\leq t\leq T}$) which satisfy $s \leq \tau \leq T$. The first player chooses a stopping time $\rho \in \mathcal{T}^{s,x}$ and the second player chooses a stopping time $\tau \in \mathcal{T}^{s,x}$, so that the first player *pays* to the second player the amount

$$J(s,x,\tau,\rho) := \mathbb{E}^{s,x}\left[\mathbb{I}_{\{\tau<\rho\}} l(\tau, X_\tau^{s,x}) + \mathbb{I}_{\{\rho\leq\tau,\rho<T\}} u(\rho, X_\rho^{s,x}) + \mathbb{I}_{\{\tau=\rho=T\}} g(X_T^{s,x})\right].$$

We are now ready to introduce the *lower value of the Dynkin game*,

$$v_*(s,x) := \sup_{\tau\in\mathcal{T}^{s,x}} \inf_{\rho\in\mathcal{T}^{s,x}} J(s,x,\tau,\rho),$$

and the *upper value of the game*,

$$v^*(s,x) := \inf_{\rho\in\mathcal{T}^{s,x}} \sup_{\tau\in\mathcal{T}^{s,x}} J(s,x,\tau,\rho).$$

The lower and the upper values satisfy

$$v_* \leq v^*,$$

and if the two functions coincide, we say that the game has a value.

Remark 2.1. We could refer here to some very general results in the theory of Dynkin games to conclude that the game has a value, i.e. $v_* = v^*$. However, the proof of such results requires the very probabilistic approach we are trying to bypass here. We will obtain this later as a result of verification without smoothness, in Theorem 2.2. At this stage, we cannot even conclude that v_* and v^* are measurable.

If the selection $X^{s,x}$ is actually Markov, we usually associate to the game of optimal stopping the *non-linear* PDE (double obstacle problem)

$$\begin{cases} F(t,x,v,v_t,v_x,v_{xx}) = 0, & \text{on } [0,T)\times\mathbb{R}^d, \\ v(T,\cdot) = g, \end{cases} \tag{2}$$

where

$$\begin{aligned} &F(t,x,v,v_t,v_x,v_{xx}) := \\ &\quad \max\{v-u, \min\{-v_t - L_t v, v-l\}\} = \min\{v-l, \max\{-v_t - L_t v, v-u\}\}, \end{aligned} \tag{3}$$

and the time dependent operator L_t is defined by

$$(L_t u)(x) := \langle b(t,x), \nabla u(t,x)\rangle + \frac{1}{2} Tr(\sigma(t,x)\sigma^T(t,x)\nabla_x^2 u(t,x)), \quad 0 \leq t < T,\ x \in \mathbb{R}^d.$$

The second equality in (3) relies on the assumption that the obstacles are ordered, $l \leq u$.

2.1. **Stochastic Perron's method.** The main *conceptual* contribution of the present note is contained below, in the proper definitions of stochastic super- and sub-solutions of the parabolic PDE (2) in the spirit of [6] and following our previous work in [1]. In order to have a workable definition of a stochastic semi-solution for the (double) obstacle problem, two very important things have to be taken into account:

(i) (conceptual) one has to account for the stochastic sub- and super-solutions along the diffusion to be sub- and super-martingales inside the continuation region(s). However, the diffusion may re-enter the continuation region *after* hitting the stopping region. In other words, the martingale property until the *first hitting time* of the stopping regions *after the initial time* may not be enough. The definition should start at *stopping times* (possibly subsequent to s) rather than at s, where s is the starting time.

(ii) (technical) semi-solutions are evaluated at *stopping times*, so the Optional Sampling Theorem has to be built into the definition of sub- and super-martingales, in case semi-solutions are *less than continuous*. This idea can eliminate an important part of the technicalities in the previous work on the linear case [1] if used directly. However, we choose below a different (and arguably better) route, where semi-solutions of (2) are *continuous*. This can be achieved due to the technical Lemma 2.4 resembling Dini's Theorem. Continuous semi-solutions together with Dini's Theorem can also eliminate the technical difficulties in [1].

Definition 2.1. The set of stochastic super-solutions for the parabolic PDE (2), denoted by $\mathcal{V}^+$, is the set of functions $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ which have the following properties:

(i) They are continuous (C) and bounded on $[0,T] \times \mathbb{R}^d$. In addition, they satisfy $v \geq l$ and the terminal condition $v(T,\cdot) \geq g$.

