

QUASI-CONFORMAL DEFORMATIONS OF NONLINEARIZABLE GERMS

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ABSTRACT. Let $f(z) = e^{2\pi i\alpha}z + O(z^2)$, $\alpha \in \mathbb{R}$, be a germ of a holomorphic diffeomorphism in \mathbb{C} . For α rational and f of infinite order, the space of conformal conjugacy classes of germs topologically conjugate to f is parametrized by the Ecalle-Voronin invariants (and in particular is infinite-dimensional). When α is irrational and f is nonlinearizable it is not known whether f admits quasi-conformal deformations. We show that if f has a sequence of repelling periodic orbits converging to the fixed point, then f embeds into an infinite-dimensional family of quasi-conformally conjugate germs, no two of which are conformally conjugate.

1. INTRODUCTION

Let $f(z) = e^{2\pi i\alpha}z + O(z^2)$, $\alpha \in \mathbb{R}/\mathbb{Z}$, be a germ of a holomorphic diffeomorphism fixing the origin in \mathbb{C} . We consider the question of when f admits quasi-conformal deformations; i.e. when do there exist germs g which are quasi-conformally but not conformally conjugate to f ? If f is *linearizable* (i.e. analytically conjugate to the rigid rotation $R_\alpha(z) = e^{2\pi i\alpha}z$), then any germ topologically conjugate to f is linearizable, in particular conformally conjugate to f , so f is rigid and does not admit nontrivial deformations. In the *nondegenerate parabolic* case (i.e. $\alpha = p/q \in \mathbb{Q}$, $f^q \neq id$), the quasi-conformal conjugacy class of f contains an infinite-dimensional family of conformal conjugacy classes parametrized by the Ecalle-Voronin invariants ([2], [6]). In the *irrationally indifferent nonlinearizable* case (α irrational, f not linearizable), it seems to be unknown whether quasi-conformal deformations are possible. We show the following:

Theorem 1.1. *Let f be an irrationally indifferent nonlinearizable germ with a sequence of repelling periodic orbits accumulating the origin. Then there is a family of quasi-conformal maps $\{H_\Xi\}_{\Xi \in \mathcal{D}}$ parametrized by an infinite polydisc $\mathcal{D} = \{\Xi = (\xi_n)_{n \geq 1} : \xi_n \in \mathbb{C}, |\xi_n| < 1\}$ such that all conjugates $G_\Xi = H_\Xi \circ f \circ H_\Xi^{-1}$ are holomorphic, and two such germs G_{Ξ_1}, G_{Ξ_2} are conformally conjugate if and only if $\Xi_1 = \Xi_2$. For fixed z , $H_\Xi(z)$ depends holomorphically on each ξ_n .*

A very rough outline of the proof is as follows: given $\Xi = (\xi_n)$ the germ G_Ξ is obtained by quasi-conformally deforming f near its repelling periodic orbits. Each such periodic orbit is attracting for f^{-1} , and it is possible to construct an

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f -invariant Beltrami differential on each basin of attraction such that the quasi-conformal map H_Ξ rectifying the Beltrami differential conjugates f to a holomorphic germ G_Ξ having multipliers at the periodic orbits, each depending on only one ξ_n , with the dependence univalent. The multipliers at periodic orbits being invariant under conformal conjugacies, the conclusion of the theorem follows.

Examples of germs satisfying the hypothesis of the theorem are germs of rational maps of degree d with α satisfying the Cremer condition of degree d (see for example Milnor [3], Ch. 8),

$$\limsup \frac{\log \log q_{n+1}}{q_n} > \log d$$

(where (p_n/q_n) are the continued fraction convergents of α), which ensures that the fixed point is accumulated by periodic orbits (only finitely many of which can be nonrepelling for a rational map).

