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LOG-CONCAVITY OF THE DUISTERMAAT-HECKMAN MEASURE FOR SEMIFREE HAMILTONIAN S^1-ACTIONS

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(Communicated by Lei Ni)

This paper is dedicated to my father

ABSTRACT. The Ginzberg-Knutson conjecture states that for any Hamiltonian Lie group G-action, the corresponding Duistermaat-Heckman measure is log-concave. It turns out that the conjecture is not true in general, but every well-known counterexample has non-isolated fixed points. In this paper, we prove that if the Hamiltonian circle action on a compact symplectic manifold (M, ω) is semifree and all fixed points are isolated, then the Duistermaat-Heckman measure is log-concave. With the same assumption, we also prove that ω and every reduced symplectic form satisfy the hard Lefschetz property.

1. INTRODUCTION

In statistical mechanics, consider the relation $S(E) = k \log W(E)$, which is called *Boltzmann's principle*, where $W(E)$ is the number of states with given values of macroscopic parameters E (like energy, temperature,$\cdots$), k is the Boltzmann's constant, and S is the entropy of the system which measures the degree of disorder in the system. For the additive values E, it is well-known that the entropy is always a concave function. (See [18] for more details.)

Now, consider a Hamiltonian G-manifold (M, ω) with the moment map $\mu : M \to \mathfrak{g}^*$. The Liouville measure m_L is defined by

$$m_L(U) := \int_U \frac{\omega^n}{n!}$$

for any open set $U \subset M$. The push-forward measure $m_{\mathrm{DH}} := \mu_* m_L$ is called the *Duistermaat-Heckman measure.* Then m_{DH} can be regarded as a measure on $\mathfrak{g}^*$ such that for any Borel subset $B \subset \mathfrak{g}^*$, $m_{\mathrm{DH}}(B) = \int_{\mu^{-1}(B)} \frac{\omega^n}{n!}$ tells us how many states of our system have momenta in B. By the Duistermaat-Heckman theorem [5], m_{DH} can be expressed in terms of the density function $\mathrm{DH}(\xi)$ with respect to the Lebesque measure on $\mathfrak{g}^*$. Hence if we consider Boltzmann's principle in our Hamiltonian system with an identification $W = \mathrm{DH}$, it is natural to ask whether the Duistermaat-Heckman measure m_{DH} is log-concave. As noted in [17], [11], and [14], V. Ginzburg and A. Knutson conjectured that for any closed Hamiltonian T-manifold, the corresponding Duistermaat-Heckman measure is log-concave.

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The log-concavity problem of the Duistermaat-Heckman measure is proved by A. Okounkov [16] when M is a co-adjoint orbit of the classical Lie groups of type A_n, B_n, or C_n with the maximal torus action. Around the same time, W. Graham [7] proved that the log-concavity property holds for any holomorphic Hamiltonian circle action on any Kähler manifold. But the counterexample was found by Y. Karshon [11]. By using Lerman's symplectic cutting method, she constructed a closed 6-dimensional semifree Hamiltonian S^1-manifold with two fixed components such that the Duistermaat-Heckman measure is not log-concave. Later, Y. Lin [14] generalized the construction of 6-dimensional Hamiltonian S^1-manifolds which do not satisfy the log-concavity of the Duistermaat-Heckman measure. But all counterexamples of Karshon and Lin are the cases when each fixed component is of codimension two (i.e. non-isolated). So, the log-concavity problem is still open for the case when (M, ω) is a Hamiltonian S^1-manifold whose fixed components are of codimension greater than two. In this paper, we will show that

Theorem 1.1. *Let (M, ω) be a closed symplectic manifold with a semifree Hamiltonian S^1-action whose fixed point set M^{S^1} consists of isolated points. Then the Duistermaat-Heckman measure is log-concave.*

The conditions "**semifree**" and "**isolated fixed points**" enable us to use the Tolman-Weitsman basis [19] of the equivariant cohomology $H^*_{S^1}(M)$. As you will see in Section 2, any semifree Hamiltonian S^1-manifold with only isolated fixed points has a lot of remarkable properties. In fact, the cohomology ring and the equivariant cohomology ring of M are the same as those of $S^2 \times \cdots \times S^2$ with a diagonal semifree circle action. In particular, (M, ω) is equivariantly symplectomorphic to the product space of S^2 copies $(S^2 \times \cdots \times S^2, \sigma)$ with some S^1-invariant Kähler structure σ when $\dim M \leq 6$. (See [12] and [6].) Therefore, we may ask whether (M, ω) satisfies the properties which the diagonal circle action on $(S^2 \times \cdots \times S^2, \sigma)$ satisfies. In this paper, we prove the following.