(ii) For each $(s,x) \in [0,T] \times \mathbb{R}^d$, and any stopping time $\tau_1 \in \mathcal{T}^{s,x}$, the function v along the solution of the SDE is a super-martingale in between τ_1 and the first (after τ_1) hitting time of the upper stopping region $\mathcal{S}^+(v) := \{v \geq u\}$. More precisely, for any $\tau_1 \leq \tau_2 \in \mathcal{T}^{s,x}$, we have

$$v(\tau_1, X^{s,x}_{\tau_1}) \geq \mathbb{E}^{s,x}\left[v(\tau_2 \wedge \rho^+, X^{s,x}_{\tau_2 \wedge \rho^+})|\mathcal{F}^{s,x}_{\tau_1}\right] \ - \mathbb{P}^{s,x} \ a.s.,$$

where the stopping time ρ^+ is defined as

$$\begin{aligned}\rho^+(v,s,x,\tau_1) :&= \inf\{t \in [\tau_1,T] : v(t,X^{s,x}) \geq u(t,X^{s,x})\} \\ &= \inf\{t \in [\tau_1,T] : X^{s,x}_t \in \mathcal{S}^+(v)\}.\end{aligned}$$

Remark 2.2. The super-solution property means that, starting at *any* stopping time $\tau_1 \geq s$, the function along the diffusion is a super-martingale *before* hitting the upper stopping region $\mathcal{S}^+(v) := \{v \geq u\}$. This takes into account the fact that the stopping time τ_1 may be subsequent to the first time the diffusion enters the stopping region $\mathcal{S}^+(v)$ after s. In other words, it accounts for the possibility of re-entering the continuation region. Building in the Optional Sampling Theorem (i.e. considering the later time τ_2 stochastic) does not make a difference, since v is continuous. However, a definition involving stopping times had to be considered anyway, since the starting time τ_1 is stochastic.

In order to simplify the presentation, we would like to explain the notation:

(i) Usually, stopping times relevant to the first (minimizer) player will be denoted by ρ (with different indexes or superscripts), and stopping times relevant to the second (maximizer) player will be denoted by τ (with different indexes or superscripts).
(ii) The superscripts $+$ and $-$ refer to hitting the upper and the lower obstacles, respectively.

Definition 2.2. The set of stochastic sub-solutions for the parabolic PDE (2), denoted by $\mathcal{V}^-$, is the set of functions $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ which have the following properties:

(i) They are continuous (C) and bounded on $[0,T] \times \mathbb{R}^d$. In addition, they satisfy $v \leq u$ and the terminal condition $v(T,\cdot) \leq g$.
(ii) For each $(s,x) \in [0,T] \times \mathbb{R}^d$ and any stopping time $\rho_1 \in \mathcal{T}^{s,x}$, the function v along the solution of the SDE is a sub-martingale in between ρ_1 and the first (after ρ_1) hitting time of the lower stopping region $\mathcal{S}^-(v) := \{v \leq l\}$. More precisely, for each $\rho_1 \leq \rho_2 \in \mathcal{T}^{s,x}$, we have

$$v(\rho_1, X^{s,x}_{\rho_1}) \leq \mathbb{E}^{s,x}\left[v(\rho_2 \wedge \tau^-, X^{s,x}_{\rho_2\wedge\tau^-})|\mathcal{F}^{s,x}_{\rho^1}\right] \ - \mathbb{P}^{s,x} \ a.s.,$$

where the stopping time τ^- is defined as

$$\begin{aligned}\tau^-(v,s,x,\rho_1) :&= \inf\{t \in [\rho_1,T] : v(t,X^{s,x}) \leq l(t,X^{s,x})\}\\ &= \inf\{t \in [\rho_1,T] : X^{s,x} \in \mathcal{S}^-(v)\}.\end{aligned}$$

An identical comment to Remark 2.2 applies to sub-solutions. The next lemma is actually obvious:

Lemma 2.1. *Assume g, l and u are bounded. Then $\mathcal{V}^+$ and $\mathcal{V}^-$ are non-empty.*

Remark 2.3. We decided to work in the framework of bounded l, u and g to minimize technicalities related to integrability. However, our method works in more general situations. If l and u are assumed unbounded, then we need the additional technical assumptions:

(i) $\mathcal{V}^+$ and $\mathcal{V}^-$ are non-empty. This is always the case if l and u are actually continuous.
(ii) For each (s,x) we need that

$$\mathbb{E}^{s,x}\left[\sup_{0\leq t\leq T}\Big(|l(t,X^{s,x}_t)| + |u(t,X^{s,x}_t)|\Big) + |g(X^{s,x}_T)|\right] < \infty.$$

This is the usual assumption made in optimal stopping or Dynkin games.

