PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 5, May 2014, Pages 1513–1526
S 0002-9939(2014)11932-0
Article electronically published on February 13, 2014

PROPER TWIN-TRIANGULAR $\mathbb{G}_a$-ACTIONS ON $\mathbb{A}^4$ ARE TRANSLATIONS

ADRIEN DUBOULOZ AND DAVID R. FINSTON

(Communicated by Harm Derksen)

Dedicated to Jim Deveney on the occasion of his retirement

Abstract. An additive group action on an affine 3-space over a complex Dedekind domain A is said to be twin-triangular if it is generated by a locally nilpotent derivation of $A[y, z_1, z_2]$ of the form $r\partial_y + p_1(y)\partial_{z_1} + p_2(y)\partial_{z_2}$, where $r \in A$ and $p_1, p_2 \in A[y]$. We show that these actions are translations if and only if they are proper. Our approach avoids the computation of rings of invariants and focuses more on the nature of geometric quotients for such actions.

Introduction

In 1968, Rentschler [16] established in a pioneering work that every algebraic action of the additive group $\mathbb{G}_a = \mathbb{G}_{a,\mathbb{C}}$ on the complex affine space $\mathbb{A}^2$ is triangular in a suitable polynomial coordinate system. Consequently, every set-theoretically free $\mathbb{G}_a$-action is a translation, in the sense that $\mathbb{A}^2$ is equivariantly isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1$ where $\mathbb{G}_a$ acts by translations on the second factor. An example due to Bass [2] in 1984 shows that in higher dimensions, $\mathbb{G}_a$-actions are no longer triangulable in general, and in 1990 Winkelmann [19] constructed a set-theoretically free $\mathbb{G}_a$-action on $\mathbb{A}^4$ which is not a translation. The question about set-theoretically free $\mathbb{G}_a$-actions on $\mathbb{A}^3$ was eventually settled affirmatively first by Deveney and the second author [5] in 1994 under the additional assumption that the action is proper, and then in general by Kaliman [14] in 2004.

For proper actions, the argument turns out to be much simpler than the general one, the crucial fact being that combined with the flatness of the algebraic quotient morphism $\pi : \mathbb{A}^3 \to \mathbb{A}^3/\!/\mathbb{G}_a = \mathrm{Spec}(\Gamma(\mathbb{A}^3, \mathcal{O}_{\mathbb{A}^3})^{\mathbb{G}_a})$ which is obtained from dimension considerations, properness implies that the action is locally trivial in the Zariski topology, i.e. that $\mathbb{A}^3$ is covered by invariant Zariski affine open subsets of the form $V_i = U_i \times \mathbb{A}^1$ on which $\mathbb{G}_a$ acts by translations on the second factor. The factoriality of $\mathbb{A}^3$ implies in turn that a geometric quotient $\mathbb{A}^3/\mathbb{G}_a$ exists as a quasi-affine open subset of $\mathbb{A}^3/\!/\mathbb{G}_a \simeq \mathbb{A}^2$ with at most finite complement, and the equality $\mathbb{A}^3/\mathbb{G}_a = \mathbb{A}^3/\!/\mathbb{G}_a$ ultimately follows by comparing Euler characteristics.

Received by the editors October 14, 2011 and, in revised form, June 8, 2012.
2010 *Mathematics Subject Classification.* Primary 14R20, 14L30.
This research was supported in part by NSF Grant OISE-0936691 and ANR Grant 08-JCJC-0130-01.

Local triviality in the Zariski topology is actually a built-in property of proper $\mathbb{G}_a$-actions on smooth algebraic varieties of dimension less than four. Indeed, recall that an action $\mu : \mathbb{G}_a \times X \to X$ on an algebraic variety X is said to be proper if the morphism $\mu \times \mathrm{pr}_2 : \mathbb{G}_a \times X \to X \times X$ is proper, in this context in fact a closed immersion since $\mathbb{G}_a$ has no nontrivial algebraic subgroup. Being in particular set-theoretically free, such an action is then locally trivial in the étale topology; i.e., there exists an étale covering $U \times \mathbb{G}_a \to X$ of X which is equivariant for the action of $\mathbb{G}_a$ on $U \times \mathbb{G}_a$ by translations on the second factor. This implies that a geometric quotient exists in the category of algebraic spaces in the form of an étale locally trivial $\mathbb{G}_a$-bundle $\rho : X \to X/\mathbb{G}_a$ over a certain algebraic space $X/\mathbb{G}_a$, the properness of μ then being equivalent to the separatedness of $X/\mathbb{G}_a$ (see e.g. [15]). Now if X is smooth of dimension at most three, then $X/\mathbb{G}_a$ is a smooth separated algebraic space of dimension at most two, whence a quasi-projective variety by virtue of Chow's Lemma. Since $\mathbb{G}_a$ is a special group, the $\mathbb{G}_a$-bundle $\rho : X \to X/\mathbb{G}_a$ is then in fact locally trivial in the Zariski topology on $X/\mathbb{G}_a$, which yields the Zariski local triviality of the $\mathbb{G}_a$-action on X.

For $\mathbb{G}_a$-actions on higher dimensional affine spaces, properness fails in general to imply Zariski local triviality and Zariski local triviality is no longer sufficient to guarantee that a proper $\mathbb{G}_a$-action is a translation. In particular, starting from dimension 5, there exist proper triangular $\mathbb{G}_a$-actions which are not Zariski locally trivial [6] and proper triangular, Zariski locally trivial actions with strictly quasi-affine geometric quotients [19]. But the question whether a proper $\mathbb{G}_a$-action on $\mathbb{A}^4$ is a translation or at least Zariski locally trivial remains open, and very little progress has been made in the study of these actions during the last decade. The only existing partial results so far concern triangular $\mathbb{G}_a$-actions: Deveney, van Rossum and the second author [9] established in 2004 that a Zariski locally trivial triangular $\mathbb{G}_a$-action on $\mathbb{A}^4$ is in fact a translation. The proof depends on the very particular structure of the ring of invariants for such actions and hence cannot be adapted to more general actions. The second positive result concerns a special type of triangular $\mathbb{G}_a$-action called *twin-triangular,* corresponding to locally nilpotent derivations of $\mathbb{C}[x, y, z_1, z_2]$ of the form $\partial = r(x)\partial_y + p_1(x, y)\partial_{z_1} + p_2(x, y)\partial z_2$ where $r(x) \in \mathbb{C}[x]$ and $p_1(x, y), p_2(x, y) \in \mathbb{C}[x, y]$. It was established by Deveney and the second author [7] that a proper twin-triangular $\mathbb{G}_a$-action corresponding to a derivation for which the polynomial $r(x)$ has simple roots is a translation. This was accomplished by explicitly computing the invariant ring $\mathbb{C}[x, y, z_1, z_2]^{\mathbb{G}_a}$ and investigating the structure of the algebraic quotient morphism $\mathbb{A}^4 \to \mathbb{A}^4/\!/\mathbb{G}_a = \mathrm{Spec}(\mathbb{C}[x, y, z_1, z_2]^{\mathbb{G}_a})$. While a result of Daigle and Freudenburg [4] gives finite generation of $\mathbb{C}[x, y, z_1, z_2]^{\mathbb{G}_a}$ for arbitrary triangular $\mathbb{G}_a$-actions, there is no a priori bound on the number of its generators, and the simplicity of the roots of $r(x)$ was crucial to achieve the computation of these rings.

Here we consider the more general case of twin-triangular actions of $\mathbb{G}_a = \mathbb{G}_{a,X} = \mathbb{G}_{a,\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} X$ on an affine space $\mathbb{A}^3_X$ over the spectrum X of a complex Dedekind domain A. Removing in particular the condition on simplicity of the roots of r, we show that a proper $\mathbb{G}_a$-action on $\mathbb{A}^3_X$ generated by an A-derivation of $A[y, z_1, z_2]$ of the form $\partial = r\partial_y + p_1(y)\partial_{z_1} + p_2(y)\partial_{z_2}$, $r \in A$, $p_1, p_2 \in A[y]$ is a translation; i.e. the geometric quotient $\mathbb{A}^3_X/\mathbb{G}_a$ is X-isomorphic to $\mathbb{A}^2_X$ and $\mathbb{A}^3_X$ is equivariantly isomorphic to $\mathbb{A}^3_X/\mathbb{G}_a \times_X \mathbb{G}_a$ where $\mathbb{G}_a$ acts by translation on the second factor. Even though finite generation of the rings of invariant for triangular A-derivations

of $A[y, z_1, z_2]$ holds in this more general setting thanks to the aforementioned result of Daigle and Freudenburg, our approach avoids the computation of these rings and focuses more on the nature of the geometric quotients $\mathbb{A}^3_X/\mathbb{G}_a$. As noted before, these quotients a priori exist only as separated algebraic spaces and the crucial step is to show that for the actions under consideration they are in fact schemes, or, equivalently, that proper twin-triangular $\mathbb{G}_a$-actions on $\mathbb{A}^3_X$ are not only locally trivial in the étale topology but also in the Zariski topology. Indeed, if so, then a straightforward generalization of the aforementioned result of Deveney, van Rossum and the second author shows that such Zariski locally trivial triangular $\mathbb{G}_a$-actions are in fact translations.