Theorem 1.2. *Let (M, ω) be a closed semifree Hamiltonian S^1-manifold whose fixed points are all isolated, and let μ be the moment map. Then ω satisfies the hard Lefschetz property. Moreover, the reduced symplectic form ω_t satisfies the hard Lefschetz property for every regular value t.*

In Section 2, we briefly review Tolman and Weitsman's work [19] which is very powerful to analyze the equivariant cohomology of the Hamiltonian S^1-manifold with isolated fixed points as we mentioned above. Especially we use the Tolman-Weitsman basis of the equivariant cohomology $H^*_{S^1}(M)$ which is constructed by using the equivariant version of Morse theory [20]. In Section 3, we express the Duistermaat-Heckman function explicitly in terms of the integration of some cohomology class on the reduced space. Then we compute the integration by using the Kirwan-Jeffrey residue formula [10]. Consequently, we will show that the log-concavity of the Duistermaat-Heckman measure is completely determined by the set of pairs $\{ (\mu(F), m_F)_F | \, F \in M^{S^1} \}$, where $\mu(F)$ is the image of the moment map of F and m_F is the product of all weights of the S^1-representation on $T_F M$ (Proposition 3.9). In Section 4, we will prove Theorem 1.1, and we will prove Theorem 1.2 in Section 5.

2. Tolman-Weitsman basis of the equivariant cohomology $H^*_{S^1}(M;\mathbb{Z})$

In this section we briefly review Tolman and Weitsman's results in [19]. Throughout this section, we assume that (M^{2n},ω) is a closed semifree Hamiltonian S^1-manifold whose fixed points are isolated. Note that for each fixed point $p \in M^{S^1}$, the index of p is the Morse index of the moment map at p, which is the same as twice the number of negative weights of the tangential S^1-representation at p.

Proposition 2.1 ([19]). *Let N_k be the number of fixed points of index $2k$. Then $N_k = \binom{n}{k}$. Hence N_k is the same as that of the standard diagonal circle action on $(S^2\times\cdots\times S^2, \omega_1\oplus\cdots\oplus\omega_n)$, where ω_i is the Fubini-Study form on S^2 of i-th factor.*

Theorem 2.2 ([19]). *Let $2^{[n]}$ be the power set of $\{1,\cdots,n\}$. Then there exists a bijection $\phi : M^{S^1} \to 2^{[n]}$ satisfying the following:*

1. *For each index-2k fixed point $x \in M^{S^1}$, $|\phi(x)| = k$.*
2. *Let u be the generator of $H^*(BS^1,\mathbb{Z})$. For each index-2k fixed point $x \in M^{S^1}$, there exists a unique cohomology class $\alpha_x \in H^{2k}_{S^1}(M;\mathbb{Z})$ such that for any $x' \in M^{S^1}$,*
    - *$\alpha_x|_{x'} = u^k$ if $\phi(x) \subset \phi(x')$.*
    - *$\alpha_x|_{x'} = 0$ otherwise.*

Here, $\alpha|_{x'}$ means the image $\pi_{x'}(i^*(\alpha))$ where $i^* : H^*_{S^1}(M;Z) \to H^*_{S^1}(M^{S^1};\mathbb{Z})$ is a homomorphism induced by an inclusion $i : M^{S^1} \hookrightarrow M$, and $\pi_{x'} : H^*_{S^1}(M^{S^1};\mathbb{Z}) \to H^*_{S^1}(x';\mathbb{Z})$ is a natural projection. Moreover $\{\alpha_x |\ x \in M^{S^1}\}$ forms a basis of $H^*_{S^1}(M;\mathbb{Z})$.

