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SIMULTANEOUS NONVANISHING OF $GL(2) \times GL(2)$ AND $GL(2)$ L-FUNCTIONS

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Abstract. Let f be a fixed holomorphic Hecke cusp form for $SL(2,\mathbb{Z})$. We prove that for each K large enough, there exists a holomorphic Hecke cusp form g of weight k with $K \leq k \leq 2K$ such that $L\left(\frac{1}{2}, g\right) L\left(\frac{1}{2}, f \times g\right) \neq 0$.

1. Introduction

The vanishing or nonvanishing of an automorphic L-function is an interesting problem in number theory. In many cases, it can encode a deep arithmetic property. For example, the Birch and Swinnerton-Dyer conjecture relates the order of vanishing of the Hasse-Weil L-function at the central point to the rank of an elliptic curve.

Let π be a unitary cuspidal automorphic representation of $GL(n)$ and let $s_0 \in \mathbb{C}$. The nonvanishing of $L(s_0, \pi)$ twists by a Dirichlet character χ has been studied by Rohrlich [Ro], Barthel-Ramakrishnan [BR], Luo [Lu], and Luo-Rudnick-Sarnak [LRS]. Especially, this problem is related to the Ramanujan conjecture (see [LRS]). The simultaneous nonvanishing of twists of automorphic L-functions was studied by many authors ([MV], [RR], [Li], [Kh], [Xu]). In her paper [Li], Li proved a simultaneous nonvanishing result for $GL(3) \times GL(2)$ and $GL(2)$ L-functions at the central point. More precisely, let F be a fixed $GL(3)$ Maass cusp form. She showed that there are infinitely many $GL(2)$ Maass-Hecke cusp forms u_j such that $L(\frac{1}{2}, F \times u_j)L(\frac{1}{2}, u_j) \neq 0$. Later Khan [Kh] proved a similar result by replacing the twists with $GL(2)$ holomorphic forms of prime level. In this paper, we will establish a simultaneous nonvanishing result for $GL(2) \times GL(2)$ and $GL(2)$ L-functions (Theorem 1).

Let H_k denote the set of holomorphic Hecke cusp forms g of weight k for $SL(2,\mathbb{Z})$, where $g(z)$ has the Fourier expansion

$$g(z) = \sum_{n=1}^{\infty} \lambda_g(n) n^{\frac{k-1}{2}} e^{2\pi i n z}$$

with $\lambda_g(1) = 1$.

Our main theorem is the following:

Theorem 1. *Let f be a fixed holomorphic Hecke cusp form for $SL(2,\mathbb{Z})$. For each K large enough, there exists $g \in H_k$ with $K \leq k \leq 2K$ such that*

$$L\left(\tfrac{1}{2}, f \times g\right) L\left(\tfrac{1}{2}, g\right) \neq 0.$$

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Note that a similar result in the Maass form case was obtained in [Xu]. Theorem 1 is an immediate corollary of the following asymptotic formula for the average of L-functions.

Theorem 2. *Let f be a fixed holomorphic Hecke cusp form for $SL(2,\mathbb{Z})$. Let $u(\xi) \in C_c^\infty(0,\infty)$. Then for K large, we have*

$$\begin{aligned}
&\sum_{k\geq 2,\ 4|k} u\left(\frac{k-1}{K}\right) \sum_{g\in H_k} \omega_g^{-1} L\left(\tfrac{1}{2}, f\times g\right) L\left(\tfrac{1}{2}, g\right) \\
&= K(\log K) L(1,f) \int_0^\infty u(\xi) d\xi \\
&\quad + K \int_0^\infty u(\xi) \left((\log \xi + \gamma - \log 4\pi) L(1,f) + \frac{1}{2} L'(1,f) \right) d\xi + O(K^\varepsilon),
\end{aligned}$$

where γ is the Euler constant and the implied constant depends on f and ε.

Remarks. (1) $L(1,f) \neq 0$ (see [JS]).

(2) We only consider $k \equiv 0 \pmod 4$, since for $k \equiv 2 \pmod 4$, the root number of $L(s,g)$ is -1 and hence $L(\frac{1}{2}, g) = 0$.

2. Preliminaries

2.1. $GL(2)$ **holomorphic cusp forms.** The following Petersson trace formula is well-known and can be found in Iwaniec's book [Iw].

Proposition 1 (Petersson trace formula)**.**

$$\sum_{g\in H_k} \omega_g^{-1} \lambda_g(m)\lambda_g(n) = \delta_{m,n} + 2\pi i^{-k} \sum_{c=1}^\infty \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where $\omega_g = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \|g\|^2$, $\delta_{m,n}$ equals 1 if $m=n$ and 0 otherwise, $S(m,n;c)$ is the Kloosterman sum defined below, and J_{k-1} is the J-Bessel function.