The next result is quite clear and represents a counterpart to something similar in [1]. However, it needs assumptions on the semi-continuity of the obstacles.

Lemma 2.2.

(i) *If u is lower semi-continuous (LSC), then for each $v \in \mathcal{V}^+$ we have $v \geq v^*$.*
(ii) *If l is upper semi-continuous (USC), then for each $v \in \mathcal{V}^-$ we have $v \leq v_*$.*

Proof. We only prove part (ii) since the other part is symmetric. Using the LSC of v (since v is actually continuous) and the USC of l, we conclude that the set $\mathcal{S}^-(v) = \{v \le l\}$ is closed. This means that, denoting by $\tau^- := \tau^-(v; s, x, s)$, we have

$$\mathbb{I}_{\{\tau^- < \rho\}} v(\tau^-, X^{s,x}_{\tau^-}) \le \mathbb{I}_{\{\tau^- < \rho\}} l(\tau^-, X^{s,x}_{\tau^-})$$

for *each* $\rho \in \mathcal{T}^{s,x}$. Applying the definition of stochastic sub-solutions between the times

$$\rho_1 := s \le \rho_2 := \rho,$$

together with the definition of the cost J and the fact that $v \le u$, we obtain that

$$v(s,x) \le \mathbb{E}^{s,x}[v(\tau^- \wedge \rho, X^{s,x}_{\tau^- \wedge \rho})] \le J(s, x, \tau^-, \rho).$$

Taking the inf over ρ, we conclude that

$$v(s,x) \le \inf_{\rho \in \mathcal{T}^{s,x}} J(s, x, \tau^-, \rho) \le v_*(s,x). \qquad \square$$

We assume now that l is USC and u is LSC. Following [1] and using Lemmas 2.1 and 2.2, we define

$$v^- := \sup_{v \in \mathcal{V}^-} v \le v_* \le v^* \le v^+ := \inf_{w \in \mathcal{V}^+} w.$$

The next result is actually the main result of the paper.

Theorem 2.1 (Stochastic Perron's method)**.**

(i) *Assume that g, l are USC and u is LSC. Assume, in addition, that*

$$\text{there exists } v \in \mathcal{V}^+ \text{ such that } v \le u. \tag{4}$$

Then, v^+ is a bounded and USC viscosity sub-solution of

$$\begin{cases} F(t, x, v, v_t, v_x, v_{xx}) \le 0 \text{ on } [0,T) \times \mathbb{R}^d, \\ v(T, \cdot) \le g. \end{cases} \tag{5}$$

(ii) *Assume g, u are LSC and l is USC. Assume, in addition, that*

$$\text{there exists } v \in \mathcal{V}^- \text{ such that } v \ge l. \tag{6}$$

Then, v^- is a bounded and LSC viscosity super-solution of

$$\begin{cases} F(t, x, v, v_t, v_x, v_{xx}) \ge 0 \text{ on } [0,T) \times \mathbb{R}^d, \\ v(T, \cdot) \ge g. \end{cases} \tag{7}$$

Remark 2.4. Assumption (4) is satisfied if the upper obstacle u is continuous, and assumption (6) is satisfied if the lower obstacle l is continuous.

We have $v^+(T, \cdot) \ge g$ and $v^-(T, \cdot) \le g$ by construction. Therefore, the terminal conditions in (5) and (7) can be replaced by equalities. The proof of Theorem 2.1 is technically different from the proof in [1]. The main reason is that we cannot prove directly that

$$v^+ \in \mathcal{V}^+, \quad v^- \in \mathcal{V}^-,$$

even if we weaken the continuity of super-solutions in $\mathcal{V}^+$ to USC and the continuity of sub-solutions in $\mathcal{V}^-$ to LSC. This technical hurdle is circumvented by the weaker lemma below, together with an approximation argument in Lemma 2.4.

Lemma 2.3.

(i) *Assume u is LSC. If $v^1, v^2 \in \mathcal{V}^+$, then $v^1 \wedge v^2 \in \mathcal{V}^+$.*
(ii) *Assume l is USC. If $v^1, v^2 \in \mathcal{V}^-$, then $v^1 \vee v^2 \in \mathcal{V}^-$.*

