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A PROBABILISTIC APPROACH TO MIXED BOUNDARY VALUE PROBLEMS FOR ELLIPTIC OPERATORS WITH SINGULAR COEFFICIENTS

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Abstract. In this paper, we establish existence and uniqueness of solutions of a class of mixed boundary value problems for elliptic operators with singular coefficients. Our approach is probabilistic. The theory of Dirichlet forms plays an important role.

1. Introduction

The pioneering work by Kakutani [20] in 1944 on representing the solution to the classical Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = \phi & \text{on } \partial D, \end{cases}$$

using Brownian motion, started a new era in the very fruitful interplay between probability theory and analysis. Here D is a bounded connected open subset of $\mathbb{R}^d$. Since then, in place of the Laplacian Δ, more general second order elliptic differential operators have been studied in connection with the probabilistic approach. In another direction, other boundary conditions (Nuemann and mixed boundary) have been investigated using probabilistic methods. For example, in [19], Hsu studied solutions to the classical Neumann boundary value problems on a bounded domain D in $\mathbb{R}^d$ with C^3 boundary:

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \mathbf{n}} = \phi & \text{on } \partial D, \end{cases}$$

using reflecting Brownian motion on $\overline{D}$, where $\mathbf{n}$ is the unit inward normal vector field of D on ∂D. We refer the reader to [9, 10, 12, 15, 19, 22] and [24] for a brief history on the related subject.

Let $d \geq 1$ be an integer and D a bounded Lipschitz domain in $\mathbb{R}^d$. Let $A = (a_{ij}) : D \to \mathbb{R}^d \times \mathbb{R}^d$ be a Borel measurable, symmetric matrix-valued function that is uniformly elliptic and bounded; that is, there are constants $\lambda_2 \geq \lambda_1 > 0$ such

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that

$$\lambda_1 \sum_{i=1}^d \xi_i^2 \le \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \le \lambda_2 \sum_{i=1}^d \xi_i^2 \qquad \text{for every } x, \xi \in \mathbb{R}^d. \tag{1.1}$$

Let $\mathbf{n}(x) = (n_1(x), ..., n_d(x))$ be the unit inward normal vector on ∂D. If $A(x)$ is defined on the boundary ∂D, it determines a conormal vector field $\overrightarrow{\gamma} = (\gamma_1, \dots, \gamma_d)$ on ∂D by $\frac{1}{2}A\mathbf{n}$; that is, $\gamma_i(x) = \frac{1}{2}\sum_{j=1}^d a_{ij}(x)n_j(x)$ for $i = 1, \dots, d$. Suppose that $\mathbf{b} = (b_1, \dots, b_d)$ and $\widehat{\mathbf{b}} = (\widehat{b}_1, \dots, \widehat{b}_d)$ are measurable $\mathbb{R}^d$-valued functions on D and q is a measurable real-valued function on D so that

$$|\mathbf{b}| \in L^{p_1}(D; dx), \quad |\widehat{\mathbf{b}}| \in L^{p_2}(D; dx) \quad \text{and} \quad q \in L^{p_3}(D; dx) \tag{1.2}$$

for some $p_1 > d$, $p_2 > d$ and $p_3 > d/2$. We always extend the definition of the functions $\mathbf{b}$, $\widehat{\mathbf{b}}$ and q to $\mathbb{R}^d$ by setting their values to be zero off D.

In this paper, we study the mixed boundary value problem

$$\begin{cases} \mathcal{L}u = 0 & \text{on } D, \\ \frac{\partial u}{\partial \gamma} - \langle \widehat{\mathbf{b}}, \mathbf{n}\rangle u = \phi & \text{on } \partial D \end{cases} \tag{1.3}$$

in the bounded Lipschitz domain D for the second order elliptic operator $\mathcal{L}$ of the following form:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\nabla \cdot (A\nabla) + \mathbf{b} \cdot \nabla - \operatorname{div}(\widehat{\mathbf{b}}\cdot) + q \\ &= \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} - \text{“}\operatorname{div}(\widehat{\mathbf{b}}\cdot)\text{''} + q(x). \end{aligned} \tag{1.4}$$

Note that “$\operatorname{div}(\widehat{\mathbf{b}}\cdot)$” in (1.4) is just a formal way of writing because the divergence really does not exist for the merely measurable vector field $\widehat{\mathbf{b}}$. It should be interpreted in the distributional sense; see Definition 1.1 below. It is exactly due to the non-differentiability of $\widehat{\mathbf{b}}$; the lower order term $\operatorname{div}(\widehat{\mathbf{b}}\cdot)$ cannot be handled by the Girsanov transform or the Feynman-Kac transform. In [9], we studied the Dirichlet boundary value problem for such an operator $\mathcal{L}$, where the lower order term $\operatorname{div}(\widehat{\mathbf{b}}\cdot)$ was tackled by a time-reversal of a Girsanov transform from the first exit time τ_D from D by the symmetric diffusion X associated with the operator $\frac{1}{2}\nabla(A\nabla)$. We show in this paper that a similar strategy works for the mixed boundary value problem (1.3), but we need to run the reflecting symmetric diffusion X on $\overline{D}$ associated with the differential operator $\frac{1}{2}\nabla(A\nabla)$ instead [3]. To the best knowledge of the authors, not only is this probabilistic approach using time-reversal to the mixed boundary value problem (1.3) new, our existence and uniqueness result given in Theorem 1.3 is also new in PDE for operators $\mathcal{L}$ of the form (1.4).

