PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 7, July 2014, Pages 2559–2568
S 0002-9939(2014)11951-4
Article electronically published on March 27, 2014

BURGHELEA-HALLER ANALYTIC TORSION OF $\mathbb{Z}_2$-GRADED ELLIPTIC COMPLEXES

GUANGXIANG SU

(Communicated by Varghese Mathai)

Abstract. In this paper, we extend the analytic torsion of $\mathbb{Z}_2$-graded elliptic complexes introduced by Mathai and Wu to the complex-valued case in the line of Burghelea and Haller. We also study properties of this generalized analytic torsion.

1. Introduction

Let E be a unitary flat vector bundle on a closed Riemannian manifold M. In [26], Ray and Singer defined an analytic torsion associated to (M, E) and proved that it does not depend on the Riemannian metric on M. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on M (cf. [16]). This conjecture was later proved in the celebrated papers of Cheeger [10] and Müller [18]. Müller generalized this result in [19] to the case when E is a unimodular flat vector bundle on M. In [1], inspired by the considerations of Quillen [24], Bismut and Zhang reformulated the above Cheeger-Müller theorem as an equality between the Reidemeister and Ray-Singer metrics defined on the determinant of cohomology, and proved an extension of it to the case of a general flat vector bundle over M. The method used in [1] is different from those of Cheeger and Müller in that it makes use of a deformation by Morse functions introduced by Witten [28] on the de Rham complex.

In [21, 22], Mathai and Wu generalized the classical Ray-Singer analytic torsion [26] to the twisted de Rham complex with an odd degree closed differential form H. Recently, Huang [14, 15] generalized the refined analytic torsion [2–4] and the Cappell-Miller holomorphic and analytic torsion [9] to the twisted de Rham complex. In [27], Su generalized the Burghelea-Haller analytic torsion [5–7] to the twisted case. On the other hand, Mathai and Wu [23] defined and studied analytic torsion of $\mathbb{Z}_2$-graded elliptic complexes which applies to a myriad of new examples, including flat superconnection complexes, twisted analytic and twisted holomorphic torsions. So it is natural to ask whether the analytic torsion of a $\mathbb{Z}_2$-graded elliptic complex can be extended to the complex-valued case. In this paper, we will do this in the line of Burghelea and Haller.

Received by the editors September 29, 2010 and, in revised form, October 13, 2010; November 16, 2011; and July 19, 2012.

2010 *Mathematics Subject Classification.* Primary 58J52.

Key words and phrases. Analytic torsion, elliptic complex, symmetric bilinear form.

The author was supported by "the Fundamental Research Funds for the Central Universities" and NSFC 11101219.

The rest of this paper is organized as follows. In Section 2, we recall the definition of the $\mathbb{Z}_2$-graded elliptic complex from [23] and define the Burghelea-Haller analytic torsion of this complex under the assumption that there exists a $\mathbb{Z}_2$-graded non-degenerate symmetric bilinear form. In Section 3, we study the dependence of the analytic torsion defined in Section 2 on the metric and the symmetric bilinear form. In Section 4, we define the relative torsion similar to that in [23] and study the dependence of it on the metric and the symmetric bilinear form. In Section 5, we discuss the case of flat superconnections.

2. $\mathbb{Z}_2$-graded elliptic complex and the construction of the torsion

Let X be a smooth closed manifold of dimension n and $\mathcal{E} = \mathcal{E}^{\bar{0}} \oplus \mathcal{E}^{\bar{1}}$ be a $\mathbb{Z}_2$-graded complex vector bundle over X. (We use $\bar{k}$ to denote the integer k modulo 2.) Suppose $D : \Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{E})$ is an elliptic differential operator which is odd with respect to the grading in $\mathcal{E}$ and satisfies $D^2 = 0$. Then D is of the form

$$D = \begin{pmatrix} 0 & D_{\bar{1}} \\ D_{\bar{0}} & 0 \end{pmatrix}$$

on $\Gamma(X, \mathcal{E}) = \Gamma(X, \mathcal{E}^{\bar{0}}) \oplus \Gamma(X, \mathcal{E}^{\bar{1}})$, where $D_{\bar{0}} : \Gamma(X, \mathcal{E}^{\bar{0}}) \to \Gamma(X, \mathcal{E}^{\bar{1}})$ and $D_{\bar{1}} : \Gamma(X, \mathcal{E}^{\bar{1}}) \to \Gamma(X, \mathcal{E}^{\bar{0}})$ are differential operators such that $D_{\bar{1}} D_{\bar{0}} = 0$ and $D_{\bar{0}} D_{\bar{1}} = 0$. Furthermore,

$$\cdots \xrightarrow{D_{\bar{1}}} \Gamma(X, \mathcal{E}^{\bar{0}}) \xrightarrow{D_{\bar{0}}} \Gamma(X, \mathcal{E}^{\bar{1}}) \xrightarrow{D_{\bar{1}}} \Gamma(X, \mathcal{E}^{\bar{0}}) \xrightarrow{D_{\bar{0}}} \cdots \tag{2.1}$$

is a $\mathbb{Z}_2$-graded elliptic complex, which we denote by $(\mathcal{E}, D)$ for short. Its cohomology groups are

$$H^{\bar{0}}(X, \mathcal{E}, D) = \ker D_{\bar{0}} / \mathrm{im} D_{\bar{1}}, \quad H^{\bar{1}}(X, \mathcal{E}, D) = \ker D_{\bar{1}} / \mathrm{im} D_{\bar{0}}.$$

And they are finite dimensional.