Proof. We prove item (ii). It is easy to see that $v := v^1 \vee v^2$ is continuous as well as $v \leq u$ and $v(T,\cdot) \leq g$. The only remaining thing to prove is for each $(s,x) \in [0,T] \times \mathbb{R}^d$, and any stopping times

$$\rho_1 \leq \rho_2 \in \mathcal{T}^{s,x},$$

$$v(\rho_1, X^{s,x}_{\rho_1}) \leq \mathbb{E}^{s,x}\left[v(\rho_2 \wedge \tau^-, X^{s,x}_{\rho_2\wedge\tau^-})|\mathcal{F}^{s,x}_{\rho_1}\right] \ -\mathbb{P}^{s,x}\ a.s., \tag{8}$$

where the stopping time τ^- is defined by $\tau^- = \tau^-(v; s, x, \rho^1)$. In other words, we want to prove the sub-martingale property of v along the process $X^{s,x}$ in between the stopping times $\rho_1 \leq \rho_2 \wedge \tau^-$. The idea of the proof relies on a sequence of stopping times $(\gamma_n)_{n\geq 0}$ defined recursively as follows: set $\gamma_0 = \rho_1$ and then, for each $n = 0, 1, 2, \ldots$:

(i) If $v(\gamma_n, X_{\gamma_n}) \leq l(\gamma_n, X_{\gamma_n})$, then $\gamma_{n+1} := \gamma_n$.

(ii) If $v(\gamma_n, X_{\gamma_n}) = v^1(\gamma_n, X_{\gamma_n}) > l(\gamma_n, X_{\gamma_n})$, then

$$\gamma_{n+1} := \inf\{t \in [\gamma_n, T] : v^1(t, X_t) \leq l(t, X_t)\}.$$

In this case, we note that $v^1(\gamma_{n+1}, X_{\gamma_{n+1}}) \leq v(\gamma_{n+1}, X_{\gamma_{n+1}})$ and $v^1(\gamma_{n+1}, X_{\gamma_{n+1}}) \leq l(\gamma_{n+1}, X_{\gamma_{n+1}})$.

(iii) If $v(\gamma_n, X_{\gamma_n}) = v^2(\gamma_n, X_{\gamma_n}) > l(\gamma_n, X_{\gamma_n})$, then

$$\gamma_{n+1} := \inf\{t \in [\gamma_n, T] : v^2(t, X_t) \leq l(t, X_t)\}.$$

In this case, we note that $v^2(\gamma_{n+1}, X_{\gamma_{n+1}}) \leq v(\gamma_{n+1}, X_{\gamma_{n+1}})$ and $v^2(\gamma_{n+1}, X_{\gamma_{n+1}}) \leq l(\gamma_{n+1}, X_{\gamma_{n+1}})$.

One can use the definition of a stochastic sub-solution for v^1 in between the times $\gamma_n \leq \gamma_{n+1}$, together with the observation following item (ii), or the definition of a stochastic sub-solution for v^2 in between $\gamma_n \leq \gamma_{n+1}$ and the observation following item (iii), to conclude that, for any $n \geq 0$, $v(t, X_t)$ satisfies the sub-martingale property in between $\gamma_n \leq \gamma_{n+1}$. Concatenating, we conclude that for each n, we have

$$v(\rho_1, X^{s,x}_{\rho_1}) \leq \mathbb{E}\left[v(\rho_2 \wedge \gamma_n, X^{s,x}_{\rho_2\wedge\gamma_n})|\mathcal{F}^{s,x}_{\rho_1}\right]. \tag{9}$$

Now, care must be taken in order to pass to the limit as $n \to \infty$. By construction, it is clear that $\gamma_n \leq \tau^-$. In the event

$$A := \{(\exists)\ n_0,\ \ \gamma_n = \tau^-,\ n \geq n_0\},$$

it is very easy to pass to the limit, since the sequence is eventually constant. However, in the complementary event

$$B := \{\gamma_n < \tau^-,\ \ (\forall)\ n\},$$

we have to be more careful. Depending on parity, for each $\omega \in B$, there exist $n_0 = n_0(\omega)$ such that

$$v^1(\gamma_{n_0+2k}, X^{s,x}_{\gamma_{n_0+2k}}) \leq l(\gamma_{n_0+2k}, X^{s,x}_{\gamma_{n_0+2k}}) \text{ for } k \geq 0$$

and

$$v^2(\gamma_{n_0+2k+1}, X^{s,x}_{\gamma_{n_0+2k+1}}) \leq l(\gamma_{n_0+2k+1}, X^{s,x}_{\gamma_{n_0+2k+1}}) \text{ for } k \geq 0.$$

This comes from the very definition of the sequence γ_n. Since both v^1 and v^2 are LSC (actually continuous) and l is USC, we can pass to the limit in both inequalities above to conclude that for $\gamma_\infty := \lim_n \gamma_n$ we have

$$v(\gamma_\infty, X^{s,x}_{\gamma_\infty}) \leq l(\gamma_\infty, X^{s,x}_{\gamma_\infty}) \text{ on } B.$$