To explain the main idea of our proof, let us assume for simplicity that $A = \mathbb{C}[x]_{(x)}$ and consider a triangular A-derivation $\partial = x^n\partial_y + p_1(y)\partial_{z_1} + p_2(y, z_1)\partial_{z_2}$ of $A[y, z_1, z_2]$ generating a proper action on $\mathbb{A}^3_X$ that we denote by $\mathbb{G}_{a,\partial}$. Being triangular, the action of $\mathbb{G}_{a,\partial}$ commutes with the action $\mathbb{G}_{a,\partial_{z_2}}$ defined by the partial derivative ∂_{z_2} and descends to an action on $\mathbb{A}^2_X = \mathrm{Spec}(A[y, z_1]) \simeq \mathbb{A}^3_X/\mathbb{G}_{a,\partial_{z_2}}$ corresponding with that generated by the derivation $x^n\partial_y + p_1(y)\partial_{z_1}$. Similarly, the action of $\mathbb{G}_{a,\partial_{z_2}}$ on $\mathbb{A}^3_X$ descends to the geometric quotient $\mathbb{A}^3_X/\mathbb{G}_{a,\partial}$. These induced actions are in general no longer set-theoretically free, but if we take the quotient of $\mathbb{A}^2_X$ by $\mathbb{G}_{a,\partial}$ as an algebraic stack $[\mathbb{A}^2_X/\mathbb{G}_{a,\partial}]$, we obtain a cartesian square

$$\begin{tikzcd} \mathbb{A}^3_X \arrow[r] \arrow[d, "\mathrm{pr}_{y,z_1}"'] & \mathbb{A}^3_X/\mathbb{G}_{a,\partial} \arrow[d] \\ \mathbb{A}^2_X \arrow[r] & {[\mathbb{A}^2_X/\mathbb{G}_{a,\partial}]} \end{tikzcd}$$

which identifies $[\mathbb{A}^2_X/\mathbb{G}_{a,\partial}]$ with the algebraic stack quotient $[(\mathbb{A}^3_X/\mathbb{G}_{a,\partial})/\mathbb{G}_{a,\partial_{z_2}}]$. In this setting, the Zariski local triviality of a proper triangular $\mathbb{G}_a$-action on $\mathbb{A}^3_X$ becomes equivalent to the statement that a separated algebraic X-space V admitting a $\mathbb{G}_a$-action with algebraic stack quotient $[V/\mathbb{G}_a]$ isomorphic to that of a triangular $\mathbb{G}_a$-action on $\mathbb{A}^2_X$ is in fact a scheme. While a direct proof (or disproof) of this equivalent characterization seems totally out of reach with existing methods, we establish that it holds at least over suitable $\mathbb{G}_{a,\partial}$-invariant principal open subsets U_1 of $\mathbb{A}^2_X = \mathrm{Spec}(A[y, z_1])$ faithfully flat over X and whose algebraic stack quotients $[U_1/\mathbb{G}_{a,\partial}]$ are in fact represented by locally separated algebraic spaces $U_1/\mathbb{G}_{a,\partial}$. So this provides at least a $\mathbb{G}_{a,\partial}$-invariant principal open subset $V_1 = \mathrm{pr}^{-1}_{x,z_1}(U_1) \simeq U_1 \times \mathrm{Spec}(\mathbb{C}[z_2])$ of $\mathbb{A}^3_X$, faithfully flat over X, and for which the Zariski open sub-space $V_1/\mathbb{G}_{a,\partial}$ of $\mathbb{A}^3_X/\mathbb{G}_{a,\partial}$ is a scheme.

This is where twin-triangularity enters the argument: indeed for such actions the symmetry between the variables z_1 and z_2 enables the same construction with respect to the other projection $\mathrm{pr}_{y,z_2} : \mathbb{A}^3_X \to \mathbb{A}^2_X = \mathrm{Spec}(A[y, z_2])$ providing a second Zariski open sub-scheme $V_2/\mathbb{G}_{a,\partial}$ of $\mathbb{A}^3_X/\mathbb{G}_{a,\partial}$ faithfully flat over X. Since the action of $\mathbb{G}_{a,\partial}$ is by definition equivariantly trivial over the complement of the closed point 0 of X, its local triviality in the Zariski topology follows, provided that the invariant affine open subsets V_1 and V_2 can be chosen so that their union covers the closed fiber of $\mathrm{pr}_X : \mathbb{A}^3_X \to X$.

With this general strategy in mind, the scheme of the proof is fairly streamlined. In the first section, we describe algebraic spaces that arise as geometric quotients of certain affine open subsets U of an affine plane $\mathbb{A}^2_X$ over a Dedekind domain

equipped with a triangular $\mathbb{G}_a$-action. Then we establish the crucial property that for such affine open subsets U, a proper lift to $U \times \mathbb{A}^1$ of the induced $\mathbb{G}_a$-action on U is equivariantly trivial with affine geometric quotient. This criterion is applied in the second section to deduce that proper twin-triangular $\mathbb{G}_a$-actions on an affine 3-space $\mathbb{A}^3_X$ over a complex Dedekind domain are locally trivial in the Zariski topology.

1. Preliminaries on triangular $\mathbb{G}_a$-actions on an affine plane over a Dedekind domain

This section is devoted to the study of certain algebraic spaces that arise as geometric quotients for triangular $\mathbb{G}_a$-actions on suitably chosen invariant open subsets in $\mathbb{A}^2_X$.

1.0.1. As a motivation for what follows, consider a $\mathbb{G}_a$-action on $\mathbb{A}^3 = \mathbb{A}^1 \times \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[x][y,z])$ generated by a triangular derivation $\partial = x^n\partial_y + p(y)\partial_z$ of $\mathbb{C}[x,y,z]$, where $n \geq 1$ and where $p(y) \in \mathbb{C}[y]$ is a nonconstant polynomial. Letting $P(y) \in \mathbb{C}[y]$ be an integral of p, the polynomials x and $t = -x^n z + P(y)$ generate the algebra of invariants $\mathbb{C}[x,y,z]^{\mathbb{G}_a} = \mathrm{Ker}\partial$. Corresponding to the fact that y/x^n is a slice for ∂ on the principal invariant open subset $\{x \neq 0\}$ of $\mathbb{A}^3$, the quotient morphism $q : \mathbb{A}^3 \to \mathbb{A}^3//\mathbb{G}_a = \mathrm{Spec}\,(\mathbb{C}\,[x]\,[t])$ restricts to a trivial principal $\mathbb{G}_a$-bundle over the open subset $\{x \neq 0\}$ of $\mathbb{A}^3//\mathbb{G}_a$. In contrast, the set-theoretic fiber of q over a point $(0, t_0) \in \mathbb{A}^3//\mathbb{G}_a$ consists of a disjoint union of affine lines in bijection with the roots of $P(y) - t_0$, each simple root corresponding in particular to an orbit of the action. Thus $\mathbb{A}^3//\mathbb{G}_a$ is in general far from being even a set-theoretic orbit space for the action. However, the observation that the inverse image by q of the line $L_0 = \{x = 0\} \subset \mathbb{A}^3//\mathbb{G}_a$ is equivariantly isomorphic to the product $L_1 \times \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[y][z])$ on which $\mathbb{G}_a$ acts via the twisted translation generated by the derivation $p(y)\partial_z$ of $\mathbb{C}[y,z]$ suggests that a better geometric object approximating an orbit space for the action should be obtained from $\mathbb{A}^3//\mathbb{G}_a$ by replacing the line L_0 by L_1, considered as the total space of the finite cover $h_0 : L_1 \to L_0$, $y \mapsto t = P(y)$.