We choose a Riemannian metric g on X and suppose there exists a $\mathbb{Z}_2$-graded non-degenerate symmetric bilinear form on $\mathcal{E}$ that is of type

$$b = \begin{pmatrix} b_{\bar{0}} & \\ & b_{\bar{1}} \end{pmatrix}$$

on $\mathcal{E} = \mathcal{E}^{\bar{0}} \oplus \mathcal{E}^{\bar{1}}$. Then there is a non-degenerate symmetric bilinear form on $\Gamma(X, \mathcal{E})$ which we still denote by b. Let $D^{\#}$ be the adjoint operator of D with respect to b. And define the Laplacian

$$L_b = D^{\#} D + D D^{\#}$$

on $\Gamma(X, \mathcal{E}) = \Gamma(X, \mathcal{E}^{\bar{0}}) \oplus \Gamma(X, \mathcal{E}^{\bar{1}})$ which is, in graded components, $L_b = \begin{pmatrix} L_{b,\bar{0}} & \\ & L_{b,\bar{1}} \end{pmatrix}$, where

$$L_{b,\bar{0}} = D_{\bar{0}}^{\#} D_{\bar{0}} + D_{\bar{1}} D_{\bar{1}}^{\#}, \quad L_{b,\bar{1}} = D_{\bar{1}}^{\#} D_{\bar{1}} + D_{\bar{0}} D_{\bar{0}}^{\#}.$$

They are elliptic operators with positive-definite self-adjoint leading symbols. Suppose the order of L_b (or that of $L_{b,\bar{0}}$ and $L_{b,\bar{1}}$) is $d > 1$. Let

$$K_b(t, x, y) = \begin{pmatrix} K_{b,\bar{0}}(t, x, y) & \\ & K_{b,\bar{1}}(t, x, y) \end{pmatrix},$$

where $t>0$, $x,y\in X$, be the kernel of $e^{-tL_b}=\begin{pmatrix} e^{-tL_{b,\bar{0}}} & \\ & e^{-tL_{b,\bar{1}}}\end{pmatrix}$. By [11, Lemma 1.8.2], when restricted to the diagonal, the heat kernel has the asymptotic expansion

$$K_b(t,x,x)\sim\sum_{l=0}^{\infty}t^{\frac{2l-n}{d}}a_{b,l}(x),$$

where $a_{b,l}(x)=\begin{pmatrix} a_{b,l,\bar{0}}(x) & \\ & a_{b,l,\bar{1}}(x)\end{pmatrix}$ can be computed locally as a combinatorial expression in the jets of the symbols.

Let λ be a generalized eigenvalue of L_b and $\Gamma(X,\mathcal{E})_{\{\lambda\}}$ be the generalized eigenspace corresponding to λ. Then by ellipticity, $\Gamma(X,\mathcal{E})_{\{\lambda\}}$ is of finite dimension. If λ,μ are both generalized eigenvalues of L_b and $\lambda\neq\mu$, we see that $\Gamma(X,\mathcal{E})_{\{\lambda\}}$ and $\Gamma(X,\mathcal{E})_{\{\mu\}}$ are b-orthogonal.

For any $0\leq a<+\infty$, set

$$\Gamma(X,\mathcal{E})_{[0,a]}=\bigoplus_{0\leq|\lambda|\leq a}\Gamma(X,\mathcal{E})_{\{\lambda\}}.$$

Then $\Gamma(X,\mathcal{E})_{[0,a]}$ is of finite dimension and one gets a non-degenerate symmetric bilinear form $b_{[0,a]}$ on it. For the $\mathbb{Z}_2$-graded finite dimensional complex

$$\cdots\xrightarrow{D_{\bar{1}}}\Gamma(X,\mathcal{E}^{\bar{0}})_{[0,a]}\xrightarrow{D_{\bar{0}}}\Gamma(X,\mathcal{E}^{\bar{1}})_{[0,a]}\xrightarrow{D_{\bar{1}}}\Gamma(X,\mathcal{E}^{\bar{0}})_{[0,a]}\xrightarrow{D_{\bar{0}}}\cdots,$$

by the canonical isomorphism

$$\det H^{\bullet}(\Gamma(X,\mathcal{E})_{[0,a]},D)\cong\det(\Gamma(X,\mathcal{E})_{[0,a]},D),$$

one gets a non-degenerate symmetric bilinear form $b_{\det H^{\bullet}(\Gamma(X,\mathcal{E})_{[0,a]},D)}$ on the line $\det H^{\bullet}(\Gamma(X,\mathcal{E})_{[0,a]},D)$.

Let $\Gamma(X,\mathcal{E})_{(a,+\infty)}$ be the b-orthogonal of $\Gamma(X,\mathcal{E})_{[0,a]}$. Consider the partial Laplacian

$$D^{\#}D=\begin{pmatrix} D_{\bar{0}}^{\#}D_{\bar{0}} & \\ & D_{\bar{1}}^{\#}D_{\bar{1}}\end{pmatrix}.$$

Since $D_{\bar{k}}D_{\bar{k}}^{\#}$ and $L_{b,\overline{k+1}}$ are equal and invertible on $\mathrm{im}D_{\bar{k}}\cap\Gamma(X,\mathcal{E}^{\overline{k+1}})_{(a,+\infty)}$, $k=0,1$, we see that

$$P_{\bar{k}}:=D_{\bar{k}}^{\#}\left(D_{\bar{k}}D_{\bar{k}}^{\#}\right)^{-1}D_{\bar{k}}=D_{\bar{k}}^{\#}(L_{b,\overline{k+1}})^{-1}D_{\bar{k}}$$

is a pseudodifferential operator of order 0 and satisfies

$$P_{\bar{k}}^2=P_{\bar{k}}.$$

By definition we have

$$\begin{aligned}\zeta\left(s,D_{\bar{k}}^{\#}D_{\bar{k}}|_{\mathrm{im}D_{\bar{k}}^{\#}\cap\Gamma(X,\mathcal{E}^{\bar{k}})_{(a,+\infty)}}\right)&=\mathrm{Tr}\left(L_{b,\bar{k}}^{-s}P_{\bar{k}}|_{\Gamma(X,\mathcal{E}^{\bar{k}})_{(a,+\infty)}}\right)\\&=\mathrm{Tr}\left(P_{\bar{k}}L_{b,\bar{k}}^{-s}|_{\Gamma(X,\mathcal{E}^{\bar{k}})_{(a,+\infty)}}\right).\end{aligned}\tag{2.2}$$

Then $\zeta\left(s,D_{\bar{k}}^{\#}D_{\bar{k}}|_{\mathrm{im}D_{\bar{k}}^{\#}\cap\Gamma(X,\mathcal{E}^{\bar{k}})_{(a,+\infty)}}\right)$ has a meromorphic extension to the whole complex plane and, by [29, Section 7], it is regular at 0. Then by [13, 29], we have the following result which is an analogue of [23, Proposition 2.1].