To begin with, this shows that $\gamma_\infty \geq \tau^-$, so $\gamma_\infty = \tau^-$ on B. However, $\gamma_\infty = \tau^-$ as well on A. Now, we let $n \to \infty$ in (9) using the Bounded Convergence Theorem to finish the proof. This proof uses essentially the continuity of v^1 and v^2. However, using more technical arguments, one could prove something similar assuming only the LSC property for stochastic sub-solutions. □

Proof of Theorem 2.1. We will only prove that v^+ is a sub-solution of (5); the other part is symmetric.

Step 1. *The interior sub-solution property.* Note that we already know that v^+ is bounded and upper semi-continuous (USC). Let

$$\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$$

be a $C^{1,2}$-test function function and assume that $v^+ - \varphi$ attains a strict local maximum (an assumption which is not restrictive) equal to zero at some parabolic interior point $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$. Assume that v^+ does not satisfy the viscosity sub-solution property, i.e.

$$\min\{\varphi(t_0,x_0) - l(t_0,x_0), \max\{-\varphi_t(t_0,x_0) - L_t\varphi(t_0,x_0), \varphi(t_0,x_0) - u(t_0,x_0)\}\} > 0.$$

According to assumption (4) (and this is actually the only place where the assumption is used), we have $v^+ \leq u$. This means

$$v^+(t_0,x_0) > l(t_0,x_0)$$

and

$$-\varphi_t(t_0,x_0) - L_t\varphi(t_0,x_0) > 0.$$

Since the coefficients of the SDE are continuous, we conclude that there exists a small enough ball $B(t_0,x_0,\varepsilon)$ such that

$$-\varphi_t - L_t\varphi > 0 \text{ on } \overline{B(t_0,x_0,\varepsilon)}$$

and

$$\varphi > v^+ \text{ on } \overline{B(t_0,x_0,\varepsilon)} - (t_0,x_0).$$

In addition, since $\varphi(t_0,x_0) = v^+(t_0,x_0) > l(t_0,x_0)$ and φ is continuous and l is USC, we conclude that if ε is small enough, then

$$\varphi - \varepsilon \geq l \text{ on } \overline{B(t_0,x_0,\varepsilon)}.$$

Since $v^+ - \varphi$ is upper semi-continuous and $\overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2)$ is compact, this means that there exists a $\delta > 0$ such that

$$\varphi - \delta \geq v^+ \text{ on } \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2).$$

The next lemma is the fundamental step in the proof by approximation (and represents a major technical difference compared to the previous paper [1]). This is the result that actually allows us to work with stochastic semi-solutions which are *continuous*.

Lemma 2.4. *Let $0 < \delta' < \delta$. Then there exists a stochastic super-solution $v \in \mathcal{V}^+$ such that*

$$\varphi - \delta' \geq v \text{ on } \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2).$$

Proof of Lemma 2.4. Using Lemma 2.3 together with the result in the Appendix of [1], we can choose a *decreasing* sequence $(v_n)_{n\geq 0} \subset \mathcal{V}^+$ of stochastic super-solutions such that

$$v_n \searrow v^+.$$

Now the proof follows the idea of Dini's Theorem. More precisely, we denote

$$A_n := \{v_n \geq \varphi - \delta'\} \cap \Big(\overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2)\Big).$$

We have that $A_{n+1} \subset A_n$ and $\bigcap_{n=0}^{\infty} A_n = \emptyset$. In addition, since v_n is USC (being continuous) and φ is continuous as well, each A_n is closed. By compactness, we get that there exists an n_0 such that $A_{n_0} = \emptyset$, which means that

$$\varphi - \delta' > v_{n_0}, \text{ on } \overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2).$$

We now choose $v := v_{n_0}$. □

We finish the proof of the main theorem as follows. Let $v \in \mathcal{V}^+$ be given by Lemma 2.4. Choosing $0 < \eta < \delta' \wedge \varepsilon$ small enough we have that the function

$$\varphi_\eta := \varphi - \eta$$

satisfies the properties

$$-\varphi_t^\eta - L_t\varphi^\eta > 0 \text{ on } \overline{B(t_0, x_0, \varepsilon)},$$
$$\varphi^\eta > v \text{ on } \overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2),$$
$$\varphi^\eta \geq l \text{ on } \overline{B(t_0, x_0, \varepsilon)},$$

and

$$\varphi^\eta(t_0, x_0) = v^+(t_0, x_0) - \eta.$$

Now, we define the new function

$$v^\eta = \begin{cases} v \wedge \varphi^\eta \text{ on } \overline{B(t_0, x_0, \varepsilon)}, \\ v \text{ outside } \overline{B(t_0, x_0, \varepsilon)}. \end{cases}$$