Now we give a precise definition of solutions to the mixed boundary value problem (1.3). Let $W^{1,2}(D) := \{u \in L^2(D; dx) : \nabla u \in L^2(D; dx)\}$ be the Sobolev space on D of order $(1, 2)$ equipped with norm

$$\|u\|_{1,2} := \left(\int_D \left(|\nabla u(x)|^2 + u(x)^2\right) dx\right)^{1/2}.$$

Define for $u, v \in W^{1,2}(D)$ a bilinear form

$$\begin{aligned} Q(u,v) \quad &:= \quad \frac{1}{2}\sum_{i,j=1}^{d}\int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx - \sum_{i=1}^{d}\int_D b_i(x)\frac{\partial u}{\partial x_i}v(x)dx \\ &\quad - \sum_{i=1}^{d}\int_D \widehat{b}_i(x)\frac{\partial v}{\partial x_i}u(x)dx - \int_D q(x)u(x)v(x)dx. \end{aligned} \tag{1.5}$$

Under the conditions of (1.1) and (1.2), the quadratic form Q on $W^{1,2}(D)$ is well defined and has the following properties: there is a constant $K \geq 1$ so that for $u, v \in W^{1,2}(D)$,

$$K^{-1}\|u\|_{1,2}^2 \leq Q_1(u,u) \leq K\|u\|_{1,2}^2 \quad \text{and} \quad |Q(u,v)| \leq K\|u\|_{1,2}\,\|v\|_{1,2}, \tag{1.6}$$

where for $\alpha > 0$, $Q_\alpha(u,v) := Q(u,v) + \alpha\int_D u(x)v(x)dx$. We then define a differential operator $\mathcal{L}$ as follows: $u \in \mathrm{Dom}(\mathcal{L})$ if and only if $u \in W^{1,2}(D)$ and there is a function $g \in L^2(\mathbb{R}^d; x)$ so that

$$Q(u,v) = -\int_{\mathbb{R}^d} g(x)v(x)\,dx \qquad \text{for every } v \in W^{1,2}(D). \tag{1.7}$$

Clearly such a function g is unique, and we define $\mathcal{L}u = g$. We call $\mathcal{L}$ the L^2-infinitesimal generator of the closed quadratic form $(Q, W^{1,2}(D))$ on $L^2(D; dx)$. An integration by parts (see [3]) implies that a function $u \in \mathrm{Dom}(\mathcal{L})$, $\mathcal{L}u$ has the expression (1.4) in D in the weak (or distributional) sense. If $a_{ij} \in W^{1,2}(D)$, then it admits a quasi-continuous version on $\overline{D}$, and so the conormal vector field $\gamma(x) := \frac{1}{2}A(x)\mathbf{n}(x)$ is quasi-everywhere defined in ∂D. Again an integration by parts (see [3]) implies that a function $u \in \mathrm{Dom}(\mathcal{L})$ satisfies the following mixed boundary condition in the distributional sense:

$$\frac{\partial u}{\partial \gamma}(x) - \langle \widehat{\mathbf{b}}, \mathbf{n}\rangle u(x) = 0 \quad \text{on} \quad \partial D. \tag{1.8}$$

Here for two vectors α and β in $\mathbb{R}^d$, we use $\alpha \cdot \beta$ or $\langle \alpha, \beta\rangle$ to denote their inner product. We remark that the condition $a_{ij} \in W^{1,2}(D)$ is not needed in many results of this paper. It is only needed when we want to interpret the boundary condition of $\mathcal{L}$ in an explicit form (1.8). One can in fact take (1.8) as the formulation of a solution to the elliptic equation $\mathcal{L}u = g$ in D satisfying the Robin boundary condition (1.8) in the distributional sense, without assuming a_{ij} is in $W^{1,2}(D)$.

Put

$$C^2(\overline{D}) = \{f|_{\overline{D}}; f \in C^2(\mathbb{R}^d)\}.$$

Definition 1.1. We say that a measurable function u is a weak solution of the boundary value problem (1.3) if $u \in W^{1,2}(D)$ and

$$Q(u,f) = \int_{\partial D} \phi(x)f(x)\sigma(dx) \qquad \text{for every } f \in C^2(\overline{D}). \tag{1.9}$$

Here Q is the quadratic form defined in (1.5) and $\sigma(dx)$ is the Lebesgue surface measure on the boundary ∂D.

To see that the above formulation of a weak solution is sensible, observe that if all functions involved are smooth, then using the Green-Gauss formula, we have for every $f \in C^2(\overline{D})$,

$$Q(u,f) = \int_D -\mathcal{L}u(x)f(x)dx + \int_{\partial D} \left(\frac{\partial u}{\partial \gamma}(x) - \langle \widehat{\mathbf{b}}, n\rangle u(x) \right) f(x)\sigma(dx).$$

It is not difficult to see that the differential operator $\mathcal{L}$ enjoys the maximum principle if and only if $-\mathrm{div}(\widehat{\mathbf{b}}) + q \le 0$ on D in the following distributional sense:

$$\sum_{i=1}^n \int_D \widehat{b}_i(x) \frac{\partial \phi}{\partial x_i} dx + \int_D q(x)\phi(x)dx \le 0, \tag{1.10}$$

for all non-negative functions ϕ in $C_c^2(\overline{D})$. We point out that we do not impose such a Markovian condition in this paper.

The aim of the present paper is twofold. The first is to give a probabilistic representation for the weak solution of the mixed boundary value problem (1.3). This is highly non-trivial because there is no longer a Markov process associated with the operator $\mathcal{L}$ due to the appearance of the lower order term $\mathrm{div}(\widehat{\mathbf{b}}\cdot)$, nor can that lower order term be handled via the Girsanov transform or the Feynman-Kac transform. Motivated by [7, 9, 22], our idea is to use the symmetric reflected diffusion process X on $\overline{D}$ associated with the divergence form operator $\frac{1}{2}\nabla(A\nabla)$, the symmetric part of $\mathcal{L}$, and treat $\mathcal{L}$ as its lower order perturbation via a combination of Girsanov and Feynman-Kac transforms and a time-reversal of a Girsanov transform. Based on the new probabilistic representation, our second aim is to establish the existence and uniqueness of the weak solution to mixed boundary problem (1.3) without the Markovian assumption (1.10). To this end, we introduce a kind of h-transformation which transforms the solution of the mixed boundary problem (1.3) to the solution of a mixed boundary value problem for operators $\widetilde{\mathcal{L}}$ that do not involve the adjoint vector field like $\widehat{\mathbf{b}}$; see (2.13). The time reversal and the theory of symmetric Dirichlet forms play essential roles throughout this paper.