Proposition 2.1. *For $k = 0, 1$, $\zeta\left(s, D^{\#}_{\bar{k}} D_{\bar{k}}|_{\mathrm{im} D^{\#}_{\bar{k}} \cap \Gamma(X, \mathcal{E}^{\bar{k}})_{(a,+\infty)}}\right)$ is holomorphic in the half plane for $\mathrm{Re}(s) > n/d$ and extends meromorphically to $\mathbb{C}$ with possible poles at $\{\frac{n-l}{d}, l = 0, 1, 2, \dots\}$ only, and it is holomorphic at $s = 0$.*

By Proposition 2.1, one can define the determinant

$$\det{}'\left(D^{\#}_{\bar{k}} D_{\bar{k}}|_{\Gamma(X, \mathcal{E}^{\bar{k}})_{(a,+\infty)}}\right) := \exp\left(-\zeta'\left(0, D^{\#}_{\bar{k}} D_{\bar{k}}|_{\mathrm{im} D^{\#}_{\bar{k}} \cap \Gamma(X, \mathcal{E}^{\bar{k}})_{(a,+\infty)}}\right)\right), \ k = 0, 1.$$

Then by the same proof as in [27, Proposition 3.2], one easily gets the following proposition.

Proposition 2.2. *The symmetric bilinear form on $\det H^{\bullet}(X, \mathcal{E}, D)$ via the isomorphism $\det H^{\bullet}(\Gamma(X, \mathcal{E})_{[0,a]}, D) \cong \det H^{\bullet}(X, \mathcal{E}, D)$ defined by*

$$\begin{aligned} &b_{\det H^{\bullet}(\Gamma(X,\mathcal{E})_{[0,a]}, D)} \\ &\cdot \det{}'\left(D^{\#}_{\bar{0}} D_{\bar{0}}|_{\Gamma(X, \mathcal{E}^{\bar{0}})_{(a,+\infty)}}\right)^{-1} \cdot \det{}'\left(D^{\#}_{\bar{1}} D_{\bar{1}}|_{\Gamma(X, \mathcal{E}^{\bar{1}})_{(a,+\infty)}}\right) \end{aligned} \tag{2.3}$$

is independent of the choice of $a \geq 0$.

Proof. Let $0 \leq a < c < +\infty$. We have

$$\left(\Gamma(X, \mathcal{E})_{[0,c]}, D\right) = \left(\Gamma(X, \mathcal{E})_{[0,a]}, D\right) \bigoplus \left(\Gamma(X, \mathcal{E})_{(a,c]}, D\right) \tag{2.4}$$

and

$$\left(\Gamma(X, \mathcal{E})_{(a,+\infty)}, D\right) = \left(\Gamma(X, \mathcal{E})_{(a,c]}, D\right) \bigoplus \left(\Gamma(X, \mathcal{E})_{(c,+\infty)}, D\right).$$

By definition of determinant, we get

$$\begin{aligned} &\det{}'\left(D^{\#}_{\bar{k}} D_{\bar{k}}|_{\Gamma(X, \mathcal{E}^{\bar{k}})_{(a,+\infty)}}\right) \\ &= \det{}'\left(D^{\#}_{\bar{k}} D_{\bar{k}}|_{\Gamma(X, \mathcal{E}^{\bar{k}})_{(a,c]}}\right) \cdot \det{}'\left(D^{\#}_{\bar{k}} D_{\bar{k}}|_{\Gamma(X, \mathcal{E}^{\bar{k}})_{(c,+\infty)}}\right). \end{aligned} \tag{2.5}$$

Applying [27, Proposition 2.1] to (2.4), we get

$$\begin{aligned} &b_{\det H^{\bullet}(\Gamma(X,\mathcal{E})_{[0,c]}, D)} \\ &= b_{\det H^{\bullet}(\Gamma(X,\mathcal{E})_{[0,a]}, D)} \cdot \det{}'\left(D^{\#}_{\bar{0}} D_{\bar{0}}|_{\Gamma(X, \mathcal{E}^{\bar{0}})_{(a,c]}}\right)^{-1} \cdot \det{}'\left(D^{\#}_{\bar{1}} D_{\bar{1}}|_{\Gamma(X, \mathcal{E}^{\bar{1}})_{(a,c]}}\right). \end{aligned} \tag{2.6}$$

Then we get the proposition. □

Definition 2.3. The symmetric bilinear form defined by (2.3) is called the Ray-Singer symmetric bilinear torsion of the $\mathbb{Z}_2$-graded elliptic complex $(\mathcal{E}, D)$ and is denoted by $b_{\det H^{\bullet}(X,\mathcal{E},D)}$.

3. Variation of the torsion with respect to the metric and symmetric bilinear form

Suppose the pair (g_u, b_u) is deformed smoothly along a one-parameter family with parameter $u \in \mathbb{R}$. Let Q be the spectral projection onto $\Gamma(X, \mathcal{E})_{[0,a]}$ and $\Pi = 1 - Q$ be the spectral projection onto $\Gamma(X, \mathcal{E})_{(a,+\infty)}$. Let

$$\alpha = *_u^{-1} \frac{\partial *_u}{\partial u} + b_u^{-1} \frac{\partial b_u}{\partial u}.$$

Then we have the $\alpha = \begin{pmatrix} \alpha_{\bar{0}} & \\ & \alpha_{\bar{1}} \end{pmatrix}$.