The Kloosterman sum is defined as

$$S(m,n;c) = \sum_{a\bar{a}\equiv 1 \pmod c} e\left(\frac{ma+n\bar{a}}{c}\right),$$

and A. Weil proved that

$$|S(m,n;c)| \leq (m,n,c)^{\frac{1}{2}} c^{\frac{1}{2}} \tau(c),$$

where $\tau(c)$ is the divisor function.

Lemma 1. *For any $\varepsilon > 0$, we have*

$$\sum_{n\leq x} |S(m,n;c)| \ll x c^{1/2+\varepsilon}, \tag{2.1}$$

where the implied constant depends only on ε.

Proof. By Weil's bound, we have

$$\begin{aligned}
\sum_{n\leq x} |S(m,n;c)| &\ll c^{1/2+\varepsilon} \sum_{n\leq x} (n,c)^{1/2} \\
&\ll c^{1/2+\varepsilon} \sum_{d|c} \sum_{\substack{n\leq x \\ d|n}} d^{1/2} \ll x c^{1/2+\varepsilon}.
\end{aligned}$$

□

For each $g \in H_k$, the L-function associated to g is defined by

$$L(s,g) = \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^s}.$$

It is entire and satisfies the functional equation

$$\Lambda(s,g) = i^k \Lambda(1-s,g),$$

where

$$\Lambda(s,g) = \pi^{-s}\Gamma\left(\tfrac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\tfrac{s+\frac{k+1}{2}}{2}\right)L(s,g).$$

Lemma 2 (Approximate functional equation)**.** *Let $k \equiv 0 \pmod 4$ and let $G(u) = e^{u^2}$. We have*

$$L\left(\tfrac{1}{2},g\right) = 2\sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^{1/2}} V(n,k), \tag{2.2}$$

where

$$V(y,k) = \frac{1}{2\pi i}\int_{(3)} y^{-u} G(u) \frac{\gamma_1(\frac{1}{2}+u,k)}{\gamma_1(\frac{1}{2},k)} \frac{du}{u} \tag{2.3}$$

and

$$\gamma_1(s,k) = \pi^{-s}\Gamma\left(\tfrac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\tfrac{s+\frac{k+1}{2}}{2}\right). \tag{2.4}$$

Proof. See [IK, Theorem 5.3]. □

Lemma 3. *1) The derivatives of $V(y,k)$ with respect to y satisfy*

$$y^a \frac{\partial^a}{\partial y^a} V(y,k) \ll \left(1+\frac{y}{k}\right)^{-A}.$$

The implied constant depends only on a and A.

2) If $1 \le y \ll k^{1+\varepsilon}$, then as $k \to \infty$

$$V(y,k) = V_1\left(\frac{y}{k}\right) + O\left(\frac{1}{k}V_1^*\left(\frac{y}{k}\right)\right),$$

where

$$V_1(x) = \frac{1}{2\pi i}\int_{(3)} x^{-u} G_0(u)\frac{du}{u}, \quad V_1^*(x) = \frac{1}{2\pi i}\int_{(3)} x^{-u} G_1(u)\frac{du}{u}$$

and $G_0(u)$, $G_1(u)$ are holomorphic functions with exponential decay as $|\Im u| \to \infty$.

Proof. 1) See [IK, Proposition 5.4].

2) By Stirling's formula

$$\begin{aligned} G(u)\frac{\gamma_1(\frac{1}{2}+u,k)}{\gamma_1(\frac{1}{2},k)} &= G(u)\left(\frac{k}{4\pi}\right)^u\left(1+O\left(\frac{p(u)}{k}\right)\right) \\ &= k^u G_0(u) + O\left(\frac{1}{k}k^u G_1(u)\right), \end{aligned}$$

where $p(u)$ is a polynomial in u, $G_0(u) = G(u)(4\pi)^{-u}$, and $G_1(u) = G(u)p(u)(4\pi)^{-u}$. The result follows by inserting this into (2.3). □

2.2. **Rankin-Selberg L-functions.** For $f \in H_l$ and $g \in H_k$, the Rankin-Selberg L-function of f and g is defined by

$$L(s, f \times g) = \zeta(2s) \sum_{m=1}^{\infty} \frac{\lambda_f(m)\lambda_g(m)}{m^s} = \sum_{m=1}^{\infty} \frac{\lambda_{f\times g}(m)}{m^s},$$

where

$$\lambda_{f\times g}(m) = \sum_{ab^2=m} \lambda_f(a)\lambda_g(a).$$

It is entire and satisfies the functional equation

$$\Lambda(s, f \times g) = \Lambda(1 - s, f \times g),$$

where

$$\Lambda(s, f \times g) = (2\pi)^{-s}\Gamma\left(s + \frac{k+l}{2} - 1\right)\Gamma\left(s + \frac{k-l}{2}\right) L(s, f \times g).$$