Let X be the symmetric reflecting diffusion on $\overline{D}$ with infinitesimal generator $\frac{1}{2}\nabla(A\nabla)$. It is known (cf. [3] and [18]) that X is a conservative Feller process on $\mathbb{R}^d$ that has a Hölder continuous transition density function which admits a two-sided Aronson's Gaussian type estimate. In general, X is a not a semimartingale, but it admits the following Fukushima decomposition (cf. [14]):

$$X_t = X_0 + M_t + N_t, \qquad t \ge 0, \tag{1.11}$$

where $M = (M, \dots, M^n)$ is a martingale additive functional of X with quadratic co-variation $\langle M^i, M^j\rangle_t = \int_0^t a_{ij}(X_s)ds$ and $N = (N^1, \dots, N^n)$ is a continuous additive functional (abbreviated CAF) of X locally of zero quadratic variations. Note that, since X has a continuous density function and each a_{ij} is bounded on D, by [13, Theorem 2], M and N can be refined to be CAFs of X in the strict sense without an exceptional set and (1.11) holds under $\mathbb{P}_x$ for every $x \in \overline{D}$. Without

loss of generality, we work on the canonical continuous path space $C([0,\infty),\overline{D})$ of X and, for $t>0$, denote by r_t the reverse operator of X from time t. Define

$$\begin{aligned} Z_t &= \exp\Bigg(\int_0^t (A^{-1}\mathbf{b})^*(X_s)dM_s + \Big(\int_0^t (A^{-1}\widehat{\mathbf{b}})^*(X_s)dM_s\Big)\circ\gamma_t \\ &\qquad -\frac{1}{2}\int_0^t (\mathbf{b}-\widehat{\mathbf{b}})A^{-1}(\mathbf{b}-\widehat{\mathbf{b}})^*(X_s)ds + \int_0^t q(X_s)ds\Bigg). \end{aligned} \tag{1.12}$$

All the vectors in this paper are row vectors, and we use $\mathbf{b}^*$ to denote the transpose of a vector $\mathbf{b}$. Let $\{T_t, t\geq 0\}$ be the semigroup associated with the quadratic form $(Q, W^{1,2}(D))$. It is established in [7] that for every $f\in L^2(D;dx)$, $T_tf(x) = \mathbb{E}_x\left[Z_tf(X_t)\right]$ for a.e. $x\in D$. Let L_t be the positive continuous additive functional (abbreviated PCAF) of X having the surface measure $\sigma(dx)$ of ∂D as its Revuz measure. The PCAF L is called the boundary local time of the reflecting diffusion X.

We introduce the following condition.

Condition (A). There is some $p>d$ so that for every $F=(F_1,\dots,F_d)\in L^p(D;dx)$ and every $v\in W^{1,2}(D)$ that satisfies

$$\int_D \sum_{i,j=1}^d a_{ij}(x)\frac{\partial v(x)}{\partial x_i}\frac{\partial\phi(x)}{\partial x_j}dx = \int_D \sum_{i,j=1}^d a_{ij}(x)F_i(x)\frac{\partial\phi(x)}{\partial x_j}dx \quad \text{for every } \phi\in W^{1,2}(D), \tag{1.13}$$

we have $v\in W^{1,p}(D) := \{f\in L^p(D;dx) : \nabla f\in L^p(D;dx)\}$.

Note that (1.13) says that v is a solution in the distributional sense to the following elliptic equation with Neumann boundary condition:

$$\begin{cases} \operatorname{div}(A\nabla v) = \operatorname{div}(AF) & \text{on } D, \\ \frac{\partial v}{\partial\gamma} = \langle F,\gamma\rangle & \text{on } \partial D. \end{cases} \tag{1.14}$$

Remark 1.2. Recall that λ_1 and λ_2 are the constants in (1.1) that describe the ellipticity of the matrix $A(x)$. By [17, Theorem 1 and Remark 1], Condition (A) is satisfied when D is a bounded C^1 domain and λ_2/λ_2 is close to 1. Moreover, by [28], Condition (A) is satisfied when each a_{ij} is C^∞ smooth and D is a bounded Lipschitz domain in $\mathbb{R}^d$.

The following is the main result of this paper.

Theorem 1.3. *Suppose that Condition* (A) *holds. Assume that* $\mathbb{E}_{x_0}\left[\int_0^\infty Z_s dL_s\right] < \infty$ *for some point* $x_0\in\overline{D}$ *and let* ϕ *be a bounded measurable function on* ∂D. *Then*

$$u(x) := \mathbb{E}_x\left[\int_0^\infty Z_s\phi(X_s)dL_s\right]$$

is bounded continuous and is the unique bounded weak solution of the mixed boundary value problem (1.3).

The remainder of the paper is organized as follows. In Section 2, we present some basic properties of the symmetric reflecting diffusion X, including the definition of the time-reversal operator r_t. A crucial h-transform is given there that transforms the differential operator $\mathcal{L}$ into $\widetilde{\mathcal{L}}$ (see (2.13) below) that does not involve the adjoint

vector field like $\mathrm{div}(\widehat{\mathbf{b}}\, u)$. In Section 3, we first show that, under Condition (A), the gauge function $g(x) := \mathbb{E}_x \left[\int_0^\infty Z_s dL_s\right]$ is either bounded or identically infinite on $\overline{D}$. We then proceed to give the proof for the main result, Theorem 1.3.

In this paper, we use ":=" as a way of definition. A statement is said to hold quasi-everywhere (q.e. for short) on some set $A \subset \mathbb{R}^d$ if there is an exceptional set N of zero capacity so that the statement holds on $A \setminus N$. For the general theory of Dirichlet forms, Markov processes and their terminology, we refer the reader to [8], [14] and [23].