We clearly have that v^η is continuous and $v^\eta(t_0, x_0) = \varphi^\eta(t_0, x_0) < v^+(t_0, x_0)$. Also, v^η satisfies the terminal condition (since ε can be chosen so that $T > t_0 + \varepsilon$ and v satisfies the terminal condition). It remains to show only that $v^\eta \in \mathcal{V}^+$ to obtain a contradiction.

We need to show that the process $(v^\eta(t, X_t^{s,x}))_{s\leq t\leq T}$ is a super-martingale on $(\Omega^{s,x}, \mathbb{P}^{s,x})$ with respect to the filtration $(\mathcal{F}_t^{s,x})_{s\leq t\leq T}$ in the *upper continuation region* $C^+ := \{v^\eta < u\}$, i.e. satisfies item (ii) in Definition 2.1.

The sketch of how to prove this can be given in three steps:

1. The process $(v^\eta(t, X_t^{s,x}))_{s\leq t\leq T}$ is a super-martingale locally in the region $\Big([s, T] \times \mathbb{R}^d - B(t_0, x_0, \varepsilon/2)\Big) \cap C^+$ because it coincides there with the process $(v(t, X_t^{s,x}))_{s\leq t\leq T}$, which is a super-martingale in C^+ (in the sense of Definition 2.1).

2. In addition, in the region $\overline{B(t_0, x_0, \varepsilon)}$ the process $(v^\eta(t, X_t^{s,x}))_{s\leq t\leq T}$ is the minimum between a local super-martingale (φ^η) and a local super-martingale as long as $v < u$. One needs to perform an *identical argument* to the sequence of stopping times in the proof of Lemma 2.3 to get that we have a super-martingale in $\overline{B(t_0, x_0, \varepsilon)}$.

3. The two items above can be easily concatenated as in the proof of Theorem 2.1 in [1]. More precisely, we exploit the fact that the two regions in items 1 and 2 overlap over the "strip" $\overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2)$, so the concatenating sequence

is the sequence of "up and down-crossings" of this "strip". This concludes the proof of the interior sub-solution property.

Step 2. *The terminal condition.* Assume that for some $x_0 \in \mathbb{R}^d$ we have $v^+(T, x_0) > g(x_0) \geq l(T, x_0)$. We want to use this information in a similar way to Step 1 to construct a contradiction. Since g and l are USC, there exists an $\varepsilon > 0$ such that

$$\max\{l(t,x), g(x)\} \leq v^+(T, x_0) - \varepsilon \quad \text{if} \quad \max\{|x - x_0|,\ T - t\} \leq \varepsilon.$$

We now use the fact that v^+ is USC to conclude it is bounded above on the compact set

$$\left(\overline{B(T, x_0, \varepsilon)} - B(T, x_0, \varepsilon/2)\right) \cap ([0,T] \times \mathbb{R}^d).$$

Choose $\eta > 0$ small enough so that

$$v^+(T, x_0) + \frac{\varepsilon^2}{4\eta} > \varepsilon + \sup_{(t,x) \in (\overline{B(T,x_0,\varepsilon)} - B(T,x_0,\varepsilon/2)) \cap ([0,T] \times \mathbb{R}^d)} v^+(t,x). \tag{10}$$

Using a (variant of) Lemma 2.4 we can find a $v \in \mathcal{V}^+$ such that

$$v^+(T, x_0) + \frac{\varepsilon^2}{4\eta} > \varepsilon + \sup_{(t,x) \in (\overline{B(T,x_0,\varepsilon)} - B(T,x_0,\varepsilon/2)) \cap ([0,T] \times \mathbb{R}^d)} v(t,x). \tag{11}$$

We now define, for $k > 0$, the following function:

$$\varphi^{\eta,\varepsilon,k}(t,x) = v^+(T, x_0) + \frac{|x - x_0|^2}{\eta} + k(T - t).$$

For k large enough (but no smaller than $\varepsilon/2\eta$), we have that

$$-\varphi_t^{\varepsilon,\eta,k} - \mathcal{L}_t \varphi^{\varepsilon,\eta,k} > 0, \quad \text{on } \overline{B(T, x_0, \varepsilon)}.$$