On the other hand, on every invariant open subset V of $\mathbb{A}^3$ on which the action restricts to a set-theoretically free $\mathbb{G}_a$-action, a geometric quotient $\rho : V \to V/\mathbb{G}_a$ exists in the form of an étale locally trivial $\mathbb{G}_a$-bundle over an algebraic space $V/\mathbb{G}_a$. By definition of ∂, the fixed points of the $\mathbb{G}_a$-action are supported on the disjoint union of lines $\{x = p(y) = 0\}$. Therefore, letting $C_0 \subset L_0 = \mathrm{Spec}(\mathbb{C}[t])$ be the complement of the branch locus of h_0 and considering $\mathbb{A}^1 \times C_0$ as an open subset of $\mathbb{A}^3//\mathbb{G}_a$, a geometric quotient exists on the open subset $V = q^{-1}(\mathbb{A}^1 \times C_0)$ of $\mathbb{A}^3$. In view of the previous discussion, the algebraic quotient morphism $q\,|_V\colon V \to V//\mathbb{G}_a \simeq \mathbb{A}^1 \times C_0 \subset \mathbb{A}^3//\mathbb{G}_a$ should thus factor through a $\mathbb{G}_a$-bundle $\rho : V \to V/\mathbb{G}_a$ over an algebraic space $V/\mathbb{G}_a$ obtained from $\mathbb{A}^1 \times C_0$ by replacing the curve $\{0\} \times C_0 \simeq C_0$ by the finite étale cover $h_0 : C_1 = h_0^{-1}(C_0) \to C_0$ of itself.

In what follows, to give precise sense to the above intuitive interpretation, we review the construction of a particular type of algebraic space $\mathfrak{S}$ obtained from a surface by "replacing a curve by a finite étale cover of itself", and we check that these spaces do indeed arise as geometric quotients for $\mathbb{G}_a$-actions on certain affine threefolds. Then, conversely, we characterize effectively which étale locally trivial $\mathbb{G}_a$-bundles $\rho : V \to \mathfrak{S}$ over such spaces have an affine total space.

1.1. **Algebraic space surfaces with an irreducible r-fold curve.** Given a smooth affine curve $X = \mathrm{Spec}(A)$, a closed point $o \in X$ and a finite étale morphism $h_0 : C_1 = \mathrm{Spec}(R_1) \to C_0 = \mathrm{Spec}(R_0)$ between smooth connected affine curves, our aim is to construct an algebraic space $\mathfrak{S} = \mathfrak{S}(X, o, h_0)$ which looks like $X \times C_0$ but with the special curve $\{o\} \times C_0 \simeq C_0$ replaced by C_1. To obtain such an $\mathfrak{S}$, one can simply define it as the quotient of $X \times C_1$ by the étale equivalence relation $(x, c_1) \sim (x', c_1') \Leftrightarrow (x = x' \neq 0 \text{ and } h_0(c_1) = h_0(c_1'))$. More formally, letting $X_* = X \setminus \{o\}$, this means that $\mathfrak{S} = X \times C_1/R$, where

$$\mathrm{diag} \sqcup j : R = X \times C_1 \sqcup (X \times C_1) \times_{X_* \times C_0} (X \times C_1) \setminus \mathrm{Diag} \longrightarrow (X \times C_1) \times (X \times C_1)$$

is the étale equivalence relation defined by the diagonal embedding $\mathrm{diag} : X \times C_1 \to (X \times C_1) \times (X \times C_1)$ and the natural immersion $j : (X \times C_1) \times_{X_* \times C_0} (X \times C_1) \setminus \mathrm{Diag} \to (X \times C_1) \times (X \times C_1)$ respectively. This equivalence relation restricts on the invariant open subset $X_* \times C_1$ to that defined by the diagonal embedding $X_* \times C_1 \to (X_* \times C_1) \times_{X_* \times C_0} (X_* \times C_1)$ which has quotient $X \times C_0$. This implies that the R-invariant morphism $\mathrm{pr}_1 \times h_0 : X \times C_1 \to X \times C_0$ descends to a morphism $\overline{\varphi} : \mathfrak{S} \to X \times C_0$ restricting to an isomorphism over $X_* \times C_0$. In contrast, since R induces the trivial equivalence relation on $\{o\} \times C_1 \simeq C_1$, one has $\overline{\varphi}^{-1}(\{o\} \times C_0) \simeq C_1$ as desired.

A disadvantage of this simple presentation of $\mathfrak{S}$ is that the equivalence relation R is quasi-finite but not finite. To construct an alternative presentation of $\mathfrak{S}$ as a quotient of a suitable scheme Z by a finite étale equivalence relation, in fact by a free action of a finite group G, we proceed as follows:

1.1.1. We let $C = \mathrm{Spec}(R)$ be the normalization of C_0 in the Galois closure of the field extension $\mathrm{Frac}(R_0) \hookrightarrow \mathrm{Frac}(R_1)$. By construction, the induced morphism $h : C \to C_0$ is a torsor under the corresponding Galois group G which factors as $h : C \xrightarrow{h_1} C_1 \xrightarrow{h_0} C_0$, where $h_1 : C \to C_1$ is a torsor under a certain subgroup H of G with index equal to the degree r of the finite morphism h_0. Now we let Z be the scheme obtained by gluing r copies $Z_{\overline{g}}$, $\overline{g} \in G/H$, of $X \times C$ by the identity outside the curves $\{o\} \times C \subset Z_{\overline{g}}$. The group G acts freely on Z by $Z_{\overline{g}} \ni (x, t) \mapsto g' \cdot (x, t) = (x, g' \cdot t) \in Z_{\overline{g' \cdot g}}$, and so a geometric quotient $\xi : Z \to \mathfrak{S} = Z/G$ exists in the category of algebraic spaces in the form of an étale G-torsor over an algebraic space $\mathfrak{S}$. The local morphisms $\mathrm{pr}_1 \times h : Z_{\overline{g}} \simeq X \times C \to X \times C_0$, $\overline{g} \in G/H$, glue to a global G-invariant morphism $\varphi : Z \to X \times C_0$ which descends in turn to a morphism $\overline{\varphi} : \mathfrak{S} = Z/G \to X \times C_0$ restricting to an isomorphism outside $\{o\} \times C_0$. In contrast, $\overline{\varphi}^{-1}(\{o\} \times C_0)$ is isomorphic as a scheme over C_0 to the quotient of $C \times (G/H)$ by the diagonal action of G, whence to $C/H \simeq C_1$.

The fact that the algebraic spaces $\mathfrak{S} = Z/G$ obtained by this construction coincide with the $X \times C_1/R$ defined above can be seen as follows. By construction, every open subset $Z_{\overline{g}} \simeq X \times C$ of Z, $\overline{g} \in G/H$, is invariant under the induced action of H, with quotient $Z_{\overline{g}}/H \simeq X \times C/H = X \times C_1$. So the morphism $X \times C \to \mathfrak{S}$ induced by restricting $\xi : Z \to \mathfrak{S}$ to any open subset $Z_{\overline{g}} \subset Z$ descends to an étale morphism $X \times C_1 = X \times C/H \to \mathfrak{S}$, and one checks that the étale equivalence relation $(\mathrm{pr}_1, \mathrm{pr}_2) : (X \times C_1) \times_{\mathfrak{S}} (X \times C_1) \rightrightarrows X \times C_1$ is precisely isomorphic to the equivalence relation $R \rightrightarrows X \times C_1$ defined above.

Remark 1.1. Note that if $h_0 : C_1 \to C_0$ is a not an isomorphism, then $\mathfrak{S}$ cannot be a scheme. Indeed, otherwise the image by ξ of a point $z_0 \in \{o\} \times C \subset Z_{\overline{g}} \subset Z$ for some $\overline{g} \in G/H$ would have a Zariski open affine neighborhood U in $\mathfrak{S}$. But then

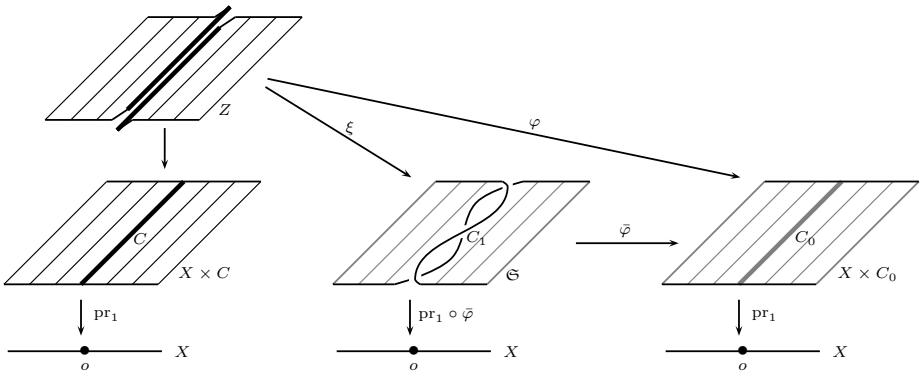

FIGURE 1.1. Construction of $\mathfrak{S}$ as a quotient of Z by a finite group action

since $\xi : Z \to \mathfrak{S}$ is a finite morphism, $\xi^{-1}(U)$ would be a G-invariant affine open neighborhood of z_0 in Z, which is absurd, as such a point does not even have a separated open neighborhood in Z.