2. Preliminaries

Let D be a bounded Lipschitz domain in $\mathbb{R}^d$. Define

$$\mathcal{E}^0(u,v) = \frac{1}{2}\sum_{i,j=1}^d \int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx \qquad \text{for } u,v \in W^{1,2}(D). \tag{2.1}$$

It is known that $(\mathcal{E}^0, W^{1,2}(D))$ is a regular Dirichlet form on $L^2(\overline{D}; dx)$. So there is a symmetric continuous Hunt process $X := \{\Omega, \mathcal{F}, X_t, \theta_t, \gamma_t, \mathbb{P}_x, x \in \overline{D}\}$ associated with it, which is called reflected diffusion on $\overline{D}$ (cf. [3]). By a similar argument as that in [2] or by Theorem 3.34 in [18], X has a jointly continuous transition density function $p(t,x,y)$ on $(0,\infty)\times\overline{D}\times\overline{D}$; moreover there are constants $c_1, c_2 \geq 1$ so that for every $t \in (0,1]$ and $x, y \in \overline{D}$,

$$c_1^{-1} t^{-d/2} \exp\left(-\frac{c_2|x-y|^2}{t}\right) \leq p(t,x,y) \leq c_1 t^{-d/2}\exp\left(-\frac{|x-y|^2}{c_2 t}\right). \tag{2.2}$$

Consequently, X can be refined to start from every point on $\overline{D}$ as a Feller process having strong Feller property. Without loss of generality, we can take Ω to be the canonical space $C([0,\infty) \to \overline{D})$ of continuous functions on $\overline{D}$ and X_t to be the coordinate projection process. Let $\{\theta_t, t \geq 0\}$ and $\{r_t, t \geq 0\}$ be the shift and reverse operators defined on Ω by

$$X_s(\theta_t(\omega)) = X_{t+s}(\omega) \ \text{ for } s,t \geq 0 \quad \text{ and } \quad X_s(r_t(\omega)) = X_{t-s}(\omega) \ \text{ for } s \leq t. \tag{2.3}$$

We will use $\mathbb{E}_x$ to denote the expectation with respect to $\mathbb{P}_x$.

For any $u \in W^{1,2}(D)$, it is well known that the following Fukushima decomposition holds:

$$u(X_t) - u(X_0) = M_t^u + N_t^u, \quad t \geq 0, \quad \mathbb{P}_x\text{-a.s.} \tag{2.4}$$

for q.e. $x \in \overline{D}$, where M_t^u is an $\{\mathcal{F}_t\}_{t\geq 0}$-square integrable continuous martingale additive functional and N_t^u is a continuous additive functional of zero energy. Here $\{\mathcal{F}_t\}_{t\geq 0}$ is the minimum augmented σ-field generated by the reflected diffusion X on $\overline{D}$. In particular, taking $u(x) = x_i$ yields that

$$X_t = x + M_t + N_t, \quad t \geq 0, \quad \mathbb{P}_x\text{-a.s.} \tag{2.5}$$

for every $x \in \overline{D}$, where $M = (M^1, ..., M^d)$ is a square integrable continuous martingale additive functional of X with

$$\langle M^i, M^j\rangle_t = \int_0^t a_{ij}(X_s)ds, \qquad t \geq 0, \tag{2.6}$$

and each component of N is a continuous additive functional of X of zero energy.

Let $\{T_t, t \geq 0\}$ be the semigroup associated with the quadratic form $(Q, W^{1,2}(D))$. It admits the following probabilistic representation, which is established in [7].

Proposition 2.1. *Let Z_t be defined as in* (1.12). *Then for every non-negative f on $\overline{D}$, $T_t f(x) = \mathbb{E}_x\left[Z_t\, f(X_t)\right]$ a.e. on D.*

For a martingale additive functional $K = \{K_t, t \geq 0\}$, recall the functional $\Gamma(K)_t$ of local zero energy defined in [25] and [11]:

$$\Gamma(K)_t = -\frac{1}{2}(K_t + K_t \circ r_t).$$

It is known (see [11]) that $\Gamma(M^u) = N^u$ for $u \in W^{1,2}(D)$.

Lemma 2.2. *Suppose that Condition* (A) *holds. Let $F = (F_1, \ldots, F_d)$ be a function on D so that $|F| \in L^2(D; dx)$ and define $\widehat{M}_t = \int_0^t F(X_s)dM_s$ for $t \geq 0$. Then there exists a function $v \in W^{1,2}(D))$ such that*

$$\Gamma(\widehat{M})_t = \Gamma(M^v)_t = N_t^v \tag{2.7}$$

and (1.13) *is satisfied. Moreover, if $|F| \in L^p(D; dx)$ for some $p > d$, then the above $v \in W^{1,p}(D)$, and therefore it has a bounded and continuous version on $\overline{D}$.*

Proof. Note that $\widehat{M}$ is a martingale additive functional of X of finite energy. Thus by Corollary 3.2 in [11], there is some $v \in W^{1,2}(D)$ and a martingale additive functional M^1 of X satisfying $\Gamma(M^1) = 0$, $e(M^1, M^\phi) := \mu_{\langle M^1, M^\phi\rangle}(\overline{D}) = 0$ for all $\phi \in W^{1,2}(D)$, and

$$\widehat{M}_t = M_t^v + M_t^1, \qquad t \geq 0. \tag{2.8}$$

Hence, we have

$$\Gamma(\widehat{M})_t = \Gamma(M^v)_t = N_t^v.$$

On the other hand, by the representation of martingale additive functionals ([14]),

$$M_t^v = \int_0^t \nabla v(X_s)dM_s, \qquad M_t^1 = \int_0^t G(X_s)dM_s$$

for some measurable vector field $G = (\,G_1(x), \ldots, G_d(x)\,) : D \longrightarrow \mathbb{R}^d$. Since $\mu_{\langle M^1, M^\phi\rangle}(\overline{D}) = 0$ for all $\phi \in W^{1,2}(D))$, it follows that

$$\int_D \sum_{i,j=1}^d a_{ij}(x)\frac{\partial \phi}{\partial x_i} G_j(x)dx = 0 \qquad \text{for every } \phi \in W^{1,2}(D). \tag{2.9}$$

On the other hand, since $\widehat{M}_t = \int_0^t F(X_s)dM_s = \int_0^t (\nabla v + G)(X_s))dM_s$ for every $t \geq 0$, we have

$$F(x) = \nabla v(x) + G(x) \qquad \text{a.e.},$$

from which we conclude that v satisfies (1.13).