If we apply Theorem 2.2 to $(S^2\times\cdots\times S^2, \omega_1\oplus\cdots\oplus\omega_n)$ with the diagonal semifree Hamiltonian circle action, we get a bijection $\psi : (S^2\times\cdots\times S^2)^{S^1} \to 2^{[n]}$ and there is a basis $\{\beta_y |\ y \in (S^2\times\cdots\times S^2)^{S^1}\}$ of $H^*_{S^1}(S^2\times\cdots\times S^2;\mathbb{Z})$ that satisfies the conditions in Theorem 2.2. Hence we have an identification map

$$\psi^{-1}\circ\phi : M^{S^1} \to (S^2\times\cdots\times S^2)^{S^1}$$

and $\psi^{-1}\circ\phi$ preserves the indices of the fixed points.

Note that $\psi^{-1}\circ\phi$ gives an identification between $H^*_{S^1}(M;\mathbb{Z})$ and $H^*_{S^1}(S^2\times\cdots\times S^2;\mathbb{Z})$ as follows. Let $a_i = \alpha_{\phi^{-1}\{i\}} \in H^2_{S^1}(M;\mathbb{Z})$ and $b_i = \beta_{\psi^{-1}\{i\}} \in H^2_{S^1}(S^2\times\cdots\times S^2;\mathbb{Z})$. The following lemma is proved by Tolman and Weitsman in [19], but we give a complete proof here to use their idea in the rest of this paper.

Lemma 2.3 ([19]). *For each $x \in M^{S^1}$, we have $\alpha_x = \prod_{j\in\phi(x)} a_j$. Similarly, we have $\beta_y = \prod_{j\in\psi(y)} b_j$ for each $y \in (S^2\times\cdots\times S^2)^{S^1}$.*

Proof. For an inclusion $i : M^{S^1} \hookrightarrow M$, we have a natural ring homomorphism $i^* : H^*_{S^1}(M) \to H^*_{S^1}(M^{S^1}) \cong H^*(M^{S^1})\otimes H*(BS^1)$. Kirwan's injectivity theorem [13] implies that i^* is an injective ring homomorphism. Hence it is enough to show that $\alpha_x|_z = (\prod_{j\in\phi(x)} a_j)|_z$ for all $x, z \in M^{S^1}$. For any $x, z \in M^{S^1}$ with $\mathrm{Ind}(x) = 2k$,

- $\alpha_x|_z = u^k$ if $\phi(x) \subset \phi(z)$.
- $\alpha_x|_z = 0$ otherwise.

On the other hand, $(\prod_{j\in\phi(x)} a_j)|_z = \prod_{j\in\phi(x)} a_j|_z$. Since $a_j|_z = u$ if and only if $j \in \phi(z)$, we have

- $(\prod_{j\in\phi(x)} a_j)|_z = u^k$ if $\phi(x) \subset \phi(z)$.
- $(\prod_{j\in\phi(x)} a_j)|_z = 0$ otherwise.

Therefore, we have $\alpha_x = \prod_{j\in\phi(x)} a_j$. The proof of the second statement is similar. $\square$

Hence the $H^*(BS^1)$-module isomorphism $f : H^*_{S^1}(M;\mathbb{Z}) \to H^*_{S^1}(S^2\times\cdots\times S^2;\mathbb{Z})$ which sends α_x to $\beta_{\psi^{-1}\circ\phi(x)}$ for each $x \in M^{S^1}$ is in fact a ring isomorphism by Lemma 2.3. To sum up, we have the following corollary.

Corollary 2.4 ([19]). *There is a ring isomorphism*

$$f : H^*_{S^1}(M;\mathbb{Z}) \to H^*_{S^1}(S^2 \times \cdots \times S^2;\mathbb{Z})$$

which sends α_x to $\beta_{\psi^{-1}\circ\phi(x)}$. Moreover, for any $\alpha \in H^*_{S^1}(M;\mathbb{Z})$ and any fixed point $x \in M^{S^1}$, we have $\alpha_x = f(\alpha)|_{\psi^{-1}\circ\phi(x)}$.

3. The Duistermaat-Heckman function and the residue formula

Let (M,ω) be a $2n$-dimensional closed Hamiltonian S^1-manifold with the moment map $\mu : M \to \mathbb{R}$. We may assume that 0 is a regular value of μ such that $\mu^{-1}(0)$ is non-empty. Choose two consecutive critical values c_1 and c_2 of μ so that the open interval (c_1, c_2) consists of regular values of μ and contains 0. By the Duistermaat-Heckman's theorem [5], $[\omega_t] = [\omega_0] - et$ where e is the Euler class of S^1-fibration $\mu^{-1}(0) \to M_0$, where M_0 is the symplectic reduction at 0 with the induced symplectic form ω_0. Hence we have

$$\mathrm{DH}(t) = \int_{M_0} \frac{1}{(n-1)!}([\omega_0] - et)^{n-1} \tag{3.1}$$

on $(c_1, c_2) \subset \mathrm{Im}\mu$.