1.2. **Geometric quotients for restricted triangular $\mathbb{G}_a$-actions on a relative affine plane.** Here we show that the algebraic spaces $\mathfrak{S} = \mathfrak{S}(X, o, h_0)$ described in the previous subsection naturally arise as geometric quotients for $\mathbb{G}_a$-actions on certain open subsets of affine planes over discrete valuation rings.

1.2.1. We let $X = \mathrm{Spec}(A)$ be the spectrum of a discrete valuation ring with uniformizing parameter x and with residue field $\mathbb{C}$. We denote by o its closed point and we let $\mathbb{A}^2_X = \mathrm{Spec}(A[y,z])$. Given an irreducible triangular locally nilpotent A-derivation $\partial = x^n\partial_y + p(y)\partial_z$ of $A[y,z]$, where $p(y) \in A[y]$, we let $P(y) \in A[y]$ be an integral of $p(y)$. Since ∂ is irreducible, $p(y)$ is not divisible by x, and so the restriction $\overline{P}$ of the morphism $P : \mathbb{A}^1_X = \mathrm{Spec}(A[y]) \to \mathbb{A}^1_X = \mathrm{Spec}(A[t])$ over the closed point of X is not constant. Its branch locus is a principal divisor $\mathrm{div}(\alpha)$ for a certain $\alpha \in \mathbb{C}[t]$, and we let $C_\partial = \mathrm{Spec}(R_0)$, where $R_0 = \mathbb{C}[t]_\alpha$, be its complement. The polynomial $-x^n z + P(y) \in A[y,z]$ defines a $\mathbb{G}_a$-invariant X-morphism $f : \mathbb{A}^2_X = \mathrm{Spec}(A[y,z]) \to \mathrm{Spec}(A[t])$, smooth over $X \times C_\partial$, and such that the induced $\mathbb{G}_a$-action on $V_\partial = f^{-1}(X \times C_\partial) \subset \mathbb{A}^2_X$ is set-theoretically free. Thus a geometric quotient exists in the category of algebraic spaces in the form of an étale locally trivial $\mathbb{G}_a$-bundle $\rho : V_\partial \to V_\partial/\mathbb{G}_a$. Clearly, the curve $C_1 = \mathrm{Spec}(R_0[y] / (\overline{P}(y) - t))$ is smooth and irreducible, and the induced morphism $h_0 : C_1 \to C_\partial$ is finite and étale. With the notation of §1.1.1 above, we have the following result:

Proposition 1.2. *The algebraic space quotient $V_\partial/\mathbb{G}_a$ is isomorphic to $\mathfrak{S}(X, o, h_0)$.*

Proof. Again, we let $h : C = \mathrm{Spec}(R) \to C_\partial$ be the Galois closure of the finite étale morphism $h_0 : C_1 \to C_\partial$. By construction, the polynomial $\overline{P}(y) - t \in R[y]$ splits as $\overline{P}(y) - t = \prod_{\overline{g} \in G/H}(y - t_{\overline{g}})$ for certain elements $t_{\overline{g}} \in R$, $\overline{g} \in G/H$, on which the Galois group G acts by permutation. Furthermore, since $h_0 : C_1 \to C_\partial$ is étale, it follows that for distinct $\overline{g}, \overline{g}' \in G/H$, one has $t_{\overline{g}}(c) \neq t_{\overline{g}'}(c)$ for every point $c \in C$.

Now a similar argument as in the proof of Theorem 3.2 in [12] implies that there exists a collection of elements $\sigma_{\overline{g}} \in A \otimes_{\mathbb{C}} R$ with respective residue classes $t_{\overline{g}} \in R$ modulo x, $\overline{g} \in G/H$, on which G acts by permutation; a polynomial $S_1 \in A \otimes_{\mathbb{C}} R[y]$ with invertible residue class modulo x; and a polynomial $S_2 \in A \otimes_{\mathbb{C}} R[y]$ such that in $A \otimes_{\mathbb{C}} R[y]$ one can write

$$P(y) - t = S_1(y) \prod_{\overline{g} \in G/H} (y - \sigma_{\overline{g}}) + x^n S_2(y).$$

This implies that $W = V_\partial \times_{C_\partial} C \simeq \operatorname{Spec}(A \otimes_{\mathbb{C}} R[y,z]/(x^n z - P(y) + t)$ is isomorphic to the sub-variety of $C \times \mathbb{A}^2_X$ defined by the equation $x^n z = \tilde{P}(y) = S_1(y) \prod_{\overline{g} \in G/H} (y - \sigma_{\overline{g}})$. Furthermore, the $\mathbb{G}_a$-action of V_∂ lifts to the set-theoretically free $\mathbb{G}_a$-action on W commuting with that of G associated with the locally nilpotent $A \otimes_{\mathbb{C}} R$-derivation $x^n \partial_y + \partial_y(\tilde{P}(y))\partial_z$. Then a standard argument (see e.g. [12] or [11]) shows that the $\mathbb{G}_a$-invariant morphism $\mathrm{pr}_{X,C} : W \to X \times C$ factors through a G-equivariant $\mathbb{G}_a$-bundle $\eta : W \to Z$ over the scheme Z as in §1.1.1 above with local trivializations $W|_{Z_{\overline{g}}} \simeq Z_{\overline{g}} \times \operatorname{Spec}(\mathbb{C}[u_{\overline{g}}])$, where $u_{\overline{g}} = x^{-n}(y - \sigma_{\overline{g}})$, $\overline{g} \in G/H$, and transition isomorphisms over $Z_{\overline{g}} \cap Z_{\overline{g}'} \simeq \operatorname{Spec}(A_x \otimes_{\mathbb{C}} R)$ of the form $u_{\overline{g}} \mapsto u_{\overline{g}'} = u_{\overline{g}} + x^{-n}(\sigma_{\overline{g}} - \sigma_{\overline{g}'})$ for every pair of distinct elements $\overline{g}, \overline{g}' \in G/H$. By construction, we have a cartesian square

$$\begin{array}{ccc} W & \longrightarrow & V_\partial \simeq V/G \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \rho} \\ Z & \longrightarrow & \mathfrak{S} = Z/G, \end{array}$$

where the horizontal arrows are G-torsors and the vertical ones are $\mathbb{G}_a$-bundles, which provides, by virtue of the universal property of categorical quotients, an isomorphism of algebraic spaces $V_\partial/\mathbb{G}_a \simeq \mathfrak{S} = \mathfrak{S}(X, o, h_0)$. □

1.3. **Criteria for affineness.** Even though Proposition 1.2 shows in particular that algebraic spaces of the form $\mathfrak{S} = \mathfrak{S}(X, o, h_0)$ may arise as a geometric quotient for $\mathbb{G}_a$-actions on affine schemes, the total space of an étale locally trivial $\mathbb{G}_a$-bundle $\rho : V \to \mathfrak{S}$ is in general neither a scheme nor even a separated algebraic space. However it is possible to characterize effectively which $\mathbb{G}_a$-bundles $\rho : V \to \mathfrak{S}$ have affine total space.