Theorem 3.1. *Under the above deformation of g and b, we have*

$$\frac{\partial}{\partial u}\log b_{\det H^\bullet(X,\mathcal{E},D)} = \mathrm{Str}(\alpha a_{b,\frac{n}{2}}).$$

Proof. The proof is similar to that in [23, Theorem 4.1]. First we have

$$\frac{\partial}{\partial u}D_u^\# = -[\alpha, D_u^\#].$$

In graded components, that is

$$\frac{\partial D_{\bar{0}}^\#}{\partial u} = -\alpha_{\bar{1}}D_{\bar{0}}^\# + D_{\bar{0}}^\#\alpha_{\bar{0}}, \quad \frac{\partial D_{\bar{1}}^\#}{\partial u} = -\alpha_{\bar{0}}D_{\bar{1}}^\# + D_{\bar{1}}^\#\alpha_{\bar{1}}.$$

Following [23], we set $P = D^\#(DD^\#)^{-1}D = D^\#L_b^{-1}D$ and

$$\begin{aligned} f(s,u) &= \int_0^{+\infty} t^{s-1}\mathrm{Str}\left(e^{-tD^\#D}P|_{\Gamma(X,\mathcal{E})_{(a,+\infty)}}\right)dt \\ &= \Gamma(s)\left(\zeta\left(s, D_{\bar{0}}^\# D_{\bar{0}}|_{\Gamma(X,\mathcal{E}^{\bar{0}})_{(a,+\infty)}}\right) - \zeta\left(s, D_{\bar{1}}^\# D_{\bar{1}}|_{\Gamma(X,\mathcal{E}^{\bar{1}})_{(a,+\infty)}}\right)\right). \end{aligned} \tag{3.1}$$

Using $PD^\# = D^\#$, $DP = P$ and $P^2 = P$, we get $D^\#DP = PD^\#D = D^\#D$ and

$$P\frac{\partial P}{\partial u}P = 0.$$

Then

$$\begin{aligned} \frac{\partial f}{\partial u} &= \int_0^{+\infty} t^{s-1}\mathrm{Str}\left(t\left[\alpha, D^\#\right]De^{-tD^\#D}\Pi + e^{-tD^\#D}\frac{\partial P}{\partial u}P\Pi\right)dt \\ &= \int_0^{+\infty} t^{s-1}\mathrm{Str}\left(t\alpha\left(e^{-tD^\#D}D^\#D - e^{-tDD^\#}DD^\#\right)\Pi + e^{-tD^\#D}P\frac{\partial P}{\partial u}\Pi\right)dt \\ &= \int_0^{+\infty} t^s\mathrm{Str}\left(\alpha e^{-tL_b}L_b\Pi\right)dt \\ &= -\int_0^{+\infty} t^s\frac{\partial}{\partial t}\mathrm{Str}\left(\alpha e^{-tL_b}\Pi\right)dt \\ &= s\left(\int_0^1 + \int_1^{+\infty}\right)t^{s-1}\mathrm{Str}\left(\alpha e^{-tL_b}(1-Q)\right)dt. \end{aligned} \tag{3.2}$$

By the asymptotic expansion of $\mathrm{Str}(\alpha e^{-tL_b})$ as $t \downarrow 0$,

$$\int_0^1 t^{s-1}\mathrm{Str}\left(\alpha e^{-tL_b}\right)dt$$

has a possible first order pole at $s = 0$ with residue $\mathrm{Str}(\alpha a_{b,\frac{n}{2}})$. On the other hand, because of the exponential decay of $\mathrm{Str}(\alpha e^{-tL_b}\Pi)$ for large t,

$$\int_1^{+\infty} t^{s-1}\mathrm{Str}\left(\alpha e^{-tL_b}\Pi\right)dt$$

is an entire function in s. So

$$\left.\frac{\partial f}{\partial u}\right|_{s=0} = -\mathrm{Str}(\alpha(Q - a_{b,\frac{n}{2}})) \tag{3.3}$$

and hence

$$\frac{\partial}{\partial u}\left(\zeta\left(0, D_{\bar{0}}^\# D_{\bar{0}}|_{\Gamma(X,\mathcal{E}^{\bar{0}})_{(a,+\infty)}}\right) - \zeta\left(0, D_{\bar{1}}^\# D_{\bar{1}}|_{\Gamma(X,\mathcal{E}^{\bar{1}})_{(a,+\infty)}}\right)\right) = 0. \tag{3.4}$$

Then from (3.3), (3.4) and

(3.5)
$$\frac{\det'(D_{\bar{1}}^{\#}D_{\bar{1}}|_{\Gamma(X,\mathcal{E}^{\bar{1}})_{(a,+\infty)}})}{\det'(D_{\bar{0}}^{\#}D_{\bar{0}}|_{\Gamma(X,\mathcal{E}^{\bar{0}})_{(a,+\infty)}})}$$
$$=\exp(\lim_{s\to 0}(f(s,u)-\frac{1}{s}(\zeta(0,D_{\bar{0}}^{\#}D_{\bar{0}}|_{\Gamma(X,\mathcal{E}^{\bar{0}})_{(a,+\infty)}})-\zeta(0,D_{\bar{1}}^{\#}D_{\bar{1}}|_{\Gamma(X,\mathcal{E}^{\bar{1}})_{(a,+\infty)}})))),$$

we get

$$\frac{\partial}{\partial u}\log\left(\frac{\det'(D_{\bar{1}}^{\#}D_{\bar{1}}|_{\Gamma(X,\mathcal{E}^{\bar{1}})_{(a,+\infty)}})}{\det'(D_{\bar{0}}^{\#}D_{\bar{0}}|_{\Gamma(X,\mathcal{E}^{\bar{0}})_{(a,+\infty)}})}\right)=-\mathrm{Str}(\alpha(Q-a_{b,\frac{n}{2}})). \tag{3.6}$$

By [27, Lemma 4.2], one easily gets

$$\frac{\partial}{\partial w}\bigg|_u\left(\frac{b_{w,\det H^\bullet(\Gamma(X,\mathcal{E})_{[0,a]},D)}}{b_{u,\det H^\bullet(\Gamma(X,\mathcal{E})_{[0,a]},D)}}\right)=\mathrm{Str}(\alpha Q). \tag{3.7}$$

Then by (3.6) and (3.7), we get the theorem. □

4. Invariance of relative torsion under deformation of metrics: The even-dimensional case

When $n=\dim X$ is even, the torsion does depend on the metric g on X and the non-degenerate symmetric bilinear form b on $\mathcal{E}$. However, we will prove that the relative analytic torsion defined similarly as in [23] is independent of the choice of g and b.