Note that a continuous function on an open interval $g : (a,b) \to \mathbb{R}$ is concave if $g(tc + (1-t)d) \geq tg(c) + (1-t)g(d)$ for any $c, d \in (a,b)$ and for any $t \in (0,1)$. We remark the basic property of a concave function as follows.

Remark 3.1. Let g be a continuous, piecewise smooth function on a connected interval $I \subset \mathbb{R}$. Then g is concave on I if and only if the derivative of g is decreasing, i.e. $g''(t) \leq 0$ for every smooth point $t \in I$ and $g'_+(c) - g'_-(c) < 0$ for every singular point $c \in I$, where $g'_+(c) = \lim_{t\to c,\ t>c} g'(t)$ and $g'_-(c) = \lim_{t\to c,\ t<c} g'(t)$.

Note that Duistermaat and Heckman proved that DH is a polynomial on a connected regular open interval $U \subset \mu(M)$. The following formula due to Guillemin, Lerman, and Sternberg describes the behavior of DH near the critical value of μ. In particular, it implies that DH is k-times differentiable at a critical value $c \in \mu(M)$ if and only if $\mu^{-1}(c)$ does not contain a fixed component whose codimension is less than $4 + 2k$.

Theorem 3.2 ([8]). *Assume that c is a critical value which corresponds to the fixed components C_i's. Then the jump of* DH(t) *at c is given by*

$$\mathrm{DH}_+ - \mathrm{DH}_- = \sum_i \frac{\mathrm{vol}(C_i)}{(d_i - 1)!\prod_j w_j}(t-c)^{d_i-1} + O((t-c)^{d_i}),$$

where the sum is over the components C_i of $M^{S^1} \cap \mu^{-1}(c)$, d_i is half the real codimension of C_i in M, and the w_j's are the weights of the S^1-representation on the normal bundle of C_i.

If c is a critical value which is not an extremum, then the codimension of the fixed point set in $\mu^{-1}(c)$ is at least 4. Therefore Theorem 3.2 implies that $\mathrm{DH}(t)$ is continuous at non-extremal critical values and $\mathrm{DH}'(t)$ jumps at c when d equals 2. In the case when $d = 2$, the two non-zero weights must have opposite signs, so the jump in the derivative is negative, i.e. $\mathrm{DH}'(t)$ decreases when it passes through the critical value with $d = 2$. Since DH is continuous, the jump in $\frac{d}{dt} \ln \mathrm{DH}(t)$ is negative at c. Combining with Remark 3.1, we have the following corollary.

Corollary 3.3. *Let (M, ω) be a closed Hamiltonian S^1-manifold with the moment map $\mu : M \to \mathbb{R}$. Then the corresponding Duistermaat-Heckman function* DH *is log-concave on $\mu(M)$ if $(\log DH(t))'' \leq 0$ for every regular value $t \in \mu(M)$.*

Note that $(\log \mathrm{DH}(t))'' = \frac{\mathrm{DH}(t) \cdot \mathrm{DH}''(t) - \mathrm{DH}(t)^2}{\mathrm{DH}(t)^2}$. Therefore $(\log \mathrm{DH}(t))'' \leq 0$ is equivalent to $\mathrm{DH}(t) \cdot \mathrm{DH}''(t) - \mathrm{DH}'(t)^2 \leq 0$. Equation (3.1) implies that

$$\mathrm{DH}(t) \cdot \mathrm{DH}''(t) = (n-1)(n-2) \int_{M_0} e^2 [\omega_t]^{n-3} \cdot \int_{M_0} [\omega_t]^{n-1} \tag{3.2}$$

and

$$\mathrm{DH}'(t)^2 = (n-1)^2 \left(\int_{M_0} e[\omega_t]^{n-2} \right)^2 . \tag{3.3}$$