1.3.1. Indeed, with the notation of §1.1.1 above, since $X \times C_0$ is affine, the affineness of V is equivalent to that of the morphism $\overline{\varphi} \circ \rho : V \to X \times C_0$. Furthermore, since $\rho : V \to \mathfrak{S}$ is an affine morphism and $\overline{\varphi} : \mathfrak{S} \to X \times C_0$ is an isomorphism outside the curve $\{o\} \times C_0$, it is enough to consider the case that $X = \operatorname{Spec}(A)$ is the spectrum of a discrete valuation ring with closed point o and uniformizing parameter x. Every $\mathbb{G}_a$-bundle $\rho : V \to \mathfrak{S}$ pulls back via the Galois cover $\xi : Z \to \mathfrak{S} = Z/G$ to a G-equivariant $\mathbb{G}_a$-bundle $\eta = \mathrm{pr}_2 : W = V \times_{\mathfrak{S}} Z \to Z$. By construction of Z, the latter becomes trivial on the canonical covering $\mathcal{U}$ of Z by the affine open subsets $Z_{\overline{g}} \simeq X \times C$, $\overline{g} \in G/H$, whence is determined up to isomorphism by a G-equivariant Čech 1-cocyle

$$\{f_{\overline{g}\overline{g}'}\} \in C^1(\mathcal{U}, \mathcal{O}_Z) \simeq \bigoplus_{\overline{g}, \overline{g}' \in G/H, \overline{g} \neq \overline{g}'} A_x \otimes_{\mathbb{C}} R.$$

With this notation, we have the following criterion:

Theorem 1.3. *For a $\mathbb{G}_a$-bundle $\rho : V \to \mathfrak{S}$, the following are equivalent:*

a) V is a separated algebraic space.

b) For every pair of distinct elements $\overline{g}, \overline{g}' \in G/H$, there exists an element $\tilde{f}_{\overline{gg}'} \in A \otimes_{\mathbb{C}} R$ with invertible residue class modulo x such that $f_{\overline{gg}'} = x^{-l} \tilde{f}_{\overline{gg}'}$ for a certain $l > 1$.

c) V is an affine scheme.

Proof. By virtue of [10, Proposition 10.1.2 and Lemma 10.1.3], b) is equivalent to the separatedness of the total space of the $\mathbb{G}_a$-bundle $\eta : W \to Z$, and this property is also equivalent to the affineness of W thanks to the generalization of the so-called Fieseler criterion for affineness [13] established in [10, Theorem 10.2.1]. Now if V is a separated algebraic space, then so is $W = V \times_{\mathfrak{S}} Z$ as the projection $\mathrm{pr}_1 : W \to V$ is a G-torsor, whence a proper morphism. Thus W is in fact an affine scheme, and so $V \simeq W/G \simeq \mathrm{Spec}(\Gamma(W, \mathcal{O}_W)^G)$ is an affine scheme as well. □

1.3.2. Given a $\mathbb{G}_a$-bundle $\rho : V \to \mathfrak{S}$ with affine total space V, we have a one-to-one correspondence between $\mathbb{G}_a$-bundles over $\mathfrak{S}$ and lifts of the $\mathbb{G}_a$-action on V to $V \times \mathbb{A}^1$. Indeed, if $\rho' : V' \to \mathfrak{S}$ is another $\mathbb{G}_a$-bundle, then the fiber product $V' \times_{\mathfrak{S}} V$ is a $\mathbb{G}_a$-bundle over V via the second projection, whence is isomorphic to the trivial one $V \times \mathbb{A}^1$ on which $\mathbb{G}_a$ acts by translation on the second factor. Via this isomorphism, the natural lift to $V' \times_{\mathfrak{S}} V$ of the $\mathbb{G}_a$-action on V defined by $t \cdot (v', v) = (v', t \cdot v)$ coincides with a lift of it to $V \times \mathbb{A}^1$ with geometric quotient $V \times \mathbb{A}^1/\mathbb{G}_a \simeq V'$. Conversely, since every lift to $V \times \mathbb{A}^1$ of the $\mathbb{G}_a$-action on V commutes with that by translations on the second factor, the equivariant projection $\mathrm{pr}_1 : V \times \mathbb{A}^1 \to V$ descends to a $\mathbb{G}_a$-bundle $\rho' : V' = V \times \mathbb{A}^1/\mathbb{G}_a \to \mathfrak{S} = V/\mathbb{G}_a$ fitting into a cartesian square

$$\begin{tikzcd} V \times \mathbb{A}^1 \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & V' = V \times \mathbb{A}^1/\mathbb{G}_a \arrow[d, "\rho'"] \\ V \arrow[r, "\rho"] & \mathfrak{S} = V/\mathbb{G}_a \end{tikzcd}$$

of $\mathbb{G}_a$-bundles. In this diagram the horizontal arrows correspond to the $\mathbb{G}_a$-actions on V and its lift to $V \times \mathbb{A}^1$ while the vertical ones correspond to the actions on $V \times \mathbb{A}^1$ by translations on the second factor and the one it induces on $V \times \mathbb{A}^1/\mathbb{G}_a$. Combined with Theorem 1.3, this correspondence leads to the following criterion:

Corollary 1.4. *Let $\rho : V \to \mathfrak{S}$ be a $\mathbb{G}_a$-bundle with affine total space over an algebraic space $\mathfrak{S}$ as in §1.1.1. Then the total space of a $\mathbb{G}_a$-bundle $\rho' : V' \to \mathfrak{S}$ is an affine scheme if and only if the corresponding lifted $\mathbb{G}_a$-action on $V \times \mathbb{A}^1$ is proper.*

Proof. Since properness of the lifted $\mathbb{G}_a$-action on $V \times \mathbb{A}^1$ is equivalent to the separatedness of the algebraic space $V' \simeq V \times \mathbb{A}^1/\mathbb{G}_a$, the assertion is a direct consequence of Theorem 1.3 above. □

2. Twin triangular $\mathbb{G}_a$-actions of affine 3-spaces over Dedekind domains

In what follows, we let X be the spectrum of a Dedekind domain A over $\mathbb{C}$, and we let $\mathbb{A}^3_X$ be the spectrum of the polynomial ring $A[y, z_1, z_2]$ in three variables over A. Algebraic actions of $\mathbb{G}_{a,X} = \mathbb{G}_a \times_{\mathrm{Spec}(\mathbb{C})} X$ on $\mathbb{A}^3_X$ are in one-to-one correspondence with locally nilpotent A-derivations of $A[y, z_1, z_2]$. Such an action is called triangular if the corresponding derivation can be written as $\partial = r\partial_y + p_1(y)\partial_{z_1} + p_2(x, y)\partial z_2$ for some $r \in A$, $p_1 \in A[y]$ and $p_2 \in A[y, z_1]$. A triangular $\mathbb{G}_{a,X}$-action on $\mathbb{A}^3_X$ is said to be *twin-triangular* if the corresponding p_2 belongs to the sub-ring $A[y]$ of $A[y, z_1]$.

2.1. **Proper twin-triangular $\mathbb{G}_a$-actions are translations.** This sub-section is devoted to the proof of the following result:

Theorem 2.1. *A proper twin-triangular $\mathbb{G}_{a,X}$-action on $\mathbb{A}^3_X$ is a translation; i.e., $\mathbb{A}^3_X/\mathbb{G}_{a,X}$ is X-isomorphic to $\mathbb{A}^2_X$ and $\mathbb{A}^3_X$ is equivariantly isomorphic to $\mathbb{A}^3_X/\mathbb{G}_a \times_X \mathbb{G}_{a,X}$, where $\mathbb{G}_{a,X}$ acts by translations on the second factor.*

2.1.1. The argument of the proof given below can be decomposed in two steps: we first establish in Proposition 2.2 that any Zariski locally trivial triangular $\mathbb{G}_{a,X}$-action on $\mathbb{A}^3_X$ is a translation. This reduces the problem to showing that a proper twin-triangular $\mathbb{G}_{a,X}$-action on $\mathbb{A}^3_X$ is not only equivariantly trivial in the étale topology, which always holds for a proper whence free $\mathbb{G}_{a,X}$-action, but also in the Zariski topology. This is done in Proposition 2.3. In the sequel, unless otherwise specified, we implicitly work in the category of schemes over X and we denote $\mathbb{G}_{a,X}$ simply by $\mathbb{G}_a$.