Let $\Pi=\pi_1(X)$ be the fundamental group of X and let $\rho:\Pi\to GL(m,\mathbb{C})$ be a representation of Π. Then ρ determines a flat bundle $\mathcal{F}_\rho$ over X given by

$$\mathcal{F}_\rho=\left(\widetilde{X}\times\mathbb{C}^m\right)/\sim,\ (x\gamma,v)\sim(x,\rho(\gamma)v),$$

where $\widetilde{X}$ is the universal cover of X. Smooth sections of $\mathcal{F}_\rho$ are smooth maps $s:\widetilde{X}\to\mathbb{C}^m$ that are Π-equivariant, i.e., $s\circ\gamma=\rho(\gamma)s$ for all $\gamma\in\Pi$. We want to extend D to an action on the sections of $\mathcal{E}_\rho=\mathcal{E}\otimes\mathcal{F}_\rho$. Since D is a differential operator, it lifts to the universal cover $\widetilde{X}$ as a Π-periodic operator $\widetilde{D}:\Gamma(\widetilde{X},\widetilde{\mathcal{E}})\to\Gamma(\widetilde{X},\widetilde{\mathcal{E}})$, where $\widetilde{\mathcal{E}}$ is the pull-back of $\mathcal{E}$ to $\widetilde{X}$. By tensoring with the identity operator on $\mathbb{C}^m$, we can extend it to $\widetilde{D}:\Gamma(\widetilde{X},\widetilde{\mathcal{E}}\otimes\mathbb{C}^m)\to\Gamma(\widetilde{X},\widetilde{\mathcal{E}}\otimes\mathbb{C}^m)$. Since for any Π-equivariant section $s\in\Gamma(\widetilde{X},\widetilde{\mathcal{E}}\otimes\mathbb{C}^m)$,

$$(\widetilde{D}s)\circ\gamma=\widetilde{D}(s\circ\gamma)=\widetilde{D}(\rho(\gamma)s)=\rho(\gamma)(\widetilde{D}s),$$

the operator $\widetilde{D}$ descends to a differential operator $D_\rho:\Gamma(X,\mathcal{E}_\rho)\to\Gamma(X,\mathcal{E}_\rho)$. If $(\mathcal{E},D)$ is a $\mathbb{Z}_2$-graded elliptic complex, then so is $(\mathcal{E}_\rho,D_\rho)$.

Now suppose X is a closed, compact, oriented Riemannian manifold and $\mathcal{E}$ is a $\mathbb{Z}_2$-graded Hermitian bundle. Let ρ_1, ρ_2 be unitary representations of Π of the same dimension m. Then the flat bundles $\mathcal{F}_{\rho_i}$ have induced non-degenerate symmetric bilinear forms from $\mathbb{C}^m$, and so $\mathcal{E}_{\rho_i}$ have non-degenerate symmetric bilinear forms. Furthermore, if $(\mathcal{E},D)$ is a $\mathbb{Z}_2$-graded elliptic complex, then so are $(\mathcal{E}_{\rho_i},D_{\rho_i})$ for $i=1,2$.

Definition 4.1. The relative Burghelea-Haller analytic torsion is the non-degenerate symmetric bilinear form on $\det H^\bullet(X,\mathcal{E}_{\rho_1},D_{\rho_1})\otimes\det H^\bullet(X,\mathcal{E}_{\rho_2},D_{\rho_2})^{-1}$ defined by

$$b_{\det H^\bullet(X,\mathcal{E}_{\rho_1},D_{\rho_1})}\otimes b^{-1}_{\det H^\bullet(X,\mathcal{E}_{\rho_2},D_{\rho_2})}.$$

To show its dependence of the metric and the non-degenerate symmetric bilinear forms, let $K_{\rho_i}(t,x,y)$ denote, for $i=1,2$, the heat kernel of the Laplacians $L_{\rho_i}=D^\#_{\rho_i}D_{\rho_i}+D_{\rho_i}D^\#_{\rho_i}$. Then by the same proof as in [23, Proposition 5.2], we have

Proposition 4.2. *In the notation above, there are positive constants C, C' such that as $t\to 0$, one has for all $x\in X$,*

$$|K_{\rho_1}(t,x,x)-K_{\rho_2}(t,x,x)|\le Ct^{-n/d}\exp\left[-C't^{-\frac{1}{d-1}}\right],$$

where d is the order of the Laplacians.

Proof. Let $\widetilde{K}(t,x,y)$ denote the heat kernel of the Laplacian $\widetilde{L}=\widetilde{D}^\#\widetilde{D}+\widetilde{D}\widetilde{D}^\#$ on $\widetilde{X}$. Then, by the Selberg principle, one has for $x,y\in\widetilde{X}$,

$$K_{\rho_j}(t,\bar{x},\bar{y})=\sum_{\gamma\in\Pi}\widetilde{K}(t,x,y\gamma)\rho_j(\gamma),$$

where $\bar{x}\in X$ stands for the projection of $x\in\widetilde{X}$. It follows that

$$K_{\rho_1}(t,\bar{x},\bar{y})-K_{\rho_2}(t,\bar{x},\bar{y})=\sum_{\gamma\in\Pi\setminus\{1\}}\widetilde{K}(t,x,y\gamma)(\rho_1(\gamma)-\rho_2(\gamma)).$$