To compute the integrals appearing in equations (3.2) and (3.3), we need the following procedures. For an inclusion $i : \mu^{-1}(0) \hookrightarrow M$, we have a ring homomorphism $\kappa : H^*_{S^1}(M; \mathbb{R}) \to H^*_{S^1}(\mu^{-1}(0); \mathbb{R}) \cong H^*(M_0; \mathbb{R})$ which is called the Kirwan map. Due to the Kirwan surjectivity [13], κ is a ring surjection. Now, consider a 2-form $\widetilde{\omega} := \omega - d(\mu\theta)$ on $M \times ES^1$ where θ is the pull-back of the connection form on ES^1 along the projection $M \times ES^1 \to ES^1$. We denote $x = \pi^* u \in H^2_{S^1}(M; \mathbb{Z})$ where $\pi : M \times_{S^1} ES^1 \to BS^1$ and u is a generator of $H^*(BS^1; \mathbb{Z})$ such that the Euler class of the Hopf bundle $ES^1 \to BS^1$ is $-u$. Some part of the following two lemmas is given in [1], but we give the complete proofs here.

Lemma 3.4. *$\widetilde{\omega}$ is S^1-invariant and closed, and $i_X \widetilde{\omega} = 0$ so that $\widetilde{\omega}$ represents a cohomology class in $H^*_{S^1}(M; \mathbb{R})$. Moreover, for any fixed component $F \in M^{S^1}$, we have $\kappa([\widetilde{\omega}]) = [\omega_0]$ and $[\widetilde{\omega}]|_F = [\omega]|_F + \mu(F)u$. In particular, if F is isolated, then $[\widetilde{\omega}]|_F = \mu(F)u$.*

Proof. For the first statement, it is enough to show that $i_X \widetilde{\omega}$ and $L_X \widetilde{\omega}$ vanish. Note that $i_X \widetilde{\omega} = i_X \omega - i_X d(\mu\theta) = -d\mu + d i_X(\mu\theta) - L_X(\mu\theta)$ by Cartan's formula. Since $i_X(\mu\theta) = \mu$ and $\mu\theta$ is invariant under the circle action, we have $i_X \widetilde{\omega} = -d\mu + d\mu = 0$. Moreover, it is obvious that $\widetilde{\omega}$ is closed by definition. Hence $L_X \widetilde{\omega} = 0$ by Cartan's formula again.

To prove the second statement, consider the following diagram:

$$\begin{array}{ccc}
\mu^{-1}(0) \times ES^1 & \hookrightarrow & M \times ES^1 \\
\downarrow & & \downarrow \\
\mu^{-1}(0) \times_{S^1} ES^1 & \hookrightarrow & M \times_{S^1} ES^1 \\
\downarrow & & \\
\mu^{-1}(0)/S^1 \cong M_{\mathrm{red}} & &
\end{array}$$

Since $d\mu$ is zero on the tangent bundle $\mu^{-1}(0) \times ES^1$, the pull-back of $\widetilde{\omega} = \omega - d\mu \wedge \theta - \mu d\theta$ to $\mu^{-1}(0) \times ES^1$ is the restriction $\omega|_{\mu^{-1}(0)\times ES^1}$ and the push-forward of $\omega|_{\mu^{-1}(0)\times ES^1}$ to $\mu^{-1}(0)/S^1$ is just a reduced symplectic form at the level 0. Hence $\kappa([\widetilde{\omega}]) = [\omega_0]$.

To show the last statement, consider $[\widetilde{\omega}]|_F = [\omega - d\mu \wedge \theta - \mu d\theta]|_F$. Since the restriction $d\mu|_{F\times ES^1}$ vanishes, we have $[\widetilde{\omega}]|_F = [\omega]|_F - \mu(F) \cdot [d\theta]|_{ES^1} = [\omega]|_F + \mu(F)u$. In particular, if F is isolated, then we have $[\widetilde{\omega}]|_F = \mu(F)u$. ☐

Lemma 3.5. *Consider a 2-form $d\theta$ on $M \times ES^1$, where θ is the pull-back of the connection form on ES^1 along the projection $M \times ES^1 \to ES^1$. Then we can push $d\theta$ forward to $M \times_{S^1} ES^1$ so that $d\theta$ represents a cohomology class in $H^*_{S^1}(M;\mathbb{R})$. Moreover, $[d\theta] = -x$ and $\kappa([d\theta]) = -\kappa(x) = e$, where e is the Euler class of the S^1-fibration $\mu^{-1}(0) \to M_{\mathrm{red}}$.*