We begin with the following generalization of Theorem 2.1 in [9]:

Proposition 2.2. *Let A be a Dedekind domain over $\mathbb{C}$ and let ∂ be a triangular A-derivation of $A[y, z_1, z_2]$ generating a Zariski locally trivial $\mathbb{G}_a$-action on $\mathbb{A}^3_X = \mathrm{Spec}(A[y, z_1, z_2])$. Then the action is equivariantly trivial with quotient isomorphic to $\mathbb{A}^2_X$.*

Proof. The hypotheses imply that $\mathbb{A}^3_X$ has the structure of a Zariski locally trivial $\mathbb{G}_a$-bundle over a quasi-affine X-scheme $\psi : Y = \mathbb{A}^3_X/\mathbb{G}_a \to X$ (see e.g. [8]). Furthermore, since each fiber, closed or not, of the invariant morphism $\mathrm{pr}_X : \mathbb{A}^3_X \to X$ is isomorphic to an affine 3-space equipped with an induced free triangular $\mathbb{G}_a$-action, it follows from [18] that all fibers of $\psi : Y \to X$ are isomorphic to affine planes over the corresponding residue fields. It is enough to show that Y is an affine X-scheme. Indeed, if so, then by virtue of [17], $\psi : Y \to X$ is in fact a locally trivial $\mathbb{A}^2$-bundle in the Zariski topology, whence a vector bundle of rank 2 by [3]. Furthermore, the affineness of Y implies that the quotient morphism $\mathbb{A}^3_X \to Y$ is a trivial $\mathbb{G}_a$-bundle. Thus $Y \times \mathbb{A}^1 \simeq \mathbb{A}^3_X$ as bundles over X, and so $\psi : Y \to X$ is the trivial bundle $\mathbb{A}^2_X$ over X by virtue of [1, IV 3.5]. The affineness of $\psi : Y \to X$ being a local question with respect to the Zariski topology on X, we may reduce to the case where A is a discrete valuation ring with uniformizing parameter x and residue field $\mathbb{C}$. Since $\Gamma(Y, \mathcal{O}_Y) \simeq A[y, z_1, z_2]^{\mathbb{G}_a}$ is finitely generated by virtue of [4], it is enough to show that the canonical morphism $\alpha : Y \to Z = \mathrm{Spec}(A[y, z_1, z_2]^{\mathbb{G}_a})$ is surjective, whence an isomorphism. If $\partial y \in A^*$ then the result is clear. Otherwise if $\partial y = 0$, then the assertion follows from [4]. We may thus assume that $\partial y \in xA \setminus \{0\}$,

and then the result follows verbatim from the argument of [9, Theorem 2.1] which shows that α is surjective over the closed point of X. $\square$

Now it remains to show the following:

Proposition 2.3. *A proper twin-triangular $\mathbb{G}_a$-action on $\mathbb{A}^3_X$ is locally trivial in the Zariski topology.*

Proof. The question is local in the Zariski topology on X. Since the corresponding derivation $\partial = r\partial_y + p_1(y)\partial_{z_1} + p_2(y)\partial z_2$ of $A[y, z_1, z_2]$ has a slice over the principal open subset D_r of X, whence is equivariantly trivial over it, we may reduce after localizing at the finitely many maximal ideals of A containing r to the case where A is a discrete valuation ring with uniformizing parameter x and $r = x^n$ for some $n \geq 1$. Then it is enough to show that the closed fiber $\mathbb{A}^3_o$ of the projection $\mathrm{pr}_X : \mathbb{A}^3_X \to X$ is contained in a union of invariant open subsets of $\mathbb{A}^3_X$ on which the induced actions are equivariantly trivial. By virtue of Lemma 2.4 below, we may assume up to a coordinate change preserving twin-triangularity that the residue classes $\overline{p}_i \in \mathbb{C}[y]$ of the p_i's modulo x are nonconstant and that the inverse images of the branch loci of the morphisms $\overline{P}_i : \mathrm{Spec}\,(\mathbb{C}\,[y]) \to \mathrm{Spec}\,(\mathbb{C}\,[t])$ defined by suitable integrals $\overline{P}_i$ of $\overline{p}_i$, $i = 1, 2$, are disjoint. The first property guarantees that the triangular derivations $\partial_i = x^n\partial_y + p_i(y)\partial_{z_i}$ of $A\,[y, z_i]$, $i = 1, 2$, are both irreducible. Furthermore, if we let V_{∂_i} be the invariant open subset of $\mathbb{A}^2_{X,i} = \mathrm{Spec}(A\,[y, z_i])$, $i = 1, 2$, equipped with $\mathbb{G}_a$-action associated with ∂_i as defined in §1.2.1 above, then the second property implies that $\mathbb{A}^3_o$ is contained in the union of the open subsets $\mathrm{pr}_{z_i}^{-1}(V_{\partial_i}) \simeq V_{\partial_i} \times \mathbb{A}^1$, where $\mathrm{pr}_{z_i} : \mathbb{A}^3_X \to \mathbb{A}^2_{X,i}$, $i = 1, 2$, are the natural projections. These projections being equivariant, the $\mathbb{G}_a$-action on $\mathbb{A}^3_{\mathrm{X}}$ restricts on $\mathrm{pr}_{z_i}^{-1}(V_{\partial_i}) \simeq V_{\partial_i} \times \mathbb{A}^1$ to a proper lift of that on V_{∂_i}, $i = 1, 2$, and so the geometric quotients $\mathrm{pr}_{z_i}^{-1}(V_{\partial_i})/\mathbb{G}_a$, $i = 1, 2$, are affine schemes by virtue of Corollary 1.4. This implies in turn that the induced actions on the open subsets $\mathrm{pr}_{z_i}^{-1}(V_{\partial_i})$, $i = 1, 2$, are equivariantly trivial and completes the proof. $\square$

In the proof of Proposition 2.3, we exploited the following crucial technical fact concerning set-theoretically free twin-triangular $\mathbb{G}_a$-actions:

Lemma 2.4. *Let A be a discrete valuation ring over $\mathbb{C}$ with uniformizing parameter x. A twin-triangular A-derivation ∂ of $A[y, z_1, z_2]$ generating a set-theoretically free $\mathbb{G}_a$-action is conjugate to one of the form $x^n\partial_y + p_1(y)\partial_{z_1} + p_2(y)\partial_{z_2}$ with the following properties:*

a) The residue classes $\overline{p}_i \in \mathbb{C}[y]$ of the polynomials $p_i \in A[y]$ modulo x, $i = 1, 2$, are both nonzero and relatively prime.

b) There exist integrals $\overline{P}_i \in \mathbb{C}[y]$ of $\overline{p}_i$, $i = 1, 2$, for which the inverse images of the branch loci of the morphisms $\overline{P}_i : \mathbb{A}^1 \to \mathbb{A}^1$, $i = 1, 2$, are disjoint.

Proof. A twin-triangular derivation $\partial = x^n\partial_y + p_1(y)\partial_{z_1} + p_2(y)\partial z_2$ generates a set-theoretically free $\mathbb{G}_a$-action if and only if x^n, $p_1(y)$ and $p_2(y)$ generate the unit ideal in $A[y, z_1, z_2]$. So $\overline{p}_1$ and $\overline{p}_2$ are relatively prime and at least one of them, say $\overline{p}_2$, is nonzero. If $\overline{p}_1 = 0$, then p_2 is necessarily of the form $p_2(y) = c + x\tilde{p}_2(y)$ for some nonzero constant c, and so changing z_1 for $z_1 + z_2$ yields a twin-triangular derivation conjugate to ∂ for which the corresponding polynomials $p_1(y)$ and $p_2(y)$ both have nonzero residue classes modulo x. More generally, changing z_2 for $\lambda z_2 + \mu z_1$ for general $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$ yields a twin-triangular derivation conjugate to ∂ and

still satisfying condition a). So it remains to show that up to such a coordinate change, condition b) can be achieved. This can be seen as follows: we consider $\mathbb{A}^2$ embedded in $\mathbb{P}^2 = \mathrm{Proj}(\mathbb{C}[u,v,w])$ as the complement of the line $L_\infty = \{w=0\}$ so that the coordinate system (u,v) on $\mathbb{A}^2$ is induced by the rational projections from the points $[0:1:0]$ and $[1:0:0]$ respectively. We let C be the closure in $\mathbb{P}^2$ of the image of the immersion $j : \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[y]) \to \mathbb{A}^2$ defined by integrals $\overline{P}_1$ and $\overline{P}_2$ of $\bar{p}_1$ and $\bar{p}_2$, and we denote by $a_1, \dots, a_r \in C$ the images by j of the points in the inverse image of the branch locus of $\overline{P}_1 : \mathbb{A}^1 \to \mathbb{A}^1$. Since the condition that a line through a fixed point in $\mathbb{P}^2$ intersects transversally a fixed curve is Zariski open, the set of lines in $\mathbb{P}^2$ passing through a point a_i and tangent to a local analytic branch of C at some point is finite. Therefore, the complement of the finitely many intersection points of these lines with L_∞ is a Zariski open subset U of L_∞ with the property that for every $q \in U$, the line through q and a_i, $i = 1, \dots, r$, intersects every local analytic branch of C transversally at every point. By construction, the rational projections from $[0:1:0]$ and an arbitrary point in $U \setminus \{[0:1:0]\}$ induce a new coordinate system on $\mathbb{A}^2$ of the form $(u, \lambda v + \mu u)$, $\lambda \neq 0$, with the property that none of the a_i, $i = 1, \dots, r$, is contained in the inverse image of the branch locus of the morphism $\lambda \overline{P}_2 + \mu \overline{P}_1 : \mathbb{A}^1 \to \mathbb{A}^1$. Hence changing z_2 for $\lambda z_2 + \mu z_1$ for a pair (λ, μ) corresponding to a general point in U yields a twin-triangular derivation conjugate to ∂ and simultaneously satisfying conditions a) and b). $\square$

2.2. **Complement: A criterion for properness of twin-triangular $\mathbb{G}_a$-actions.** In contrast with the set-theoretic freeness of a $\mathbb{G}_a$-action on an affine variety, which can be easily decided in terms of the corresponding locally nilpotent derivation ∂ of its coordinate ring, it is difficult in general to give effective conditions on ∂ which would guarantee that the action is proper. However, for twin-triangular derivations, we derive below from our previous descriptions a criterion that can be checked algorithmically.