Lemma 4 (Approximate functional equation)**.** *Let $G(u) = e^{u^2}$. We have*

$$L\left(\tfrac{1}{2}, f \times g\right) = 2 \sum_{m=1}^{\infty} \frac{\lambda_{f\times g}(n)}{n^{1/2}} U(m, k), \tag{2.5}$$

where

$$U(y, k) = \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \frac{\gamma_2(\frac{1}{2} + u, k)}{\gamma_2(\frac{1}{2}, k)} \frac{du}{u} \tag{2.6}$$

and

$$\gamma_2(s, k) = (2\pi)^{-s}\Gamma\left(s + \frac{k+l}{2} - 1\right)\Gamma\left(s + \frac{k-l}{2}\right). \tag{2.7}$$

Proof. See [IK, Theorem 5.3]. □

Lemma 5. *1) The derivatives of $U(y, k)$ with respect to y satisfy*

$$y^a \frac{\partial^a}{\partial y^a} U(y, k) \ll \left(1 + \frac{y}{k^2}\right)^{-A}.$$

The implied constant depends only on a and A.

2) If $1 \le y \ll k^{2+\varepsilon}$, then as $k \to \infty$

$$U(y, k) = U_1\left(\frac{y}{k^2}\right) + O\left(\frac{1}{k} U_1^*\left(\frac{y}{k^2}\right)\right),$$

where

$$U_1(x) = \frac{1}{2\pi i} \int_{(3)} x^{-u} G_2(u) \frac{du}{u}, \ U_1^*(x) = \frac{1}{2\pi i} \int_{(3)} x^{-u} G_3(u) \frac{du}{u}$$

and $G_0(u)$, $G_1(u)$ are holomorphic functions with exponential decay as $|\Im u| \to \infty$.

Proof. 1) See [IK, Proposition 5.4].

2) By Stirling's formula

$$\begin{aligned} G(u) \frac{\gamma_2(\frac{1}{2} + u, k)}{\gamma_2(\frac{1}{2}, k)} &= G(u) \left(\frac{k^2}{16\pi^2}\right)^u \left(1 + O\left(\frac{p(u)}{k}\right)\right) \\ &= k^{2u} G_2(u) + O\left(\frac{1}{k} k^{2u} G_3(u)\right), \end{aligned}$$

where $p(u)$ is a polynomial in u, $G_2(u) = G(u)(16\pi^2)^{-u}$, and $G_3(u) = G(u)p(u)(16\pi^2)^{-u}$. The result follows by inserting this into (2.6). $\square$

3. Proof of Theorem 2

By the approximate functional equations (2.2), (2.5) and the Petersson trace formula (Proposition 1), we have

$$\begin{aligned}
&\sum_{g\in H_k} \omega_g^{-1} L\left(\tfrac{1}{2}, f\times g\right) L\left(\tfrac{1}{2}, g\right) \\
&= 4\sum_{g\in H_k} \omega_g^{-1} \sum_{m=1}^{\infty} \frac{\lambda_{f\times g}(m)}{m^{1/2}} U(m,k) \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^{1/2}} V(n,k) \\
&= 4\sum_{g\in H_k} \omega_g^{-1} \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{\lambda_f(a)\lambda_g(a)}{(ab^2)^{1/2}} U(ab^2,k) \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^{1/2}} V(n,k) \\
&= 4\sum_{n=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{n^{1/2}} V(n,k) \frac{\lambda_f(n)}{(nb^2)^{1/2}} U(nb^2,k) \\
&\quad +8\pi \sum_{n=1}^{\infty}\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{n^{1/2}} V(n,k) \frac{\lambda_f(a)}{(ab^2)^{1/2}} U(ab^2,k) \sum_{c=1}^{\infty} \frac{S(n,a;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{na}}{c}\right) \\
&:= 4D_k + 8\pi E_k.
\end{aligned}$$

3.1. **Diagonal terms.** We first deal with the diagonal terms, and these will contribute to the main term. Let

$$D := 4 \sum_{k\geq 2,\ 4|k} u\left(\frac{k-1}{K}\right) D_k.$$

Lemma 6. *We have*

$$D_k = (\log k + \gamma - \log 4\pi) L(1,f) + \tfrac{1}{2} L'(1,f) + O(k^{-1}) \tag{3.1}$$

and

$$\begin{aligned}
D &= K(\log K) L(1,f) \int_0^{\infty} u(\xi)\, d\xi \\
&\quad + K \int_0^{\infty} u(\xi) \left((\log \xi + \gamma - \log 4\pi) L(1,f) + \tfrac{1}{2} L'(1,f)\right) d\xi + O(1). \qquad (3.2)
\end{aligned}$$

Proof.