Proof. Note that $i_X d\theta = L_X\theta - d i_X\theta = 0$. Hence we can push $d\theta$ forward to $M \times_{S^1} ES^1$. For any fixed point $p \in M^{S^1}$, the restriction $[d\theta]|_p$ is the Euler class of $p \times ES^1 \to BS^1$. Hence $[d\theta] = -u \cdot 1 = -x$. Here, the multiplication "$\cdot$" comes from the $H^*(BS^1)$-module structure on $H^*_{S^1}(M)$. By the diagram in the proof of Lemma 3.4, $\kappa([d\theta])$ is just the Euler class of the S^1-fibration $\mu^{-1}(0) \to \mu^{-1}(0)/S^1$. Therefore $\kappa([d\theta]) = -\kappa(x) = e$. ☐

Combining equations (3.2), (3.3), Lemma 3.4 and Lemma 3.5, we have the following corollary.

Corollary 3.6. $\mathrm{DH}(0) \cdot \mathrm{DH}''(0) - \mathrm{DH}'(0)^2 \leq 0$ *if and only if*

$$(n-2)\int_{M_0} \kappa([d\theta]^2[\widetilde{\omega}]^{n-3}) \cdot \int_{M_0} \kappa([\widetilde{\omega}]^{n-1}) - (n-1)\left(\int_{M_0} \kappa([d\theta][\widetilde{\omega}]^{n-2})\right)^2 \leq 0.$$

To compute the above integrals $\int_{M_0} \kappa([d\theta]^2[\widetilde{\omega}]^{n-3})$, $\int_{M_0} \kappa([\widetilde{\omega}]^{n-1})$, and $\int_{M_0} \kappa([d\theta][\widetilde{\omega}]^{n-2})$, we need the residue formula due to Jeffrey and Kirwan. (See [10] and [9].)

Theorem 3.7 ([10]). *Let $\nu \in H^*_{S^1}(M;\mathbb{R})$. Then*

$$\int_{M_0} \kappa(\nu) = \sum_{F\in M^{S^1},\ \mu(F)>0} \mathrm{Res}_{u=0}\left(\frac{\nu|_F}{e_F}\right).$$

Here, e_F is the equivariant Euler class of the normal bundle to F so that we can regard $\nu|_F$ and e_F as polynomials with one variable u. $\mathrm{Res}_{u=0}(f)$ means a residue of f where f is a rational function with one variable u.

Now, let's compute $\int_{M_0} \kappa([d\theta]^2[\widetilde{\omega}]^{n-3})$. By Theorem 3.7,

$$\int_{M_0} \kappa([d\theta]^2[\widetilde{\omega}]^{n-3}) = \sum_{F\in M^{S^1},\ \mu(F)>0} \mathrm{Res}_{u=0}\left(\frac{[d\theta]^2[\widetilde{\omega}]^{n-3}|_F}{e_F}\right).$$

Since $[\widetilde{\omega}]|_z = \mu(z)u$ and $[d\theta]|z = -u$ by Lemma 3.4 and 3.5, we have

$$\begin{aligned}\int_{M_0} \kappa([d\theta]^2[\widetilde{\omega}]^{n-3}) &= \textstyle\sum_{F\in M^{S^1},\ \mu(F)>0} \mathrm{Res}_{u=0}\left(\frac{\mu(F)^{n-3}u^{n-1}}{e_F}\right)\\ &= \textstyle\sum_{F\in M^{S^1},\ \mu(F)>0} \mathrm{Res}_{u=0}\left(\frac{\mu(F)^{n-3}u^{n-1}}{m_F u^n}\right)\\ &= \textstyle\sum_{F\in M^{S^1},\ \mu(F)>0} \frac{1}{m_F}\mu(F)^{n-3},\end{aligned}$$

where m_F is the product of all weights of tangential S^1-representation at F. Similarly, if $\xi \in \mathbb{R}$ is a regular value of μ, then we let $\widetilde{\mu} = \mu - \xi$ be the new moment map. By the same argument, we have the following lemma.