Perez-Marco has shown ([5]) for any nonlinearizable germ f the existence of a unique monotone one-parameter family $(K_t)_{t>0}$ of full, totally invariant continua called *hedgehogs* containing the fixed point. In [1] it is proved that any conformal mapping in a neighbourhood of a hedgehog K of a germ f_1 mapping K to a hedgehog of a germ f_2 necessarily conjugates f_1 to f_2 . As a corollary of Theorem 1.1 we have

Theorem 1.2. *There exists a holomorphic motion $\phi : \mathbb{D} \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ of $\hat{\mathbb{C}}$ over \mathbb{D} and a hedgehog K such that all the sets $\phi(t, K)$ are hedgehogs, all of which are quasi-conformal images of K , but for $s \neq t$, $\phi(s, K)$ cannot be conformally mapped to $\phi(t, K)$.*

2. DEFORMATIONS

We fix a germ $f(z) = e^{2\pi i \alpha} z + O(z^2)$, $\alpha \in \mathbb{R} - \mathbb{Q}$, and a neighbourhood U of the origin such that f and f^{-1} are univalent on a neighbourhood V of \overline{U} . By a *periodic orbit* or *cycle* of f of order $q \geq 1$ we mean a finite set $\mathcal{O} = \{z_1, \dots, z_q\} \subset U$ such that $f(z_i) = z_{i+1}$, $1 \leq i \leq q-1$, and $f(z_q) = z_1$. The *multiplier* at the periodic orbit is defined to be $\lambda = f'(z_1)f'(z_2) \dots f'(z_q) = (f^q)'(z_i)$. The periodic orbit is called *attracting*, *indifferent* or *repelling* according as $|\lambda| < 1$, $|\lambda| = 1$ or $|\lambda| > 1$ respectively. A periodic orbit for f of multiplier λ is a periodic orbit for f^{-1} of multiplier λ^{-1} . The basin of attraction of an attracting periodic cycle is defined by $\mathcal{A}(\mathcal{O}, f) := \{z \in U : f^n(z) \rightarrow \mathcal{O} \text{ as } n \rightarrow +\infty\}$.

Lemma 2.1. *Given $q \geq 1$, the set of multipliers of cycles of order q of f in U is finite.*

Proof. Suppose not, and let \mathcal{O}_n be a sequence of cycles of f of order q such that the multipliers λ_n of \mathcal{O}_n are all distinct. Choose $z_n \in \mathcal{O}_n$; then passing to a subsequence we may assume $z_n \rightarrow \xi \in \overline{U}$. Since $f(z_n) \in U$ for all n , it follows that $f(\xi) \in \overline{U}$; similarly it follows that f can be iterated q times near ξ . Let f^q be the germ defined near ξ given by iterating q times f near ξ ; then f^q has a sequence of fixed points z_n accumulating ξ . Hence $f^q \equiv id$, so $(f^q)' \equiv 1$ and $\lambda_n = 1$ for all n large, a contradiction. \square

Since f is asymptotic to an irrational rotation, i.e. $f(z)/z \rightarrow e^{2\pi i \alpha}$ as $z \rightarrow 0$, it follows that if f has small cycles, then the orders of the cycles must go to infinity.

2.1. Deforming repelling periodic orbits. Let \mathbb{H} denote the upper half-plane and let $E : \mathbb{H} \rightarrow \{|z| > 1\}$ denote the universal covering defined by $E(z) = e^{-2\pi iz}$. For $r > 0$ and $\tau \in \mathbb{H}$, let $B(\tau, r)$ denote the ball of radius r around τ with respect to the hyperbolic metric on \mathbb{H} . Given a repelling periodic orbit \mathcal{O} of f with multiplier λ , choose $\tau \in \mathbb{H}$ such that $E(\tau) = \lambda$ and $0 < r < 1$ such that E is injective on $B(\tau, r)$. For $\tau' \in B(\tau, r)$, we deform the multiplier of f quasi-conformally from λ to $\lambda' = E(\tau')$ by constructing a corresponding f -invariant Beltrami differential $\mu = \mu(\mathcal{O}, \tau, \tau')$ on the basin of attraction $\mathcal{A}(\mathcal{O}, f^{-1})$ as follows:

By K oenigs linearization theorem (see [3], Ch. 6), for any repelling periodic orbit \mathcal{O} of f with multiplier λ there exists a unique holomorphic map ϕ defined on a neighbourhood of \mathcal{O} such that $\phi(z_i) = 0, \phi'(z_i) = 1, i = 1, \dots, n$, and $\phi(f^q(z)) = \lambda\phi(z)$. The conformal isomorphism $L : \mathbb{C}^* \rightarrow \mathbb{C}/\mathbb{Z}, w \mapsto \xi = -\frac{1}{2\pi i} \log w$ conjugates the linear map $w \mapsto \lambda w$ on \mathbb{C}^* to the translation $\xi \mapsto \xi + \tau$ on \mathbb{C}/\mathbb{Z} .

Let $\tau' \in B(\tau, r)$ and let K be the real linear map on \mathbb{C} defined by $K(1) = 1, K(\tau) = \tau'$. Then K commutes with the translation by one and hence gives a quasi-conformal orientation preserving (since $\Im \tau, \Im \tau' > 0$) homeomorphism $\tilde{K} : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$. The Beltrami differential of \tilde{K} is constant and invariant under translations of \mathbb{C}/\mathbb{Z} , and \tilde{K} conjugates the translation $\xi \mapsto \xi + \tau$ on \mathbb{C}/\mathbb{Z} to $\xi \mapsto \xi + \tau'$.

We let μ be the Beltrami differential of $\tilde{K} \circ L \circ \phi$ restricted to a small neighbourhood D of $z_1 \in \mathcal{O}$. The map $\tilde{K} \circ \phi$ conjugates f^q to the translation $\xi \mapsto \xi + \tau'$. So at points $z, z' = f^q(z)$ in D , μ satisfies the invariance condition

$$\mu(z') \overline{(f^q)'(z)} = \mu(z) (f^q)'(z).$$

So μ extended to the neighbourhood $V = D \cup f(D) \cup \dots \cup f^{q-1}(D)$ of \mathcal{O} by putting

$$\mu(f^j(z)) = \mu(z) \frac{(f^j)'(z)}{\overline{(f^j)'(z)}}$$

for $z \in D, j = 1, \dots, q-1$ is an f -invariant Beltrami differential. Similarly the above equation allows us to extend μ to an f -invariant Beltrami differential $\mu(\mathcal{O}, \tau, \tau')$ on $\mathcal{A}(\mathcal{O}, f^{-1}) = \bigcup_{n \geq 1} f^n(V)$. Note that since the hyperbolic distance between τ and τ' is bounded by one, μ has L^∞ norm less than κ for some fixed constant $0 < \kappa < 1$.

Lemma 2.2. *Any quasi-conformal homeomorphism h with a Beltrami coefficient equal to $\mu = \mu(\mathcal{O}, \tau, \tau')$ on a neighbourhood of \mathcal{O} conjugates f to a map $g = h \circ f \circ h^{-1}$ with a periodic orbit $h(\mathcal{O})$ of multiplier $\lambda' = E(\tau')$. The dependence of $\mu(\mathcal{O}, \tau, \tau')$ on τ' is holomorphic.*

Proof. For $i = 1, \dots, q$ we let ψ_i be the branch of ϕ^{-1} sending 0 to z_i . By construction the map k defined by $k = \psi_i \circ L^{-1} \circ \tilde{K} \circ L \circ \phi(z)$ for z in a neighbourhood of z_i has a Beltrami coefficient equal to μ and conjugates f on a neighbourhood of \mathcal{O} to a holomorphic map $f_1 = k \circ f \circ k^{-1}$ with periodic orbit $k(\mathcal{O})$ and multiplier $\lambda' = E(\tau')$. Since h and k have the same Beltrami coefficient, the map $h \circ k^{-1}$ is holomorphic and conjugates f_1 to g , so the multipliers of f_1 and g are equal. The Beltrami differential of \tilde{K} is constant equal to $\frac{\tau - \tau'}{2\Re \tau + (\tau' - \tau)}$, which depends holomorphically on τ' , so the Beltrami differential $\mu(\mathcal{O}, \tau, \tau')$ of $\tilde{K} \circ L \circ \phi$ depends holomorphically on τ' . \square