$$\begin{aligned}
D_k &= \sum_{n=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{n^{1/2}} V(n,k) \frac{\lambda_f(n)}{(nb^2)^{1/2}} U(nb^2,k) \\
&= \frac{1}{(2\pi i)^2} \int_{(3)}\int_{(3)} \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1+u+w}} \sum_{b=1}^{\infty} \frac{1}{b^{1+2w}} \frac{\gamma_1(\frac{1}{2}+u,k)}{\gamma_1(\frac{1}{2},k)} \frac{\gamma_2(\frac{1}{2}+w,k)}{\gamma_2(\frac{1}{2},k)} \\
&\qquad \times G(u)G(w) \frac{du}{u}\frac{dw}{w} \\
&= \frac{1}{(2\pi i)^2} \int_{(3)}\int_{(3)} L(1+u+w,f)\zeta(1+2w) \frac{\gamma_1(\frac{1}{2}+u,k)}{\gamma_1(\frac{1}{2},k)} \frac{\gamma_2(\frac{1}{2}+w,k)}{\gamma_2(\frac{1}{2},k)} \\
&\qquad \times G(u)G(w) \frac{du}{u}\frac{dw}{w}.
\end{aligned}$$

Moving the line of integration to $\Re u = -2$ and $\Re w = -\frac{1}{2}$, we pick up a simple pole at $u = 0$ and a double pole at $w = 0$. By the residue theorem, we have

$$\begin{aligned} D_k &= \left(\frac{1}{2}\frac{\gamma_2'(\frac{1}{2},k)}{\gamma_2(\frac{1}{2},k)} + \gamma\right) L(1,f) + \frac{1}{2}L'(1,f) \\ &+ \frac{1}{2\pi i}\int_{(-2)} \left(\left(\frac{1}{2}\frac{\gamma_2'(\frac{1}{2},k)}{\gamma_2(\frac{1}{2},k)} + \gamma\right) L(1+u,f) + \frac{1}{2}L'(1+u,f)\right) \\ &\times \frac{\gamma_1(\frac{1}{2}+u,k)}{\gamma_1(\frac{1}{2},k)} G(u)\frac{du}{u} \\ &+ \frac{1}{2\pi i}\int_{(-\frac{1}{2})} L(1+w,f)\zeta(1+2w)\frac{\gamma_2(\frac{1}{2}+w,k)}{\gamma_2(\frac{1}{2},k)} G(w)\frac{dw}{w} \\ &+ \frac{1}{(2\pi i)^2}\int_{(-2)}\int_{(-\frac{1}{2})} L(1+u+w,f)\zeta(1+2w)\frac{\gamma_1(\frac{1}{2}+u,k)}{\gamma_1(\frac{1}{2},k)}\frac{\gamma_2(\frac{1}{2}+w,k)}{\gamma_2(\frac{1}{2},k)} \\ &\quad \times G(u)G(w)\frac{du}{u}\frac{dw}{w}. \end{aligned}$$

By Stirling's formula, we have

$$\frac{\gamma_2'(\frac{1}{2},k)}{\gamma_2(\frac{1}{2},k)} = 2\log k - 4\log 2 - 2\log\pi + O(k^{-1}),$$

$$\frac{\gamma_1(\frac{1}{2}+u,k)}{\gamma_1(\frac{1}{2},k)} \ll k^{-2}$$

for $\Re u = -2$, and

$$\frac{\gamma_2(\frac{1}{2}+w,k)}{\gamma_2(\frac{1}{2},k)} \ll k^{-1}$$

for $\Re w = -\frac{1}{2}$. (3.1) follows from using the above estimates. For (3.2), by the Poisson summation formula, we have

$$4\sum_{k\geq 2,\ 4|k} u\left(\frac{k-1}{K}\right) = K\int_0^\infty u(\xi)d\xi + O(K^{-A}) \tag{3.3}$$

for any $A > 0$. By writing $\log k = \log(\frac{k-1}{K}K) + O(k^{-1})$ and using (3.3) for $u(\xi)\log\xi$, we obtain (3.2). □

3.2. **Off-diagonal terms.** We will prove that the off-diagonal terms only contribute to the error terms. Let

$$\begin{aligned} E &:= \sum_{4|k} u\left(\frac{k-1}{K}\right) E_k \\ &= \sum_{n=1}^\infty\sum_{a=1}^\infty\sum_{b=1}^\infty \frac{1}{n^{1/2}}\frac{\lambda_f(a)}{(ab^2)^{1/2}}\sum_{c=1}^\infty \frac{S(n,a;c)}{c}\sum_{4|k} u\left(\frac{k-1}{K}\right) \\ &\quad \times V(n,k)U(ab^2,k)J_{k-1}\left(\frac{4\pi\sqrt{na}}{c}\right). \end{aligned}$$

Lemma 7. *For any $\varepsilon > 0$, we have*

$$E \ll K^\varepsilon.$$

Proof. By taking a smooth dyadic subdivision, it suffices to estimate

$$\begin{aligned} E_{N,M} &= \sum_{n=1}^{\infty}\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} w\left(\frac{n}{N}\right) h\left(\frac{ab^2}{M}\right) \frac{1}{n^{1/2}} \frac{\lambda_f(a)}{(ab^2)^{1/2}} \sum_{c=1}^{\infty} \frac{S(n,a;c)}{c} \\ &\quad \times \sum_{4|k} u\left(\frac{k-1}{K}\right) V(n,k)U(ab^2,k) J_{k-1}\left(\frac{4\pi\sqrt{na}}{c}\right) \end{aligned}$$

for $N \ll K^{1+\varepsilon}$ and $M \ll K^{2+\varepsilon}$, where $w(\xi)$ and $h(\xi)$ are fixed smooth functions with support contained in $[1,2]$.