Now assume that $F \in L^p(D; dx)$ for some $p > d$. Condition (A) implies that $v \in W^{1,p}(D)$. Now by the Sobolev embedding theorem, v has a bounded and continuous version on D. □

The function v in Lemma 2.2 allows us to drop the time-reversal operator r_t in the expression of Z in (1.12) and rewrite it as a combination of an h-transform,

a Girsanov transform and a Feynman-Kac transform. Since $\widehat{\mathbf{b}} \in L^p(D;dx)$, by Lemma 2.2 there is some $v \in W^{1,2}(D)$ with a bounded continuous version so that

$$\begin{aligned}\int_0^t (A^{-1}\widehat{\mathbf{b}})^*(X_s)dM_s \circ r_t &= N_t^v - \int_0^t (A^{-1}\widehat{\mathbf{b}})^*(X_s)dM_s \\ &= v(X_t) - v(X_0) - \int_0^t \big(\nabla v + A^{-1}\widehat{\mathbf{b}}\big)^*(X_s)dM_s.\end{aligned} \tag{2.10}$$

Hence the multiplicative function Z in (1.12) can be expressed as

$$Z_t = \frac{e^{v(X_t)}}{e^{v(X_0)}}\,\widetilde{Z}_t, \tag{2.11}$$

where

$$\begin{aligned}\widetilde{Z}_t = &\exp\bigg(\int_0^t \big(A^{-1}(\mathbf{b}-\widehat{\mathbf{b}}) - \nabla v\big)^*(X_s)dM_s \\ &\quad - \frac{1}{2}\int_0^t (\mathbf{b}-\widehat{\mathbf{b}})A^{-1}(\mathbf{b}-\widehat{\mathbf{b}})^*(X_s)ds + \int_0^t q(X_s)ds\bigg) \\ = &\exp\bigg(\int_0^t \big(A^{-1}(\mathbf{b}-\widehat{\mathbf{b}}) - \nabla v\big)^*(X_s)dM_s \\ &\quad - \frac{1}{2}\int_0^t \big(\mathbf{b}-\widehat{\mathbf{b}} - A\nabla v\big)\cdot A^{-1}\big(\mathbf{b}-\widehat{\mathbf{b}} - A\nabla v\big)(X_s)ds \\ &\quad + \int_0^t \Big(\frac{1}{2}\nabla v\cdot A\nabla v - \langle \mathbf{b}-\widehat{\mathbf{b}}, \nabla v\rangle + q\Big)(X_s)ds\bigg).\end{aligned} \tag{2.12}$$

Let $\widetilde{T}_t$ be defined by

$$\widetilde{T}_t f(x) = \mathbb{E}_x\left[\widetilde{Z}_t f(X_t)\right] (= e^{-v(x)}T_t(e^v f)(x)).$$

Observe that $\widetilde{Z}_t$ is a combination of a Girsanov transform followed by a Feynman-Kac transform, and so $\{\widetilde{T}_t; t \geq 0\}$ is the semigroup associated with differential operator

$$\widetilde{\mathcal{L}}u := \frac{1}{2}\nabla\cdot(A\nabla u) + (\mathbf{b}-\widehat{\mathbf{b}} - A\nabla v)\cdot\nabla u + \Big(\frac{1}{2}\nabla v\cdot A\nabla v - \langle \mathbf{b}-\widehat{\mathbf{b}}, \nabla v\rangle + q\Big)u. \tag{2.13}$$

3. Mixed boundary value problem

In this section, we consider the boundary value problem (1.3). We assume throughout this section that Condition (A) holds.

Recall that Z_t is the multiplicative functional of X defined by (1.12) and $L = \{L_s, s \geq 0\}$ is the boundary local time of the reflected diffusion process $\{X_t, t \geq 0\}$ on D, i.e. the continuous additive functional associated with the smooth measure $\sigma(dx)$.

Theorem 3.1. *Assume that Condition* (A) *holds. The function*

$$x \mapsto \mathbb{E}_x\left[\int_0^\infty Z_s dL_s\right] < \infty \tag{3.1}$$

is either bounded or identically infinite on $\overline{D}$.

Proof. In view of (2.11), the function $\mathbb{E}_x\left[\int_0^\infty Z_s dL_s\right]$ is comparable to $\tilde{u}(x) := \mathbb{E}_x\left[\int_0^\infty \widetilde{Z}_s dL_s\right]$ and so it suffices to show that either u is bounded or identically infinite on $\overline{D}$. In view of (2.2), one can deduce by a similar argument as that in [10, Theorem 3.1] together with Khasminskii's lemma that

$$\sup_{x\in\overline{D}} \mathbb{E}_x\left[\widetilde{Z}_t^2\right] < \infty \quad \text{and} \quad \lim_{t\to 0}\sup_{x\in\overline{D}} \mathbb{E}_x\left[\widetilde{Z}_t^2\right] = 1.$$

Moreover,

$$\inf_{0\le s\le t} \widetilde{Z}_s \ge \exp\Bigg(-\sup_{s\le t}\Big|\int_0^t \big(A^{-1}(\mathbf{b}-\widehat{\mathbf{b}}) - \nabla v\big)^*(X_s)dM_s\Big| \\ - \frac{1}{2}\int_0^t (\mathbf{b}-\widehat{\mathbf{b}})A^{-1}(\mathbf{b}-\widehat{\mathbf{b}})^*(X_s)ds - \int_0^t |q(X_s)|ds\Bigg).$$

In addition,

$$\mathbb{E}_x\left[L_t\right] = \int_0^t \int_{\partial D} p(t,x,y)\sigma(dy)ds.$$

It follows from (2.2) that there exist constants $c > 0$ and $t_0 > 0$ so that

$$\mathbb{E}_x\left[L_t\right] \le c\sqrt{t} \qquad \text{for every } x \in \overline{D} \text{ and } t \le t_0.$$

With these ingredients, one can show that $\tilde{u}$ is either bounded or identically infinite on $\overline{D}$ in a similar way as in [19, Theorem 2.3]; thus its proof is omitted here. □