Since ρ_i $(i=1,2)$ are unitary representations, one has

$$|K_{\rho_1}(t,\bar{x},\bar{y})-K_{\rho_2}(t,\bar{x},\bar{y})|\le\sum_{\gamma\in\Pi\setminus\{1\}}2\left|\widetilde{K}(t,x,y\gamma)\right|.$$

Since the principal symbol of $\widetilde{L}$ is the same as the operator considered in [23, Proposition 5.2], then by [12] and the proof of [8, Proposition 1], one can get the following off-diagonal Gaussian estimate for the heat kernel on $\widetilde{X}$:

$$\left|\widetilde{K}(t,x,y)\right|\le C_1t^{-n/d}\exp\left[-C_2t^{-\frac{1}{d-1}}d(x,y)^{\frac{d}{d-1}}\right],$$

where $d(x,y)$ is the Riemannian distance between $x,y\in\widetilde{X}$. Therefore

$$|K_{\rho_1}(t,\bar{x},\bar{x})-K_{\rho_2}(t,\bar{x},\bar{x})|\le 2C_1t^{-n/d}\sum_{\gamma\in\Pi\setminus\{1\}}\exp\left[-C_2t^{-\frac{1}{d-1}}d(x,y)^{\frac{d}{d-1}}\right].$$

By Milnor's theorem [17], there is a positive constant C_3 such that $d(x,x\gamma)\ge C_3l(\gamma)$, where l denotes a word metric on Π. Moreover, the number of elements in the sphere S_l of radius l in Π satisfies $\#S_l\le C_4e^{C_5l}$ for some positive constants C_4,C_5. Therefore

$$\begin{aligned}&\sum_{\gamma\in\Pi\setminus\{1\}}\exp\left[-C_2t^{-\frac{1}{d-1}}d(x,x\gamma)^{\frac{d}{d-1}}\right]\\ &\le\sum_{\gamma\in\Pi\setminus\{1\}}\exp\left[-C't^{-\frac{1}{d-1}}l(\gamma)^{\frac{d}{d-1}}\right]\le\sum_{l=1}^{\infty}\left[-C't^{-\frac{1}{d-1}}l^{\frac{d}{d-1}}\right]C_4e^{C_5l}\\ &\le C_4\exp\left[-C't^{-\frac{1}{d-1}}\right]\sum_{l=1}^{\infty}\exp\left[-C'\left(l^{\frac{d}{d-1}}-1\right)+C_5l\right]\end{aligned}\tag{4.1}$$

for all t such that $0 < t \leq 1$ for some positive constant C'. Since $\frac{d}{d-1} > 1$, the infinite sum over l converges and then we get the result. $\square$

Theorem 4.3. *Let X be a closed oriented manifold of even dimension. Let ρ_1, ρ_2 be unitary representations of $\pi_1(X)$ of the same dimension. Then the relative analytic torsion $b_{\det H^\bullet(X,\mathcal{E}_{\rho_1},D_{\rho_1})} \otimes b^{-1}_{\det H^\bullet(X,\mathcal{E}_{\rho_2},D_{\rho_2})}$ is independent of the choice of the metric on X and the non-degenerate symmetric bilinear form on $\mathcal{E}$.*

Proof. By Theorem 3.1, under a one-parameter deformation of the metric and the non-degenerate symmetric bilinear form,

$$\frac{\partial}{\partial u} \log b_{\det H^\bullet(X,\mathcal{E}_{\rho_i},D_{\rho_i})} = \mathrm{Str}(\alpha a^{\rho_i}_{\frac{n}{2}})$$

for $i = 1, 2$. By Proposition 4.2, we have $a^{\rho_1}_{\frac{n}{2}} = a^{\rho_2}_{\frac{n}{2}}$. Then we get the theorem. $\square$

5. The case of flat superconnections

Let X be a smooth manifold and $\mathcal{F} = \mathcal{F}^{\bar{0}} \oplus \mathcal{F}^{\bar{1}}$, a $\mathbb{Z}_2$-graded complex vector bundle over X. Then the space $\Omega(X, \mathcal{F})$ of $\mathcal{F}$-valued differential forms has a $\mathbb{Z}_2$-grading with

$$\Omega(X,\mathcal{F})^{\bar{0}} = \Omega^{\bar{0}}(X,\mathcal{F}^{\bar{0}}) \oplus \Omega^{\bar{1}}(X,\mathcal{F}^{\bar{1}}),\ \Omega(X,\mathcal{F})^{\bar{1}} = \Omega^{\bar{0}}(X,\mathcal{F}^{\bar{1}}) \oplus \Omega^{\bar{1}}(X,\mathcal{F}^{\bar{0}}).$$

A superconnection (cf. [20, 25]) is a first-order differential operator $\mathbb{A}$ on $\Omega(X, \mathcal{F})$ that is odd with respect to the $\mathbb{Z}_2$-grading and satisfies

$$\mathbb{A}(\alpha \wedge s) = (\mathbb{A}\alpha) \wedge s + (-1)^{|\alpha|}\alpha \wedge s$$

for any $\alpha \in \Omega(X)$ and $s \in \Omega(X, \mathcal{F})$. The bundle $\mathrm{End}(\mathcal{F})$ is also $\mathbb{Z}_2$-graded and $\mathbb{A}$ extends to $\Omega(X, \mathrm{End}(\mathcal{F}))$. If X is closed, compact, oriented Riemannian manifold, $\mathbb{A}$ is a flat superconnection and there exists a $\mathbb{Z}_2$-graded non-degenerated symmetric bilinear form on $\mathcal{F}$, then we can define a symmetric bilinear form $b_{\det H^\bullet(X,\mathcal{F},\mathbb{A})}$ as in Section 2 and Theorem 3.1 also holds for $b_{\det H^\bullet(X,\mathcal{F},\mathbb{A})}$.