Let

$$\begin{aligned} R_1 &= \sum_{n=1}^{\infty}\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} w\left(\frac{n}{N}\right) h\left(\frac{ab^2}{M}\right) \frac{1}{n^{1/2}} \frac{\lambda_f(a)}{(ab^2)^{1/2}} \sum_{c\geq 16\pi\sqrt{Na}} \frac{S(n,a;c)}{c} \\ &\quad \times \sum_{4|k} u\left(\frac{k-1}{K}\right) V(n,k)U(ab^2,k) J_{k-1}\left(\frac{4\pi\sqrt{na}}{c}\right) \end{aligned}$$

and

$$\begin{aligned} R_2 &= \sum_{n=1}^{\infty}\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} w\left(\frac{n}{N}\right) h\left(\frac{ab^2}{M}\right) \frac{1}{n^{1/2}} \frac{\lambda_f(a)}{(ab^2)^{1/2}} \sum_{c< 16\pi\sqrt{Na}} \frac{S(n,a;c)}{c} \\ &\quad \times \sum_{4|k} u\left(\frac{k-1}{K}\right) V(n,k)U(ab^2,k) J_{k-1}\left(\frac{4\pi\sqrt{na}}{c}\right). \end{aligned}$$

Then $E_{N,M} = R_1 + R_2$. We will show R_1 is negligible and $R_2 \ll K^{\varepsilon}$ in the next two sections. □

3.3. **Estimate of R_1.** For $c \geq 16\pi\sqrt{Na}$ and $N \leq n \leq 2N$, we have

$$x = \frac{4\pi\sqrt{na}}{c} \leq \frac{1}{4}\left(\frac{n}{N}\right)^{1/2} < e^{-1}.$$

Using Weil's bound for Kloosterman sums, the bound $J_{k-1}(x) \ll x^{k-1}$ for $0 < x < 1$ and

$$\sum_{n\leq x} |\lambda_f(n)| \ll x, \tag{3.4}$$

we have

$$\begin{aligned} R_1 &\ll \sum_{n=1}^{\infty}\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \left|w\left(\frac{n}{N}\right)\right| \left|h\left(\frac{ab^2}{M}\right)\right| \frac{1}{n^{1/2}} \frac{|\lambda_f(a)|}{(ab^2)^{1/2}} \sum_{c\geq 16\pi\sqrt{Na}} c^{-1/2+\varepsilon}(n,a,c)^{1/2} \\ &\quad \times \sum_{4|k} |u\left(\frac{k-1}{K}\right)| \left(\frac{4\pi\sqrt{na}}{c}\right)^{k-1} \\ &\ll \sum_{N\leq n\leq 2N} \frac{1}{n^{1/2}} \sum_{M\leq ab^2\leq 2M} \frac{|\lambda_f(a)|}{(ab^2)^{1/2}} \sum_{c\geq 16\pi\sqrt{Na}} c^{\varepsilon} \left(\frac{4\pi\sqrt{na}}{c}\right)^2 \sum_{4|k,\ k\sim K} e^{-(k-3)} \\ &\ll K^{-A} \end{aligned}$$

for any $A > 0$. □

4. Estimate of R_2

Lemma 8. *Fix a function $g(\xi) \in C_c^\infty(0,\infty)$. We have*

$$\sum_{4|k} g(k-1)J_{k-1}(x) = -\frac{1}{2}F_1(x) - \frac{i}{2}F_2(x),$$

where

$$F_1(x) = \int_{-\infty}^{\infty} \widehat{g}(t)\sin(x\cos(2\pi t))dt,$$
$$F_2(x) = \int_{-\infty}^{\infty} \widehat{g}(t)\sin(x\sin(2\pi t))dt$$

and

$$\widehat{g}(t) = \int_{-\infty}^{\infty} g(\xi)e(t\xi)d\xi$$

is the Fourier transform of $g(\xi)$.