2.2.1. For a set-theoretically free twin-triangular $\mathbb{G}_a$-action on the affine space $\mathbb{A}^3_X = \mathrm{Spec}(A[y, z_1, z_2])$ over a Dedekind domain A, properness is equivalent to the separatedness of the algebraic space quotient $Y = \mathbb{A}^3_X/\mathbb{G}_a$. Since X is affine, the separatedness of Y is equivalent to that of the morphism $\theta : Y = \mathbb{A}^3_X/\mathbb{G}_a \to X$ induced by the invariant projection $\mathrm{pr}_X : \mathbb{A}^3_X \to X$. The question being local in the Zariski topology on X, we may reduce again to the case where A is a discrete valuation ring with uniformizing parameter x.

We may further assume that the corresponding twin-triangular A-derivation $\partial = x^n \partial_y + p_1(y)\partial_{z_1} + p_2(y)\partial z_2$ of $A[y, z_1, z_2]$ satisfies the hypotheses of Lemma 2.4. If $n = 0$, then ∂ generates an equivariantly trivial, whence proper, $\mathbb{G}_a$-action with y as an obvious global slice. So we may assume from now on that $n \geq 1$. Our assumptions guarantee that similarly to §1.2.1 above, an integral $P_i \in A[y]$ of p_i defines a morphism $P_i : \mathbb{A}^1_X \to \mathbb{A}^1_X = \mathrm{Spec}(A[t])$ whose restriction $\overline{P}_i$ over the closed point of X is nonconstant. Passing to the Galois closure $C_i = \mathrm{Spec}(R_i)$ of the finite étale morphism obtained by restricting $\overline{P}_i$ over the complement $C_{0,i} \subset \mathrm{Spec}(\mathbb{C}[t])$ of its branch locus enables, as in the proof of Proposition 1.2, the expression of $P_i(y) - t \in A \otimes_{\mathbb{C}} R_i\,[y]$ as

$$P_i(y) - t = S_{1,i}(y) \prod_{\overline{g} \in G_i/H_i} (y - \sigma_{\overline{g},i}) + x^n S_{2,i}(y) \tag{2.1}$$

for suitable elements $\sigma_{\overline{g},i} \in A \otimes_{\mathbb{C}} R_i$, $\overline{g} \in G_i/H_i$ and polynomials $S_{1,i}, S_{2,i} \in A \otimes_{\mathbb{C}} R_i\,[y]$. Then we have the following criterion:

Proposition 2.5. *With the assumption and notation above, the following are equivalent:*

a) ∂ generates a proper $\mathbb{G}_a$-action on $\mathbb{A}^3_X$.

b) For every $i \neq j$ in $\{1,2\}$ and every pair of distinct elements $\overline{g}, \overline{g}' \in G_i/H_i$, $P_j(\sigma_{\overline{g},i}) - P_j(\sigma_{\overline{g}',i}) \in A \otimes_{\mathbb{C}} R_i$ can be written as $x^{n-k}\tilde{f}_{ij,\overline{g}\overline{g}'}$ where $1 \leq k \leq n$ and where $\tilde{f}_{ij,\overline{g}\overline{g}'} \in A \otimes_{\mathbb{C}} R_i$ has invertible residue class modulo x.

Proof. The hypothesis on ∂ guarantees that the A-derivations $\partial_i = x^n\partial_y + p_i(y)\partial_{z_i}$ of $A[y, z_i]$ are both irreducible. Letting V_{∂_i} be the invariant open subset of $\mathbb{A}^2_X = \mathrm{Spec}(A[y, z_i])$ associated to ∂_i as defined in §1.2.1, it follows from the construction given in the proof of Proposition 1.2 that $W_i = V_{\partial_i} \times_{C_{0,i}} C_i$ is the total space of a $\mathbb{G}_a$-bundle $\eta_i : W_i \to Z_i$ over an appropriate scheme Z_i. The $\mathbb{G}_a$-action on $V_{\partial_i} \times \mathrm{Spec}(\mathbb{C}[z_j]) \subset \mathbb{A}^3_X$, $j \neq i$, induced by the restriction of ∂, lifts to one on $W_i \times \mathbb{A}^1$, commuting with that by translations on the second factor, and so the quotient $W_i' = W_i \times \mathbb{A}^1/\mathbb{G}_a$ has the structure of a $\mathbb{G}_a$-bundle $\eta_i' : W_i' \to Z_i$ over Z_i. Since ∂ satisfies the conditions of Lemma 2.4, it follows from Corollary 1.4 and the proof of Proposition 2.3 that the properness of ∂ is equivalent to the separatedness of the schemes W_i', $i = 1, 2$. So it is enough to show that in our case condition b) above is equivalent to that in Theorem 1.3. We give the argument only for W_1', the case of W_2' being similar. With the notation of the proof of Proposition 1.2, $\eta_1 : W_1 \to Z = Z_1$ is the $\mathbb{G}_a$-bundle with local trivializations $W_1 \mid_{Z_{\overline{g}}} \simeq Z_{\overline{g}} \times \mathrm{Spec}(\mathbb{C}[u_{\overline{g}}])$, where $u_{\overline{g}} = x^{-n}(y - \sigma_{\overline{g},1})$, $\overline{g} \in G_1/H_1$, and transition isomorphism over $Z_{\overline{g}} \cap Z_{\overline{g}'} \simeq \mathrm{Spec}(A_x \otimes_{\mathbb{C}} R_1)$ given by $u_{\overline{g}} \mapsto u_{\overline{g}'} = u_{\overline{g}} + x^{-n}(\sigma_{\overline{g},1} - \sigma_{\overline{g}',1})$ for every pair of distinct elements $\overline{g}, \overline{g}' \in G_1/H_1$. The lift to $W_1 \times \mathbb{A}^1$ of the induced $\mathbb{G}_a$-action on $V_{\partial_1} \times \mathrm{Spec}(\mathbb{C}[z_2]) \subset \mathbb{A}^3_X$ coincides with the one defined locally on the open covering $\{W_1 \mid_{Z_{\overline{g}}} \simeq Z_{\overline{g}} \times \mathbb{A}^1,\ \overline{g} \in G_1/H_1\}$ of $W_1 \times \mathbb{A}^1$ by the derivations $\partial_{\overline{g}} = \partial_{u_{\overline{g}}} + \varphi_2(u_{\overline{2}})\partial_{z_2}$ of $A \otimes_{\mathbb{C}} R_1[u_{\overline{g}}, z_2]$, where $\varphi_2(u_{\overline{g}}) = p_2(x^n u_{\overline{g}} + \sigma_{\overline{g},1})$, $\overline{g} \in G_1/H_1$. Letting $\Phi_2(t) \in A \otimes_{\mathbb{C}} R_1\,[t]$ be an integral of $\varphi_2(t) \in A \otimes_{\mathbb{C}} R_1\,[t]$, a direct computation of invariants shows that $\eta_1' : W_1' = W_1 \times \mathbb{A}^1/\mathbb{G}_a \to Z$ is the $\mathbb{G}_a$-bundle with local trivializations $W_1' \mid_{Z_{\overline{g}}} \simeq Z_{\overline{g}} \times \mathrm{Spec}(\mathbb{C}[v_{\overline{g}}])$ where $v_{\overline{g}} = z_2 - \Phi_2(u_{\overline{g}})$, $\overline{g} \in G_1/H_1$, and transition isomorphisms

$$v_{\overline{g}} \mapsto v_{\overline{g}'} = v_{\overline{g}} + \Phi_2(u_{\overline{g}}) - \Phi_2(u_{\overline{g}'}) = v_g + x^{-n}(P_2(\sigma_{\overline{g},1}) - P_2(\sigma_{\overline{g}',1})).$$