Suppose $G \in \Omega(X, \mathrm{End}(\mathcal{F}))^{\bar{0}}$ is point-wise invertible. Then $\mathbb{A}' = G^{-1}\mathbb{A}G$ is another flat superconnection on $\mathcal{F}$; we say that $\mathbb{A}'$ is gauge equivalent to $\mathbb{A}$. There is an isomorphism of cohomology groups $H^\bullet(X, \mathcal{F}, \mathbb{A}) \cong H^\bullet(X, \mathcal{F}, \mathbb{A}')$, and hence of the corresponding determinant lines, induced by G. Now suppose $\mathbb{A}$ is deformed to $\mathbb{A}_v$ along a path parameterized by v so that each $\mathbb{A}_v$ is gauge equivalent to $\mathbb{A}$ via G_v. Let $\phi_v : H^\bullet(X, \mathcal{F}, \mathbb{A}) \to H^\bullet(X, \mathcal{F}, \mathbb{A}_v)$ and $\phi_v : \det H^\bullet(X, \mathcal{F}, \mathbb{A}) \to \det H^\bullet(X, \mathcal{F}, \mathbb{A}_v)$ be the induced isomorphisms. Then we can compare the symmetric bilinear forms $\phi_v^* b_{\det H^\bullet(X,\mathcal{F},\mathbb{A}_v)}$ and $b_{\det H^\bullet(X,\mathcal{F},\mathbb{A})}$.

Let

$$\beta_v = G_v^{-1}\frac{\partial G_v}{\partial v} \in \Omega(X, \mathrm{End}(\mathcal{F}))^{\bar{0}}.$$

Since $\mathbb{A}_v = G_v^{-1}\mathbb{A}G_v$, then we get

$$\begin{aligned} \frac{\partial \mathbb{A}_v}{\partial v} &= \frac{\partial G_v^{-1}}{\partial v}\mathbb{A}G_v + G_v^{-1}\mathbb{A}\frac{\partial G_v}{\partial v} \\ &= -G_v^{-1}\frac{\partial G_v}{\partial v}G_v^{-1}\mathbb{A}G_v + G_v^{-1}\mathbb{A}\frac{\partial G_v}{\partial v} \\ &= -\beta_v\mathbb{A}_v + \mathbb{A}_v\beta_v = -[\beta_v, \mathbb{A}_v] \end{aligned} \tag{5.1}$$

and

$$\frac{\partial \mathbb{A}_v^\#}{\partial v} = [\beta_v^\#, \mathbb{A}_v^\#].$$

Theorem 5.1. *Under deformation of $\mathbb{A}$ by gauge equivalence and the natural identification of determinant lines, we have*

$$\frac{\partial}{\partial v}\log\left(\frac{\phi_v^* b_{\det H^\bullet(X,\mathcal{F},\mathbb{A}_v)}}{b_{\det H^\bullet(X,\mathcal{F},\mathbb{A})}}\right) = -2\mathrm{Str}(\beta_v a_{b,\frac{n}{2}}).$$

If $\dim X = n$ *is odd, then the above is zero. In this case, the analytic torsion* $b_{\det H^\bullet(X,\mathcal{F},\mathbb{A})}$ *is invariant under gauge equivalence.*

Proof. Set $P_v = \mathbb{A}_v^\#(\mathbb{A}_v\mathbb{A}_v^\#)^{-1}\mathbb{A}_v$ and

$$f(s,v) = \int_0^{+\infty} t^{s-1}\mathrm{Str}\left(e^{-t\mathbb{A}_v^\#\mathbb{A}_v}P_v|_{\Omega(X,\mathcal{F})_{(a,+\infty)}}\right)dt.$$

Then as in the proof of Theorem 3.1, using the fact that $\mathrm{Tr}(A) = \mathrm{Tr}(A^\#)$ for the trace class operator A, we get

$$\frac{\partial f}{\partial v}|_{s=0} = 2\mathrm{Str}(\beta_v(Q - a_{b,\frac{n}{2}})).$$

Then one gets

$$\frac{\partial}{\partial v}\log\left(\frac{\det'(\mathbb{A}_{v,\bar{1}}^\#\mathbb{A}_{v,\bar{1}}|_{\Omega(X,\mathcal{F})^{\bar{1}}_{(a,+\infty)}})}{\det'(\mathbb{A}_{v,\bar{0}}^\#\mathbb{A}_{v,\bar{0}}|_{\Omega(X,\mathcal{F})^{\bar{0}}_{(a,+\infty)}})}\right) = 2\mathrm{Str}(\beta_v(Q - a_{b,\frac{n}{2}})). \tag{5.2}$$

On the other hand, for small v, we have $\phi_v : \Omega(X,\mathcal{F},\mathbb{A})_{[0,a]} \to \Omega(X,\mathcal{F},\mathbb{A}_v)_{[0,a]}$ is an isomorphism of complexes, and the induced symmetric bilinear form on the $\det H^\bullet(\Omega(X,\mathcal{F},\mathbb{A})_{[0,a]})$ is

$$(\phi_v^* b_{\det H^\bullet(X,\mathcal{F},\mathbb{A}_v)_{[0,a]}})(\cdot,\cdot) = b_{\det H^\bullet(X,\mathcal{F},\mathbb{A}_v)}(\phi_v\cdot,\phi_v\cdot).$$

Since $G_v^{-1}\mathbb{A} = \mathbb{A}_v G_v^{-1}$, one easily gets

$$\frac{\partial}{\partial v}\log\left(\frac{\phi_v^* b_{\det H^\bullet(X,\mathcal{F},\mathbb{A}_v)_{[0,a]}}}{b_{\det H^\bullet(X,\mathcal{F},\mathbb{A})_{[0,a]}}}\right) = -2\mathrm{Str}(\beta_v Q). \tag{5.3}$$

Then from (5.2) and (5.3) we get the theorem. □

If $\dim X$ is even, a relative version of $b_{\det H^\bullet(X,\mathcal{F},\mathbb{A})}$ is invariant under gauge equivalence.