Proof. See [Iw, pp. 85-87]. □

Let $g(\xi) = u(\frac{\xi}{K})V(n,\xi+1)U(ab^2,\xi+1)$. By Lemma 8, $R_2 = -\frac{1}{2}H_1 - \frac{i}{2}H_2$, where

$$H_1 = \sum_{n=1}^{\infty}\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} w\left(\frac{n}{N}\right) h\left(\frac{ab^2}{M}\right)\frac{1}{n^{1/2}}\frac{\lambda_f(a)}{(ab^2)^{1/2}}\sum_{c<16\pi\sqrt{Na}}\frac{S(n,a;c)}{c}F_1\left(\frac{4\pi\sqrt{na}}{c}\right)$$

and

$$H_2 = \sum_{n=1}^{\infty}\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} w\left(\frac{n}{N}\right) h\left(\frac{ab^2}{M}\right)\frac{1}{n^{1/2}}\frac{\lambda_f(a)}{(ab^2)^{1/2}}\sum_{c<16\pi\sqrt{Na}}\frac{S(n,a;c)}{c}F_2\left(\frac{4\pi\sqrt{na}}{c}\right).$$

Let

$$g_0(\xi) = u(\xi)V(n,K\xi+1)U(ab^2,K\xi+1).$$

An easy change of variables gives $\widehat{g}(t) = K\widehat{g_0}(Kt)$ and

$$F_1(x) = \int_{-\infty}^{\infty} \widehat{g_0}(t)\sin\left(x\cos\left(\frac{2\pi t}{K}\right)\right)dt,$$

$$F_2(x) = \int_{-\infty}^{\infty} \widehat{g_0}(t)\sin\left(x\sin\left(\frac{2\pi t}{K}\right)\right)dt.$$

4.1. Estimate of H_1.

Lemma 9. *For any $\varepsilon > 0$, we have*

$$H_1 \ll K^{-1+\varepsilon}.$$

Proof. Let

$$W_1(x) = \int_{-\infty}^{\infty} \widehat{g_0}(t)e\left(\frac{x}{2\pi}\left(1-\cos\frac{2\pi t}{K}\right)\right)dt;$$

then

$$F_1(x) = \frac{-e(-\frac{x}{2\pi})W_1(x) + e(\frac{x}{2\pi})W_1(-x)}{2i}.$$

Since the contribution to $W_1(x)$ from $|t| \geq K^\varepsilon$ is negligible, we only need to consider $|t| \leq K^\varepsilon$. Now we expand $\cos(\frac{2\pi t}{K})$ into the Taylor series to get

$$W_1(x) = \int_{-\infty}^{\infty} \widehat{g_0}(t) e\left(\frac{\pi x t^2}{K^2}\right) dt + O\left(\int_{-\infty}^{\infty} |\widehat{g_0}(t)| \frac{|x||t|^4}{K^4} dt\right).$$

Recall that $\widehat{g_0^{(a)}}(t) = (-2\pi i t)^a \widehat{g_0}(t)$. Hence

$$W_1(x) = \int_{-\infty}^{\infty} \widehat{g_0}(t) e\left(\frac{\pi x t^2}{K^2}\right) dt + O\left(\frac{|x|}{K^4}\right). \tag{4.1}$$

Using (2.1) and (3.4), the O-term in (4.1) contributes at most

$$\begin{aligned}
&\sum_{N \leq n \leq 2N} \sum_{M \leq ab^2 \leq 2M} \frac{1}{n^{1/2}} \frac{|\lambda_f(a)|}{(ab^2)^{1/2}} \sum_{c < 16\pi\sqrt{Na}} \frac{|S(n,a;c)|}{c} \frac{\sqrt{na}}{c} K^{-4} \\
&\ll K^{-4} \sum_{M \leq ab^2 \leq 2M} \frac{|\lambda_f(a)|}{b} \sum_{c < 16\pi\sqrt{Na}} \frac{1}{c^2} \sum_{N \leq n \leq 2N} |S(n,a;c)| \\
&\ll NMK^{-4} \ll K^{-1+\varepsilon}.
\end{aligned}$$

By Parseval and [GR, 3.691, part 1],

$$\begin{aligned}
\int_{-\infty}^{\infty} \widehat{g_0}(t) e\left(\frac{\pi x t^2}{K^2}\right) dt &= \frac{1+i}{2} \int_0^{\infty} g_0(t) \frac{1}{\sqrt{\frac{\pi|x|}{K^2}}} e\left(\frac{-t^2}{4\frac{\pi x}{K^2}}\right) dt \\
&= \frac{1+i}{2\sqrt{\pi}} \frac{K}{\sqrt{|x|}} \int_0^{\infty} g_0(t) e\left(-\frac{K^2 t^2}{4\pi x}\right) dt \\
&\ll \left(\frac{K^2}{|x|}\right)^{-A},
\end{aligned}$$

for any $A > 0$ (by using integration by parts).

Note that for $|x| = \frac{4\pi\sqrt{na}}{c} \ll N^{1/2} M^{1/2}$,

$$\frac{K^2}{|x|} \gg K^2 N^{-1/2} M^{-1/2} \gg K^{1/2-\varepsilon}.$$

Hence the contribution from the integral in (4.1) to H_1 is negligible. □

4.2. **Estimate of H_2.** To estimate H_2, we need the following Voronoi formula on $GL(2)$ ([DI]).