So condition b) above for $i = 1$ and $j = 2$ is precisely equivalent to that of Theorem 1.3. □

Remark 2.6. With the notation of §2.2.1, for every regular value λ_i of $\overline{P}_i : \mathbb{A}^1 \to \mathbb{A}^1$, the expression 2.1 specializes to one of the form

$$P_i(y) - \lambda_i = \overline{S}_{1,i}(y) \prod_{\overline{g} \in G_i/H_i} (y - \overline{\sigma}_{\overline{g},i}) + x^n \overline{S}_{2,i}(y)$$

for elements $\overline{\sigma}_{\overline{g},i} \in A$, $\overline{g} \in G_i/H_i$, reducing modulo x to the distinct roots of $\overline{P}_i(y) - \lambda_i \in \mathbb{C}[y]$, and polynomials $\overline{S}_{1,i}, \overline{S}_{2,i} \in A\,[y]$. One checks that condition b) in Proposition 2.5 can be equivalently rephrased in this context as the fact that for every $i \neq j$ in $\{1, 2\}$, every regular value λ_i of $\overline{P}_i$, and every pair of distinct elements $\overline{g}, \overline{g}' \in G_i/H_i$, $P_j(\overline{\sigma}_{\overline{g},i}) - P_j(\overline{\sigma}_{\overline{g}',i}) \in A \setminus x^n A$. This alternative form enables us to quickly decide that certain twin-triangular derivations give rise

to improper $\mathbb{G}_a$-actions. For instance, consider the family of derivations $D_n = x\partial_y + 2y\partial_{z_1} + (1+y^n)\,\partial_{z_2}$, $n \geq 1$, of $\mathbb{C}[x]_{(x)}[y, z_1, z_2]$. If $n = 2m$, one has $P_1 = y^2$ and $P_2 = y\left(y^{2m} + 2m + 1\right)/(2m+1)$. At the regular value 0 of $\overline{P}_2$, the $2m$ nonzero roots of P_2 come in pairs $\pm\alpha_k \in \mathbb{C}^*$, $k = 1, \ldots, m$, and so $P_1(\alpha_k) - P_1(-\alpha_k) = 0$ for every k. It follows that the corresponding action is improper. In contrast, if n is odd, then the criterion is satisfied at the regular value 0 of $\overline{P}_2$. Actually, for all odd n, it was established in [7] by different methods that the corresponding $\mathbb{G}_a$-action is a translation.

For a triangular derivation $\partial = x^n\partial_y + p_1(y)\partial_{z_1} + p_2(y, z_1)\partial_{z_2}$ of $A[y, z_1, z_2]$ generating a set-theoretically free $\mathbb{G}_a$-action and such that the induced derivation $x^n\partial_y + p_1(y)\partial_{z_1}$ of $A[y, z_1]$ is irreducible, on can still deduce from Theorem 1.3 a more general version of the above criterion which is again a necessary condition for properness. While more cumbersome than the twin-triangular case, the criterion can be used to construct improper actions and has the potential for being used to study arbitrary proper triangular actions.

References

[1] Hyman Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968. MR0249491 (40 #2736)

[2] Hyman Bass, *A nontriangular action of $\mathbf{G}_a$ on $\mathbf{A}^3$*, J. Pure Appl. Algebra **33** (1984), no. 1, 1–5, DOI 10.1016/0022-4049(84)90019-7. MR750225 (85j:14086)

[3] H. Bass, E. H. Connell, and D. L. Wright, *Locally polynomial algebras are symmetric algebras*, Invent. Math. **38** (1976/77), no. 3, 279–299. MR0432626 (55 #5613)

[4] Daniel Daigle and Gene Freudenburg, *Locally nilpotent derivations over a UFD and an application to rank two locally nilpotent derivations of $k[X_1, \cdots, X_n]$*, J. Algebra **204** (1998), no. 2, 353–371, DOI 10.1006/jabr.1998.7465. MR1624439 (99c:13011)

[5] James K. Deveney and David R. Finston, *G_a actions on $\mathbf{C}^3$ and $\mathbf{C}^7$*, Comm. Algebra **22** (1994), no. 15, 6295–6302, DOI 10.1080/00927879408825190. MR1303005 (95j:13004)

[6] James K. Deveney and David R. Finston, *A proper $\mathbf{G}_a$ action on $\mathbf{C}^5$ which is not locally trivial*, Proc. Amer. Math. Soc. **123** (1995), no. 3, 651–655, DOI 10.2307/2160782. MR1273487 (95j:14065)

[7] James K. Deveney and David R. Finston, *Regular G_a invariants*, Osaka J. Math. **39** (2002), no. 2, 275–282. MR1914293 (2003e:14037)

[8] James K. Deveney, David R. Finston, and Mai Gehrke, *$\mathbf{G}_a$ actions on $\mathbf{C}^n$*, Comm. Algebra **22** (1994), no. 12, 4977–4988, DOI 10.1080/00927879408825115. MR1285720 (95e:14038)

[9] James K. Deveney, David R. Finston, and Peter van Rossum, *Triangular $\mathbf{G}_a$ actions on $\mathbf{C}^4$*, Proc. Amer. Math. Soc. **132** (2004), no. 10, 2841–2848 (electronic), DOI 10.1090/S0002-9939-04-07500-8. MR2063101 (2005d:14064)

[10] A. Dubouloz, *Sur une classe de schémas avec actions de fibrés en droites (on a class of schemes with line bundles actions)*, Ph.D. thesis, Université Joseph-Fourier, 2004.

[11] A. Dubouloz, *Affine open subsets in $\mathbb{A}^3$ without the cancellation property*, Commutative Algebra and Algebraic Geometry (CAAG-2010): Proceedings of the Bangalore conference in honour of B. Singh, U. Storch and R. Gurjar, Ramanujan Mathematical Society Lecture Notes Series, no. 17, 2013. (ISBN: 9789380416106)

[12] Adrien Dubouloz and Pierre-Marie Poloni, *On a class of Danielewski surfaces in affine 3-space*, J. Algebra **321** (2009), no. 7, 1797–1812, DOI 10.1016/j.jalgebra.2008.12.009. MR2494748 (2010g:14088)

[13] Karl-Heinz Fieseler, *On complex affine surfaces with $\mathbf{C}^+$-action*, Comment. Math. Helv. **69** (1994), no. 1, 5–27, DOI 10.1007/BF02564471. MR1259603 (95b:14027)

[14] Shulim Kaliman, *Free $\mathbf{C}_+$-actions on $\mathbf{C}^3$ are translations*, Invent. Math. **156** (2004), no. 1, 163–173, DOI 10.1007/s00222-003-0336-1. MR2047660 (2005b:14102)

[15] Herbert Popp, *Moduli theory and classification theory of algebraic varieties*, Lecture Notes in Mathematics, Vol. 620, Springer-Verlag, Berlin, 1977. MR0466143 (57 #6024)

[16] Rudolf Rentschler, *Opérations du groupe additif sur le plan affine* (French), C. R. Acad. Sci. Paris Sér. A-B **267** (1968), A384–A387. MR0232770 (38 #1093)

[17] A. Sathaye, *Polynomial ring in two variables over a DVR: a criterion*, Invent. Math. **74** (1983), no. 1, 159–168, DOI 10.1007/BF01388536. MR722731 (85j:14098)

[18] Dennis M. Snow, *Triangular actions on* $\mathbf{C}^3$, Manuscripta Math. **60** (1988), no. 4, 407–415, DOI 10.1007/BF01258660. MR933471 (89e:32043)

[19] Jörg Winkelmann, *On free holomorphic* $\mathbf{C}$*-actions on* $\mathbf{C}^n$ *and homogeneous Stein manifolds*, Math. Ann. **286** (1990), no. 1-3, 593–612, DOI 10.1007/BF01453590. MR1032948 (90k:32094)

Institut de Mathématiques de Bourgogne, Université de Bourgogne, 9 avenue Alain Savary - BP 47870, 21078 Dijon cedex, France

E-mail address: `Adrien.Dubouloz@u-bourgogne.fr`

Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003

Current address: Department of Mathematics, Brooklyn College CUNY, 2900 Bedford Avenue, Brooklyn, New York 11210

E-mail address: `dfinston@brooklyn.cuny.edu`