References

[1] Jean-Michel Bismut and Weiping Zhang, *An extension of a theorem by Cheeger and Müller* (English, with French summary), Astérisque **205** (1992), 235. With an appendix by François Laudenbach. MR1185803 (93j:58138)

[2] Maxim Braverman and Thomas Kappeler, *Refined analytic torsion*, J. Differential Geom. **78** (2008), no. 2, 193–267. MR2394022 (2009a:58041)

[3] Maxim Braverman and Thomas Kappeler, *Refined analytic torsion as an element of the determinant line*, Geom. Topol. **11** (2007), 139–213, DOI 10.2140/gt.2007.11.139. MR2302591 (2008a:58031)

[4] Maxim Braverman and Thomas Kappeler, *Ray-Singer type theorem for the refined analytic torsion*, J. Funct. Anal. **243** (2007), no. 1, 232–256, DOI 10.1016/j.jfa.2006.10.008. MR2291437 (2008e:58042)

[5] Dan Burghelea and Stefan Haller, *Torsion, as a function on the space of representations*, C^*-algebras and elliptic theory II, Trends Math., Birkhäuser, Basel, 2008, pp. 41–66, DOI 10.1007/978-3-7643-8604-7_2. MR2408135 (2009i:58045)

[6] Dan Burghelea and Stefan Haller, *Complex-valued Ray-Singer torsion*, J. Funct. Anal. **248** (2007), no. 1, 27–78, DOI 10.1016/j.jfa.2007.03.027. MR2329682 (2008b:58035)

[7] Dan Burghelea and Stefan Haller, *Complex valued Ray-Singer torsion II*, Math. Nachr. **283** (2010), no. 10, 1372–1402, DOI 10.1002/mana.200910122. MR2744135 (2011j:58057)

[8] Jochen Brüning and Toshikazu Sunada, *On the spectrum of periodic elliptic operators*, Nagoya Math. J. **126** (1992), 159–171. MR1171598 (93f:58235)

[9] Sylvain E. Cappell and Edward Y. Miller, *Complex-valued analytic torsion for flat bundles and for holomorphic bundles with* $(1,1)$ *connections*, Comm. Pure Appl. Math. **63** (2010), no. 2, 133–202, DOI 10.1002/cpa.20307. MR2588459 (2011a:58061)

[10] Jeff Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. (2) **109** (1979), no. 2, 259–322, DOI 10.2307/1971113. MR528965 (80j:58065a)

[11] Peter B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, 2nd ed., Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR1396308 (98b:58156)

[12] Peter Greiner, *An asymptotic expansion for the heat equation*, Arch. Rational Mech. Anal. **41** (1971), 163–218. MR0331441 (48 #9774)

[13] Gerd Grubb and Robert T. Seeley, *Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems*, Invent. Math. **121** (1995), no. 3, 481–529, DOI 10.1007/BF01884310. MR1353307 (96k:58216)

[14] Rung-Tzung Huang, *Refined analytic torsion for twisted de Rham complexes*, Comm. Anal. Geom. **19** (2011), no. 2, 401–449. MR2835884 (2012h:58044)

[15] Rung-Tzung Huang, *Twisted Cappell-Miller holomorphic and analytic torsions*, Pacific J. Math. **251** (2011), no. 1, 81–107, DOI 10.2140/pjm.2011.251.81. MR2794616 (2012f:58076)

[16] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), 358–426. MR0196736 (33 #4922)

[17] J. Milnor, *A note on curvature and fundamental group*, J. Differential Geometry **2** (1968), 1–7. MR0232311 (38 #636)

[18] Werner Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. in Math. **28** (1978), no. 3, 233–305, DOI 10.1016/0001-8708(78)90116-0. MR498252 (80j:58065b)

[19] Werner Müller, *Analytic torsion and R-torsion for unimodular representations*, J. Amer. Math. Soc. **6** (1993), no. 3, 721–753, DOI 10.2307/2152781. MR1189689 (93m:58119)

[20] Varghese Mathai and Daniel Quillen, *Superconnections, Thom classes, and equivariant differential forms*, Topology **25** (1986), no. 1, 85–110, DOI 10.1016/0040-9383(86)90007-8. MR836726 (87k:58006)

[21] Varghese Mathai and Siye Wu, *Analytic torsion for twisted de Rham complexes*, J. Differential Geom. **88** (2011), no. 2, 297–332. MR2838268 (2012j:58049)

[22] Varghese Mathai and SiYe Wu, *Twisted analytic torsion*, Sci. China Math. **53** (2010), no. 3, 555–563, DOI 10.1007/s11425-010-0053-3. MR2608312 (2011h:58053)

[23] Varghese Mathai and Siye Wu, *Analytic torsion of* $\mathbb{Z}_2$*-graded elliptic complexes*, Noncommutative geometry and global analysis, Contemp. Math., vol. 546, Amer. Math. Soc., Providence, RI, 2011, pp. 199–212, DOI 10.1090/conm/546/10790. MR2815136 (2012d:58054)

[24] D. Quillen, *Determinants of Cauchy-Riemann operators over a Riemann surface,* Funct. Anal. Appl. **14** (1985), 31–34.

[25] Daniel Quillen, *Superconnections and the Chern character*, Topology **24** (1985), no. 1, 89–95, DOI 10.1016/0040-9383(85)90047-3. MR790678 (86m:58010)

[26] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971), 145–210. MR0295381 (45 #4447)

[27] Guangxiang Su, *Burghelea-Haller analytic torsion for twisted de Rham complexes*, Pacific J. Math. **250** (2011), no. 2, 421–437, DOI 10.2140/pjm.2011.250.421. MR2794608 (2012d:58055)

[28] Edward Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), no. 4, 661–692 (1983). MR683171 (84b:58111)

[29] M. Wodzicki, *Local invariants of spectral asymmetry*, Invent. Math. **75** (1984), no. 1, 143–177, DOI 10.1007/BF01403095. MR728144 (85g:58089)

Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, People's Republic of China

E-mail address: guangxiangsu@nankai.edu.cn