Remark 3.2. We mention here without proof a sufficient condition for the gauge function $u(x) := \mathbb{E}_x\left[\int_0^\infty Z_s dL_s\right]$ to be finite. As mentioned in the proof of Theorem 3.1, $u(x)$ is comparable to $\widetilde{u}(x) := \mathbb{E}_x\left[\int_0^\infty \widetilde{Z}_s dL_s\right]$, where $\widetilde{Z}_t$ is given by (2.12). If

$$\kappa(x) := -\frac{1}{2}\nabla v \cdot A\nabla v + \langle \mathbf{b}-\widehat{\mathbf{b}}, \nabla v\rangle - q \ge 0 \quad \text{a.e. in } D,$$

then there is a reflected diffusion process $\widetilde{X}$ on $\overline{D}$ with killing rate κ in D whose transition semigroup is given by $\{\widetilde{T}_t; t \ge 0\}$. In fact, in this case $\widetilde{X}$ is obtained from the symmetric reflected diffusion X via supermartingale multiplicative functional Z_t. Suppose further that $\widetilde{X}$ is transient, which is the case when $\kappa(x) > 0$ on a Borel subset of D having positive Lebesgue measure. Let $\widetilde{G}(x,y)$ denote the Green function of $\widetilde{X}$. Then $\widetilde{u}(x) = \int_{\partial D} \widetilde{G}(x,y)\sigma(dy) < \infty$ for every $x \in \overline{D}$. In summary, if $\kappa(x) \ge 0$ on D and $\{x \in D : \kappa(x) > 0\}$ has positive Lebesgue measure, then $u(x) < \infty$ for every $x \in \overline{D}$. We plan to investigate the characterization of the finiteness of the gauge function $u(x)$ in a future work. For the analytic characterization for the gauge function of Feynman-Kac transforms, we refer the reader to [4] and [5].

Proof of Theorem 1.3. By Theorem 3.1, gauge function $\mathbb{E}_x[\int_0^\infty Z_s dL_s]$ is bounded on $\overline{D}$; thus so is u. For any $t > 0$, by the Markov property of X, we have

$$u(x) = \mathbb{E}_x\left[\int_0^t Z_s\phi(X_s)dL_s\right] + T_t u(x).$$

Since $\widetilde{T}_t$ is the semigroup for the differential operator $\widetilde{\mathcal{L}}$ in (2.13), $|\mathbf{b}-\widehat{\mathbf{b}}-A\nabla v|\in L^p(D;dx)$ for some $p>d$ and $\frac{1}{2}\nabla v\cdot A\nabla v-\langle \mathbf{b}-\widehat{\mathbf{b}},\nabla v\rangle+q\in L^q(D;dx)$ for some $q>d/2$, one can derive in a similar way as in [10] that $\widetilde{T}_t f$ is a bounded continuous function on $\overline{D}$ for every $f\in L^2(D;dx)$ and $t>0$. Consequently, $T_t u(x)=e^{-v(x)}\widetilde{T}_t(e^v u)(x)$ is a bounded continuous function on $\overline{D}$ for every $t>0$. On the other hand, by the proof of [19, Lemma 2.1],

$$\lim_{t\to 0}\mathbb{E}_x\left[\int_0^t Z_s\phi(X_s)dL_s\right]=0 \qquad \text{uniformly on } \overline{D}.$$

It follows that $u(x)$ is continuous on $\overline{D}$. Under Condition (A), by Lemma 2.2, there is some $v\in W^{1,2}(D)\cap C(\overline{D})$ so that (2.10) holds. Let Z and $\widetilde{Z}$ be defined by (1.12) and (2.12). Then in view of (2.11), $u=e^{-v(x)}u_0(x)$ with

$$u_0(x):=\mathbb{E}_x\left[\int_0^\infty \widetilde{Z}_s e^{v(X_s)}\phi(X_s)dL_s\right].$$

Let

$$\begin{aligned}\widetilde{T}_t g(x) &:= \mathbb{E}_x[\widetilde{Z}_t g(X_t)],\\ \widetilde{R}_\beta g(x) &:= \int_0^\infty e^{-\beta t}\widetilde{T}_t g(x)dt=\mathbb{E}_x\left[\int_0^\infty e^{-\beta t}\widetilde{Z}_t g(X_t)dt\right].\end{aligned} \tag{3.2}$$

From the expression (2.12), it is easy to see that the quadratic form $(\widetilde{Q},\mathrm{Dom}(\widetilde{Q}))$ associated with the semigroup $\{\widetilde{T}_t, t\ge 0\}$ is given by

$$\begin{aligned}\widetilde{Q}(f,g)=&\frac{1}{2}\int_D\langle A\nabla f,\nabla g\rangle dx-\int_D\langle \mathbf{b}-\widehat{\mathbf{b}}-A\nabla v,\,\nabla f\rangle\, g\,dx\\ &-\int_D\left(\frac{1}{2}\langle A\nabla,\nabla v\rangle-\langle \mathbf{b}-\widehat{\mathbf{b}},\nabla v\rangle+q\right) f\,g\,dx,\end{aligned} \tag{3.3}$$

and $\mathrm{Dom}(\widetilde{Q})=W^{1,2}(D)$. To prove that $u\in W^{1,2}(D)$, it is sufficient to show that $u_0\in W^{1,2}(D)$. For notational simplicity, let $f(x)=e^{v(x)}\phi(x)$. By the Markov property of X and Fubini's theorem,

$$\begin{aligned}\widetilde{R}_\beta u_0(x) &= \mathbb{E}_x\left[\int_0^\infty e^{-\beta t}\widetilde{Z}_t\mathbb{E}_x\left[(\int_0^\infty \widetilde{Z}_s f(X_s)dL_s)\circ\theta_t\middle|\mathcal{F}_t\right]dt\right]\\ &= \mathbb{E}_x\left[\int_0^\infty e^{-\beta t}\mathbb{E}_x\left[\int_t^\infty \widetilde{Z}_s f(X_s)dL_s\middle|\mathcal{F}_t\right]dt\right]\\ &= \mathbb{E}_x\left[\int_0^\infty \widetilde{Z}_s f(X_s)dL_s\int_0^s e^{-\beta t}dt\right]\\ &= \frac{1}{\beta}u_0(x)-\frac{1}{\beta}\mathbb{E}_x\left[\int_0^\infty e^{-\beta s}\widetilde{Z}_s f(X_s)dL_s\right].\end{aligned} \tag{3.4}$$