We would like to emphasize that, for convenience, we work here with the norm

$$\|(t,x)\| = \max\{|t|, \|x\|\}, (t,x) \in \mathbb{R} \times \mathbb{R}^d,$$

where $\|x\|$ is the Euclidean norm on $\mathbb{R}^d$. With this particular norm, we can use (11) to obtain

$$\varphi^{\varepsilon,\eta,k} \geq \varepsilon + v \text{ on } (\overline{B(T, x_0, \varepsilon)} - B(T, x_0, \varepsilon/2)) \cap ([0,T] \times \mathbb{R}^d).$$

Also,

$$\varphi^{\varepsilon,\eta,k}(T,x) \geq v^+(T, x_0) \geq g(x) + \varepsilon \text{ for } |x - x_0| \leq \varepsilon$$

and

$$\varphi^{\varepsilon,\eta,k}(t,x) \geq v^+(T, x_0) \geq l(t,x) + \varepsilon \ \text{ for } \max\{|x - x_0|, T - t\} \leq \varepsilon.$$

Now, we can choose $\delta < \varepsilon$ and define

$$v^{\varepsilon,\eta,k,\delta} = \begin{cases} v \wedge \left(\varphi^{\varepsilon,\eta,k} - \delta\right) \text{ on } \overline{B(T, x_0, \varepsilon)}, \\ v \text{ outside } \overline{B(T, x_0, \varepsilon)}. \end{cases}$$

Again using the ideas in items 1-3 in Step 1 of the proof, we can show that $v^{\varepsilon,\eta,k,\delta} \in \mathcal{V}^+$ but $v^{\varepsilon,\eta,k,\delta}(T, x_0) = v^+(T, x_0) - \delta < v^+(T, x_0)$, leading to a contradiction. $\square$

2.2. **Verification by comparison.**

Definition 2.3. We say that the viscosity comparison principle holds for equation (2) if, whenever we have a bounded, upper semi-continuous (USC) sub-solution v of (5) and a bounded lower semi-continuous (LSC) super-solution w of (7), then $v \leq w$.

Theorem 2.2. *Let l, u, g be bounded such that l is USC, u is LSC and g is continuous. Assume that conditions* (4) *and* (6) *hold. Assume also that the comparison principle is satisfied. Then there exists a unique bounded and continuous viscosity solution v to* (2) *which equals both the lower and the upper values of the game, which means*

$$v_* = v = v^*.$$

In addition, for each $(s, x) \in [0, T] \times \mathbb{R}^d$, the stopping times

$$\rho^*(s, x) = \rho^+(v, s, x, s) \text{ and } \tau^*(s, x) = \tau^-(v, s, x, s)$$

are optimal for the two players.

Proof. It is clear that the unique and continuous viscosity solution of (2) is

$$v^- = v_* = v^* = v^+.$$

The only thing to be proved is that

$$\rho^*(s, x) = \rho^+(v, s, x, s), \quad \tau^*(s, x) = \tau^-(v, s, x, s)$$

are optimal strategies. Let v_n be an increasing sequence such that $v_n \in \mathcal{V}^-$ and $v^- = \sup_n v_n$. According to the proof of Lemma 2.2, if we define

$$\tau_n^- := \tau^-(v_n, s, x, s),$$

then for each $\rho \in \mathcal{T}^{s,x}$ we have

$$v_n(s, x) \leq \mathbb{E}\left[v_n(\tau_n^- \wedge \rho, X^{s,x}_{\tau_n^- \wedge \rho})\right] \leq J(s, x, \tau_n^-, \rho).$$

Using the definition of J, together with the fact that the lower obstacle l is USC, $l \leq u$ and $l(T, \cdot) \leq g \leq u(T, \cdot)$, we can pass to the limit to conclude

$$v^-(s, x) \leq J(s, x, \tau_\infty^-, \rho), \quad \text{for all } \rho \in \mathcal{T}^{s,x},$$

where

$$\tau_\infty^- := \lim_n \tau_n^- \leq \tau^*.$$

The inequality $\tau_n^- \leq \tau^*$ is obvious, taking into account the definition of the two stopping times and the fact that $v_n \leq v$. Now, since v_n are LSC (actually continuous) and converge increasingly to the *continuous* function $v = v^-$, we can use Dini's Theorem to conclude that the convergence is uniform on compacts $|x| \leq N$. Now, we have

$$v_n(\tau_n^-, X^{s,x}_{\tau_n^-}) \leq l(\tau_n^-, X^{s,x}_{\tau_n^-}) \text{ on } \{\tau_n^- < T\} \supset \{\tau_\infty^- < T\}.$$