2.2. Deforming germs with small cycles. We use the Beltrami differentials $\mu(\mathcal{O}, \tau, \tau')$ to deform a germ with small cycles as follows:

Let f be an irrationally indifferent germ with a sequence of repelling small cycles of orders (q_n) (which we may assume to be strictly increasing). Let U be a neighbourhood of the origin such that f and f^{-1} are univalent on a neighbourhood of \overline{U} . Let $L_n \subset \{|z| > 1\}$ be the finite set of multipliers of all repelling period q_n orbits of f in U . Choose a finite set $T_n \subset \mathbb{H}$ and $0 < r_n < 1$ such that E maps T_n bijectively onto L_n , the balls $B(\tau, r_n), \tau \in T_n$, are disjoint, and E is injective on the union of these balls.

Let $I = \{\iota = (n, \tau) : n \geq 1, \tau \in T_n\}$, and for each $\iota = (n, \tau) \in I$, choose a conformal map S_ι of the unit disk \mathbb{D} onto $B(\tau, r_n)$ such that $S_\iota(0) = \tau$. Let \mathcal{D}_1 be the infinite polydisk $\mathcal{D}_1 = \{\Omega = (\omega_\iota)_{\iota \in I} : \omega_\iota \in \mathbb{D}\}$. The basins of attraction $\mathcal{A}(\mathcal{O}, f^{-1})$ of distinct repelling cycles of f are disjoint, so for any $\Omega = (\omega_\iota)_{\iota \in I} \in \mathcal{D}_1$ we can define an f -invariant Beltrami differential μ_Ω on U as follows:

We put, for z belonging to $\mathcal{A}(\mathcal{O}, f^{-1})$, where \mathcal{O} is a repelling cycle for f of order q_n and multiplier $E(\tau)$ where $\tau \in T_n$,

$$\mu_\Omega(z) := \mu(\mathcal{O}, \tau, S_\iota(\omega_\iota))(z),$$

where $\iota = (n, \tau)$, and put $\mu_\Omega(z) = 0$ otherwise.

Note that since the hyperbolic distance between τ and $S_\iota(\omega_\iota)$ is bounded by 1, μ_Ω has L^∞ norm less than κ for a fixed constant $0 < \kappa < 1$. Let h_Ω be the unique quasi-conformal homeomorphism with Beltrami coefficient μ_Ω fixing $0, 1, \infty$ given by the Measurable Riemann Mapping Theorem. Then $g_\Omega = h_\Omega \circ f \circ h_\Omega^{-1}$ is a holomorphic germ fixing the origin with small cycles. By Naishul's theorem [4] the multiplier at an indifferent fixed point is a topological conjugacy invariant, so $g'_\Omega(0) = e^{2\pi i \alpha}$. Note that if \mathcal{O} is any repelling cycle for g_Ω of order q_n contained in $h_\Omega(U)$, then the multiplier of g_Ω at \mathcal{O} is $E(S_\iota(\omega_\iota))$ for some $\iota = (n, \tau) \in I$.