By Lemma 4.1 in [9], we know that

$$\mathbb{E}_x\left[\int_0^\infty e^{-\beta s}\widetilde{Z}_s f(X_s)dL_s\right]=\widetilde{U}_\beta(f\sigma)(x),$$

where $\widetilde{U}_\beta(f\sigma)$ is the β-potential of the smooth measure $f\sigma$ associated with the quadratic form $\widetilde{Q}$; namely, it satisfies that for $h\in W^{1,2}(D)$,

$$\widetilde{Q}_\beta(\widetilde{U}_\beta(f\sigma),h)=\int_{\partial D}h(x)f(x)\sigma(dx), \tag{3.5}$$

where $\widetilde{Q}_\beta(\cdot,\cdot) = \widetilde{Q}(\cdot,\cdot) + \beta(\cdot,\cdot)$. Rearrange (3.4) to obtain

$$u_0(x) - \beta\widetilde{R}_\beta u_0(x) = \widetilde{U}_\beta(f\sigma)(x).$$

For $\beta > 0$, denote by $\widehat{R}_\beta^{(1)}$ the adjoint resolvent operator of $\widetilde{R}_\beta$ on $L^2(D, dx)$. Then,

$$\begin{aligned}\beta(u_0 - \beta\widetilde{R}_\beta u_0, u_0) &= \beta(U_\beta^{(1)}(f\sigma), u_0) = \beta\widetilde{Q}_\beta(\widetilde{U}_\beta(f\sigma), \widehat{R}_\beta u_0)\\ &= \beta\int_{\partial D}(\widehat{R}_\beta u_0)(x)f(x)\sigma(dx).\end{aligned}$$

The adjoint resolvent operator of $\widetilde{R}_\beta$ is generated by a semigroup that has the same form as the semigroup in (1.12). Hence we have

$$\sup_\beta \beta|\widehat{R}_\beta u_0(x)| \le C\|u_0\|_\infty.$$

This yields that

$$\sup_\beta\{\beta(u_0 - \beta\widetilde{R}_\beta u_0, u_0)\} < \infty.$$

Therefore, we conclude that $u_0 \in D(\widetilde{Q}) = W^{1,2}(D)$ (see [21]). Moreover, for $h \in W^{1,2}(D) \cap C(\overline{D})$, as $\lim_{\beta\to\infty}\beta\widetilde{R}_\beta h(x) = h(x)$ and $\sup_\beta \beta|\widetilde{R}_\beta h(x)| \le C\|h\|_\infty$, we have

$$\begin{aligned}\widetilde{Q}(u_0, h) &= \lim_{\beta\to\infty}\beta(u_0 - \beta\widetilde{R}_\beta u_0, h)\\ &= \lim_{\beta\to\infty}\beta\int_{\partial D}(\widehat{R}_\beta^{(1)}h)(x)f(x)\sigma(dx)\\ &= \int_{\partial D}h(x)f(x)\sigma(dx). \qquad (3.6)\end{aligned}$$

Next we show that u satisfies (1.9). For $g \in C^2(\overline{D})$, we have

$$\begin{aligned}Q(u,g) &= \frac{1}{2}\sum_{i,j=1}^d\int_D a_{ij}(x)\frac{\partial(u_0e^{-v})}{\partial x_i}\frac{\partial g}{\partial x_j}dx - \sum_{i=1}^d\int_D b_i(x)\frac{\partial(u_0e^{-v})}{\partial x_i}gdx\\ &\quad -\sum_{i=1}^d\int_D \widehat{b}_i(x)\frac{\partial g}{\partial x_i}u_0e^{-v}dx - \int_D q(x)u_0(x)e^{-v}gdx\\ &= \frac{1}{2}\sum_{i,j=1}^d\int_D a_{ij}(x)\frac{\partial u_0}{\partial x_i}\frac{\partial(e^{-v}g)}{\partial x_j}dx - \frac{1}{2}\sum_{i,j=1}^d\int_D a_{ij}(x)\frac{\partial u_0}{\partial x_i}\frac{\partial e^{-v}}{\partial x_j}gdx\\ &\quad +\frac{1}{2}\sum_{i,j=1}^d\int_D a_{ij}(x)\frac{\partial e^{-v}}{\partial x_i}\frac{\partial g}{\partial x_j}u_0dx - \sum_{i=1}^d\int_D b_i(x)\frac{\partial u_0}{\partial x_i}e^{-v}gdx\\ &\quad -\sum_{i=1}^d\int_D b_i(x)\frac{\partial e^{-v}}{\partial x_i}u_0gdx - \sum_{i=1}^d\int_D \widehat{b}_i(x)\frac{\partial g}{\partial x_i}u_0e^{-v}dx\\ &\quad -\int_D q(x)u_0(x)e^{-v}gdx. \qquad (3.7)\end{aligned}$$

Applying (3.6) to $h = ge^{-v}$ we obtain that

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j=1}^{d}\int_D a_{ij}(x)\frac{\partial u_0}{\partial x_i}\frac{\partial(ge^{-v})}{\partial x_j}dx \\
= &\int_D \langle \mathbf{b}(x) - \widehat{\mathbf{b}}(x) - (A\nabla v)(x),\, \nabla u_0(x)\rangle g(x)e^{-v}dx \\
&+\int_D q(x)u_0(x)ge^{-v}dx - \int_D \langle \mathbf{b} - \widehat{\mathbf{b}}, \nabla v\rangle(x)u_0(x)ge^{-v}dx \\
&+\frac{1}{2}\int_D (\nabla v)A(\nabla v)^*(x)u_0ge^{-v}dx + \int_{\partial D} ge^{-v}e^{v}\phi(x)\sigma(dx). \qquad (3.8)
\end{aligned}$$

Substituting this expression in (3.7), it follows after cancellations that

$$\begin{aligned}
Q(u,g) = &-\int_D \langle \widehat{\mathbf{b}}, u_0 g\rangle e^{-v}dx - \int_D \langle A\nabla v), \nabla u_0\rangle ge^{-v}dx \\
&+\int_D \langle \widehat{\mathbf{b}}, \nabla v\rangle gu_0e^{-v}dx + \frac{1}{2}\int_D \langle \nabla v, A\nabla v\rangle\, u_0ge^{-v}dx \\
&-\frac{1}{2}\int_D \langle A\nabla u,\, \nabla e^{-v}\rangle gdx + \frac{1}{2}\int_D \langle A\nabla g,\, \nabla e^{-v}\rangle u_0dx \\
&+\int_{\partial D} g(x)\phi(x)\sigma(dx). \qquad (3.9)
\end{aligned}$$