Proposition 2. *Let $F(\xi)$ be a smooth and compactly supported function on $\mathbb{R}^+$. For any integers $c \geq 1$ and $(c,d) = 1$ we have*

$$\sum_n \lambda_f(n) e\left(\frac{dn}{c}\right) F(n) = \sum_r \lambda_f(r) e\left(\frac{-\bar{d}r}{c}\right) \check{F}(r),$$

where $d\bar{d} \equiv 1 \pmod{c}$ and $\check{F}(r)$ is the Hankel-type transform

$$\check{F}(y) = 2\pi i^l c^{-1} \int_0^{\infty} F(\xi) J_{l-1}\left(\frac{4\pi\sqrt{\xi y}}{c}\right) d\xi,$$

and where $J_\nu(z)$ is the usual Bessel function.

Lemma 10. *For any $\varepsilon > 0$, we have*

$$H_2 \ll K^\varepsilon.$$

Proof. Let

$$W_2(x) = \int_{-\infty}^{\infty} \widehat{g_0}(t) e\left(\frac{x}{2\pi} \sin \frac{2\pi t}{K}\right) dt;$$

then

$$F_2(x) = \frac{W_2(x) - W_2(-x)}{2i}.$$

Since the contribution to $W_2(x)$ from $|t| \geq K^{\varepsilon}$ is negligible, we only need to consider $|t| \leq K^{\varepsilon}$. Now expanding $\sin(\frac{2\pi t}{K})$ into the Taylor series and using $\widehat{\widehat{g_0}}(x) = g_0(-x)$, we get

$$\begin{aligned} W_2(x) &= \int_{-\infty}^{\infty} \widehat{g_0}(t) e\left(\frac{xt}{K}\right) dt + O\left(\int_{-\infty}^{\infty} |\widehat{g_0}(t)| \frac{|x|}{K^3} |t|^3 dt\right) \\ &= g_0\left(-\frac{x}{K}\right) + O\left(\frac{|x|}{K^3}\right). \end{aligned} \tag{4.2}$$

Using (2.1) and (3.4), the O-term in (4.2) contributes at most

$$K^{-3} NM \ll K^{\varepsilon}.$$

Since $g_0(\xi)$ has compact support, $g_0\left(-\frac{x}{K}\right)$ is nonzero only when $|x| \sim K$. For $|x| = \frac{4\pi\sqrt{na}}{c}$, $g_0\left(-\frac{x}{K}\right)$ is nonzero only when $c \sim \frac{4\pi\sqrt{na}}{K} \sim \frac{\sqrt{NM}}{Kb}$ and the contribution to H_2 is

$$\sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w\left(\frac{n}{N}\right) h\left(\frac{ab^2}{M}\right) \frac{1}{n^{1/2}} \frac{\lambda_f(a)}{(ab^2)^{1/2}} \sum_{c \sim \frac{\sqrt{NM}}{Kb}} \frac{S(n,a;c)}{c} g_0\left(\frac{4\pi\sqrt{na}}{Kc}\right).$$

Note that

$$g_0\left(\frac{4\pi\sqrt{na}}{Kc}\right) = u\left(\frac{4\pi\sqrt{na}}{Kc}\right) V\left(n, \frac{4\pi\sqrt{na}}{c} + 1\right) U\left(ab^2, \frac{4\pi\sqrt{na}}{c} + 1\right).$$

By Lemma 3 and Lemma 5, we only need to consider the contribution from

$$u\left(\frac{4\pi\sqrt{na}}{Kc}\right) V_1\left(\frac{n}{\frac{4\pi\sqrt{na}}{c} + 1}\right) U_1\left(\frac{ab^2}{(\frac{4\pi\sqrt{na}}{c} + 1)^2}\right)$$

in (4.3), since the O-term is smaller. Let

$$\begin{aligned} \widetilde{H_2} &= \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w\left(\frac{n}{N}\right) h\left(\frac{ab^2}{M}\right) \frac{1}{n^{1/2}} \frac{\lambda_f(a)}{(ab^2)^{1/2}} \\ &\quad \times \sum_{c \sim \frac{\sqrt{NM}}{Kb}} \frac{S(n,a;c)}{c} u\left(\frac{4\pi\sqrt{na}}{Kc}\right) V_1\left(\frac{n}{\frac{4\pi\sqrt{na}}{c} + 1}\right) U_1\left(\frac{ab^2}{(\frac{4\pi\sqrt{na}}{c} + 1)^2}\right). \end{aligned}$$