Lemma 2.3. *For $\Omega_1 = (\omega_{\iota,1})_{\iota \in I}, \Omega_2 = (\omega_{\iota,2})_{\iota \in I} \in \mathcal{D}_1$, if the germs $g_1 = g_{\Omega_1}, g_2 = g_{\Omega_2}$ are holomorphically conjugate, then for all n large there exists $\iota = (n, \tau_n) \in I$ such that $\omega_{\iota,1} = \omega_{\iota,2}$.*

Proof. Let ϕ be a holomorphic germ conjugating g_1 to g_2 . Let $h_1 = h_{\Omega_1}, h_2 = h_{\Omega_2}$. Choose a small neighbourhood $W \subset U$ of the origin such that ϕ is univalent on $h_1(W)$ and $\phi(h_1(W)) \subset h_2(U)$. For all n large, f has a repelling cycle \mathcal{O}_n of order q_n contained in W , with multiplier $\lambda_n = E(\tau_n)$ for some $\tau_n \in T_n$. For such an n , $h_1(\mathcal{O}_n)$ is a repelling cycle of g_1 of order q_n with multiplier $E(S_\iota(\omega_{\iota,1}))$, where $\iota = (n, \tau_n) \in I$. Now $\phi(h_1(\mathcal{O}_n))$ is a repelling cycle of g_2 of order q_n contained in $h_2(U)$, and hence must have multiplier $E(S_{\iota'}(\omega_{\iota',2}))$ for some $\iota' = (n, \tau'_n) \in I$, so it follows that $E(S_\iota(\omega_{\iota,1})) = E(S_{\iota'}(\omega_{\iota',2}))$. Since E is injective on the union of the balls $B(\tau, r_n), \tau \in T_n$, we must have $S_\iota(\omega_{\iota,1}) = S_{\iota'}(\omega_{\iota',2})$, and since these balls are disjoint it follows that $\tau'_n = \tau_n, \iota' = \iota, \omega_{\iota,1} = \omega_{\iota,2}$. \square

We can now prove Theorem 1.1.

Proof. Let \mathcal{D} be the infinite polydisc $\mathcal{D} = \{\Xi = (\xi_m)_{m \geq 1} : \xi_m \in \mathbb{D}\}$. Let $(p_m)_{m \geq 1}$ be an infinite sequence of distinct primes, and define a map $R : \mathcal{D} \rightarrow \mathcal{D}_1$ as follows: $R(\Xi) := \Omega$, where $\omega_\iota = \xi_m$ if $\iota = (n, \tau)$ where n is a power of p_m , and $\omega_\iota = 0$ otherwise. Define $H_\Xi := h_{R(\Xi)}$. Then the conjugates $G_\Xi = H_\Xi \circ f \circ H_\Xi^{-1} = g_{R(\Xi)}$ are holomorphic.

Suppose for $\Xi_1, \Xi_2 \in \mathcal{D}$, the germs G_{Ξ_1}, G_{Ξ_2} are holomorphically conjugate. Then $\Omega_1 = R(\Xi_1), \Omega_2 = R(\Xi_2)$ satisfy the conclusion of Lemma 2.3 above, and it follows easily from the definition of R that $\Xi_1 = \Xi_2$.

It follows from Lemma 2.2 that μ_Ω restricted to any basin $A = \mathcal{A}(\mathcal{O}, f^{-1})$ of a repelling cycle \mathcal{O} for f of order some q_n depends holomorphically (as an element of $L^\infty(A)$) on each ω_ι for ι of the form (n, τ) . It then follows easily from the definition of the map R that $\mu_{R(\Xi)}$ depends holomorphically (as an element of $L^\infty(\hat{\mathbb{C}})$) on each ξ_m , and hence by the Measurable Riemann Mapping Theorem so does $H_\Xi(z)$ for fixed z . \square

We now prove Theorem 1.2.

Proof. Let f be as above and $K \subset U$ be a hedgehog of f . For $t \in \mathbb{D}$ we let Ξ_t be the constant sequence $(\xi_m = t)$. Then it is easy to see (by considering the restriction to basins of attraction) that $\mu_{R(\Xi_t)}$ depends holomorphically on t , and $\phi : (t, z) \mapsto H_{\Xi_t}(z)$ gives the required holomorphic motion. \square

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