In the sequel, we write $\mathrm{div}(\cdot)$ for the divergence in the sense of distributions. Now,

$$\begin{aligned}
&-\int_D \langle \widehat{\mathbf{b}}, \nabla(u_0g)\rangle e^{-v}dx \\
&= \int_D \mathrm{div}(e^{-v}\,\widehat{\mathbf{b}})u_0gdx - \int_{\partial D} e^{-v}\langle \widehat{\mathbf{b}}, \mathbf{n}\rangle u_0g\sigma(dx) \qquad (3.10) \\
&= \int_D \mathrm{div}(\widehat{\mathbf{b}})e^{-v}u_0gdx - \int_D \langle \widehat{\mathbf{b}}, \nabla v\rangle gu_0e^{-v}dx - \int_{\partial D} e^{-v}\langle \widehat{\mathbf{b}}, \mathbf{n}\rangle u_0g\sigma(dx). \qquad (3.11)
\end{aligned}$$

Furthermore, since (1.13) holds for v with $\widehat{\mathbf{b}}$ in place of F, we have

$$\begin{aligned}
&\int_D \mathrm{div}(\widehat{\mathbf{b}})e^{-v}u_0gdx = -\frac{1}{2}\int_D \mathrm{div}(A\nabla v)e^{-v}u_0gdx \\
= &\frac{1}{2}\int_D \langle A\nabla v, \nabla(e^{-v}u_0g)\rangle dx - \frac{1}{2}\int_{\partial D}\langle A\nabla v, n\rangle e^{-v}u_0gdx \\
= &-\frac{1}{2}\int_D \langle A\nabla v, \nabla v\rangle e^{-v}u_0gdx + \frac{1}{2}\int_D \langle A\nabla v, \nabla u_0\rangle e^{-v}gdx \\
&+\frac{1}{2}\int_D \langle A\nabla v, \nabla g\rangle e^{-v}u_0gdx + \int_{\partial D} e^{-v}\langle \widehat{\mathbf{b}}, n\rangle u_0g\sigma(dx). \qquad (3.12)
\end{aligned}$$

Combining (3.7)-(3.12) we arrive at

$$\begin{aligned}
Q(u,g) = &-\int_D \langle A\nabla v, \nabla u_0\rangle ge^{-v}dx - \frac{1}{2}\int_D \langle A\nabla u_0, \nabla e^{-v}\rangle\, g\, dx \\
&+\frac{1}{2}\int_D \langle A\nabla v, \nabla u_0\rangle e^{-v}gdx + \int_{\partial D} g\phi(x)\sigma(dx) \\
= &\int_{\partial D} g\phi(x)\sigma(dx), \qquad (3.13)
\end{aligned}$$

which completes the proof of the existence. Now we show the uniqueness. Let u be any bounded, continuous weak solution to the boundary value problem (1.3). Define $u_0(x) = e^{v(x)}u(x)$. Then it is easy to verify that u_0 satisfies (3.6). Under assumption (3.1), using a similar proof to that of [19, Theorem 3.3], we can show that u_0 is uniquely determined; thus so is u. This completes the proof of uniqueness.

We finish the paper with a probabilistic characterization of weak solutions of the mixed boundary value problem.

Theorem 3.3. *Assume that Condition* (A) *holds. Suppose* $u \in W^{1,2}(D)$ *is bounded and that* $\mathbb{E}_{x_0}\left[\int_0^\infty Z_s dL_s\right] < \infty$ *for some* $x_0 \in D$. *Then the following statements are equivalent:*

(i) u *is a weak solution of the boundary value problem* (1.3)*;*

(ii) $u(X_t)Z_t - u(X_0) + \int_0^t Z_s\phi(X_s)dL_s$ *is a* $\mathbb{P}_x$*-martingale for a.e.* $x \in D$.

Proof. We first prove (i)⇒(ii). By Theorem 1.3, the continuous version of the unique bounded weak solution of (1.3) is given by $u(x) = \mathbb{E}_x\left[\int_0^\infty Z_s\phi(X_s)dL_s\right]$. Using this version of u, we get a martingale

$$K_t := \mathbb{E}_x\left[\int_0^\infty Z_s\phi(X_s)dL_s\Big|\mathcal{F}_t\right] = \int_0^t Z_s\phi(X_s)dL_s + Z_t u(X_t)$$

by the Markov property of X and that Z is a multiplicative functional of X. It follows that $u(X_t)Z_t - u(X_0) + \int_0^t Z_s\phi(X_s)dL_s = K_t - u(X_0)$ is a $\mathbb{P}_x$-martingale for every $x \in \overline{D}$, so the asserted implication (i)⇒(ii) follows.

Next we show that (ii)⇒(i). Suppose that u is a bounded function in $W^{1,2}(D)$ so that $u(X_t)Z_t - u(X_0) + \int_0^t Z_s\phi(X_s)dL_s$ is a $\mathbb{P}_x$-martingale for a.e. $x \in D$. Let $v(x) := \mathbb{E}_x\left[\int_0^\infty Z_s\phi(X_s)dL_s\right]$, which, by Theorem 1.3, is a continuous bounded weak solution for (1.3). From above, we know that $\int_0^t Z_s\phi(X_s)dL_s + Z_t v(X_t)$ is a $\mathbb{P}_x$-martingale for every $x \in \overline{D}$. It follows then that $(u(X_t) - v(X_t))Z_t$ is a $\mathbb{P}_x$-martingale for a.e. $x \in \overline{D}$. Consequently, $u(x) - v(x) = \mathbb{E}_x\left[(u(X_t) - v(X_t))Z_t\right] = T_t(u - v)(x)$. Since $u - v \in W^{1,2}(D)$, this implies that $Q(u - v, f) = 0$ for every $f \in C^2(\overline{D})$. In other words,

$$Q(u, f) = Q(v, f) = \int_{\partial D} f(x)\phi(x)\sigma(dx) \qquad \text{for every } f \in C^2(\overline{D}),$$

and so u is a weak solution of (1.3). □

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