Since $v_n \nearrow v$ uniformly on compacts and l is USC, we can pass to the limit, first in n, to conclude that

$$v(\tau_\infty^-, X^{s,x}_{\tau_\infty^-}) \leq l(\tau_\infty^-, X^{s,x}_{\tau_\infty^-}) \text{ on } \{\tau_\infty^- < T\} \cap \{|X_t^{s,x}| \leq N \ (\forall)\ t\},$$

and then let $N \to \infty$ to obtain

$$v(\tau_\infty^-, X^{s,x}_{\tau_\infty^-}) \leq l(\tau_\infty^-, X^{s,x}_{\tau_\infty^-}) \text{ on } \{\tau_\infty^- < T\}.$$

This shows that $\tau^* \leq \tau_\infty^-$, so $\tau^* = \tau_\infty^-$. To sum up, we have that

$$v(s,x) \leq J(s,x,\tau^*,\rho), \quad \text{for all } \rho \in \mathcal{T}^{s,x}.$$

Similarly, we can prove

$$v(s,x) \geq J(s,x,\tau,\rho^*), \quad \text{for all } \tau \in \mathcal{T}^{s,x},$$

and the proof is complete, showing that ρ^* and τ^* are optimal strategies for the two players. □

Remark 2.5. The regularity needed in our main results, Theorems 2.1 and 2.2, is the minimal regularity needed to insure that the stopping regions are closed and the continuation regions are open. However, unless one has a direct way to check (4) and (6), the opposite semi-continuity of the obstacles has to be assumed, according to Remark 2.4. Theorem 2.2 can therefore be applied in general when the obstacles are continuous.

3. Optimal stopping and (single) obstacle problems

The case of optimal stopping can be easily treated as above, assuming, *formally*, that the upper obstacle is infinite: $u = +\infty$. However, in order to have meaningful results, Definitions 2.1 and 2.2 have to be modified accordingly, together with the main Theorems 2.1 and 2.2, noting that $\rho^- = T$; i.e. the minimizer player never stops the game before maturity.

Having this in mind, a stochastic super-solution is, therefore, a *continuous excessive* function in the lingo of optimal stopping for Markov processes. We remind the reader that we did *not* assume the selection $X^{s,x}$ to be Markov, though.

One could show easily that $v^+ \in \mathcal{V}^+$ if the continuity in the definition of $\mathcal{V}^+$ were relaxed to USC. Under this USC (only) framework, the corresponding Theorem 2.1, part (i), can be translated as: *the least excessive function is a viscosity sub-solution of the obstacle problem.* While it is known in the Markov case that the value function is the least excessive function (see [4]), we *never* analyze the value function *directly* in our approach. We cannot show directly that $v^- \in \mathcal{V}^-$. Viscosity comparison is necessary to close the argument.

All the proofs of the modified Theorems 2.1 and 2.2 work identically, up to obvious modifications.

References

[1] E. Bayraktar and M. Sîrbu, *Stochastic Perron's method and verification without smoothness using viscosity comparison: the linear case*, Proceedings of the AMS **140** (2012), no. 10, 3645–3654. MR2929032

[2] E. B. Dynkin, *A game-theoretic version of an optimal stopping problem*, Dokl. Akad. Nauk SSSR **185** (1969), 16–19 (Russian). MR0241121 (39 #2466)

[3] G. Peskir, *Optimal stopping games and Nash equilibrium*, Teor. Veroyatn. Primen. **53** (2008), no. 3, 623–638 (English, with Russian summary); English transl., Theory Probab. Appl. **53** (2009), no. 3, 558–571, DOI 10.1137/S0040585X97983821. MR2759714 (2012g:91042)

[4] A. N. Shiryaev, *Optimal stopping rules*, Stochastic Modelling and Applied Probability, vol. 8, Springer-Verlag, Berlin, 2008. Translated from the 1976 Russian second edition by A. B. Aries. Reprint of the 1978 translation. MR2374974 (2008m:60003)

[5] Daniel W. Stroock and S. R. Srinivasa Varadhan, *Multidimensional diffusion processes*, Classics in Mathematics, Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition. MR2190038 (2006f:60005)

[6] D. Stroock and S. R. S. Varadhan, *On degenerate elliptic-parabolic operators of second order and their associated diffusions*, Comm. Pure Appl. Math. **25** (1972), 651–713. MR0387812 (52 #8651)

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, Michigan 48109

E-mail address: erhan@umich.edu.

Department of Mathematics, University of Texas at Austin, 1 University Station C1200, Austin, Texas 78712

E-mail address: sirbu@math.utexas.edu.