After opening the Kloosterman sum and applying Proposition 2 to sum over a with

$$F(\xi) = h\left(\frac{\xi b^2}{M}\right) u\left(\frac{4\pi\sqrt{n\xi}}{Kc}\right) V_1\left(\frac{n}{\frac{4\pi\sqrt{n\xi}}{c} + 1}\right) U_1\left(\frac{\xi b^2}{(\frac{4\pi\sqrt{n\xi}}{c} + 1)^2}\right) \xi^{-1/2},$$

we have

$$\widetilde{H_2} = \sum_{n=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c \sim \frac{\sqrt{NM}}{Kb}} \sideset{}{^*}\sum_{d \,(\mathrm{mod}\ c)} w\left(\frac{n}{N}\right) \frac{1}{n^{1/2} bc} e\left(\frac{n\bar{d}}{c}\right) \sum_{r=1}^{\infty} \lambda_f(r) e\left(-\frac{r\bar{d}}{c}\right) \check{F}(r).$$

Note that $F(\xi)$ has support contained in $[\frac{M}{b^2}, 2\frac{M}{b^2}]$ and

$$F^{(i)}(\xi) \ll \left(\frac{M}{b^2}\right)^{-1/2-i}. \tag{4.3}$$

Using the recurrence formula

$$\frac{d}{dz}(z^\nu J_\nu(z)) = z^\nu J_{\nu-1}(z)$$

and the bound $J_\nu(z) \ll (1+z)^{-1/2}$, we have

$$\check{F}(y) \ll c^{-1} y^{-7/4} \frac{N^{7/4} M^{1/2}}{K^{7/2} b} \ll y^{-7/4} \frac{N^{3/4}}{K^{5/2}} \tag{4.4}$$

by integration by parts three times and (4.3). By (4.4) and Deligne's bound $\lambda(r) \ll r^\varepsilon$, we have

$$\begin{aligned}\widetilde{H_2} &\ll \sum_{N \le n \le 2N} \sum_{b=1}^{\infty} \sum_{c \sim \frac{\sqrt{NM}}{Kb}} \sideset{}{^*}\sum_{d \,(\mathrm{mod}\ c)} \frac{1}{n^{1/2} bc} \sum_{r=1}^{\infty} r^{-7/4+\varepsilon} \frac{N^{3/4}}{K^{5/2}} \\ &\ll N^{7/4} M^{1/2} K^{-7/2} \ll K^{-3/4+\varepsilon}. \end{aligned}$$

□

5. A note on simultaneous nonvanishing of $GL(3) \times GL(2)$ and $GL(2)$ L-functions

Let F be a fixed $GL(3)$ Maass-Hecke cusp form. Li [Li] and Khan [Kh] proved that there are *infinitely* many $GL(2)$ cusp forms g such that $L(\frac{1}{2}, F \times g) L(\frac{1}{2}, g) \neq 0$. Their proofs were to establish an asymptotic formula like Theorem 2 which is very delicate in their cases. However if we just want to prove that there *exists one* g such that $L(\frac{1}{2}, F \times g) L(\frac{1}{2}, g) \neq 0$ for the special case $F = \mathrm{sym}^2 f$, where f is a $GL(2)$ cusp form, it is much easier. Actually, it is a consequence of Watson's formula [Wa]. More precisely, we have the following result.

Theorem 3. *Let f be a fixed holomorphic Hecke cusp form of weight k for $SL(2, \mathbb{Z})$. Then there exists $g \in H_{2k}$ such that*

$$L(\tfrac{1}{2}, \mathrm{sym}^2 f \times g) L(\tfrac{1}{2}, g) \neq 0. \tag{5.1}$$

Proof. Since f^2 is a holomorphic cusp of weight $2k$, we can express it as

$$f^2 = \sum_{g \in H_{2k}} \left\langle f^2, \frac{g}{||g||} \right\rangle \frac{g}{||g||}.$$

Using Watson's formula [Wa, Theorem 3]

$$\frac{|\langle f^2, g\rangle|^2}{\langle f, f\rangle^2 \langle g, g\rangle} = \frac{\Lambda(\frac{1}{2}, f \times f \times g)}{4\Lambda(1, \mathrm{sym}^2 f)^2 \Lambda(1, \mathrm{sym}^2 g)} \tag{5.2}$$

(here $\Lambda(s, f \times f \times g), \Lambda(s, \mathrm{sym}^2 f)$ and $\Lambda(s, \mathrm{sym}^2 g)$ are completed L-functions) and the factorization $L(s, f \times f \times g) = L(s, \mathrm{sym}^2 f \times g) L(s, g)$, we deduce that

$$0 \neq \langle f^2, f^2\rangle = \sum_{g \in H_{2k}} \frac{|\langle f^2, g\rangle|^2}{\langle g, g\rangle} = C_{k,f} \sum_{g \in H_{2k}} \frac{L(\frac{1}{2}, \mathrm{sym}^2 f \times g) L(\frac{1}{2}, g)}{L(1, \mathrm{sym}^2 g)}, \tag{5.3}$$

where $C_{k,f}$ is a nonzero constant which depends on k and f. The theorem follows from (5.3). □

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