Lemma 3.8. *For a regular value ξ of the moment map μ, we have the following:*

(1)
$$\int_{M_0} \kappa([d\theta]^2[\widetilde{\omega}]^{n-3}) = \sum_{\substack{F \in M^{S^1}, \\ \mu(F) > \xi}} \frac{1}{m_F}(\mu(F) - \xi)^{n-3}.$$

(2)
$$\int_{M_0} \kappa([d\theta][\widetilde{\omega}]^{n-2}) = \sum_{\substack{F \in M^{S^1}, \\ \mu(F) > \xi}} \frac{-1}{m_F}(\mu(F) - \xi)^{n-2}.$$

(3)
$$\int_{M_0} \kappa([\widetilde{\omega}]^{n-1}) = \sum_{\substack{F \in M^{S^1}, \\ \mu(F) > \xi}} \frac{1}{m_F}(\mu(F) - \xi)^{n-1}.$$

Combining Corollary 3.6 and Lemma 3.8, we have the following proposition.

Proposition 3.9. *Let (M, ω) be a closed Hamiltonian S^1-manifold with the moment map $\mu : M \to \mathbb{R}$. Assume that M^{S^1} consists of isolated fixed points. Then a density function of the Duistermaat-Heckman measure with respect to μ is log-concave if and only if*

$$\sum_{\substack{F \in M^{S^1}, \\ \mu(F) > \xi}} \frac{1}{m_F}(\mu(F) - \xi)^{n-3} \cdot \sum_{\substack{F \in M^{S^1}, \\ \mu(F) > \xi}} \frac{1}{m_F}(\mu(F) - \xi)^{n-1} - \left(\sum_{\substack{F \in M^{S^1}, \\ \mu(F) > \xi}} \frac{1}{m_F}(\mu(F) - \xi)^{n-2} \right)^2 \leq 0$$

for every regular value $\xi \in \mu(M)$, where m_F is the product of all weights of the S^1-representation on $T_F M$. In particular, the log-concavity of the Duistermaat-Heckman measure is completely determined by the set $\{(\mu(F), m_F)_F | F \in M^{S^1}\}$.

Corollary 3.10. *Let (M^{2n}, ω) and (N^{2n}, σ) be two closed Hamiltonian S^1-manifolds with the moment maps μ_1 and μ_2, respectively. Assume there exists a bijection $\phi : M^{S^1} \to N^{S^1}$ which satisfies*

(1) *for each $F \in M^{S^1}$, $m_F = m_{\phi(F)}$, and*
(2) *for each $F \in M^{S^1}$, $\mu_1(F) = \mu_2(\phi(F))$,*

where m_F is the product of all weights of the tangential S^1-representation at F. If N satisfies the log-concavity of the Duistermaat-Heckman measure with respect to μ_2, then so does M with respect to μ_1.

Remark 3.11. The integration formulae (1) and (3) in Lemma 3.8 are proved by Wu by using the stationary phase method. See Theorem 5.2 in [21] for the details.

4. Proof of Theorem 1.1

As noted in the introduction, if a Hamiltonian S^1-action on the Kähler manifold is holomorphic, then the corresponding Duistermaat-Heckman function is log-concave by [7]. Let (M^{2n}, ω) be a closed semifree Hamiltonian S^1-manifold with the moment map μ. Assume that all fixed points are isolated. Let DH be the corresponding Duistermaat-Heckman function with respect to μ. We will show that there is a Kähler form $\omega_1 \oplus \cdots \oplus \omega_n$ on $S^2 \times \cdots \times S^2$ with the standard diagonal holomorphic semifree circle action such that a bijection $\psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}$ given in Section 2 satisfies the conditions in Corollary 3.10, which implies the log-concavity of DH. Now we start with the lemma below.

Lemma 4.1. *Let (M^{2n}, ω) be a closed semifree Hamiltonian circle action with the moment map μ. Assume that all fixed points are isolated. Then $\{(\mu(F), m_F)_F \mid F \in M^{S^1}\}$ is completely determined by $\mu(p_0^1), \mu(p_1^1), \cdots, \mu(p_1^n)$, where the p_k^j's are the fixed points of index $2k$ for $j = 1, \cdots, \binom{n}{k}$.*

Proof. Consider an equivariant symplectic 2-form $\widetilde{\omega}$ on $M \times_{S^1} ES^1$ which is given in Section 3. Since the set $\{x, a_1, \cdots, a_n\}$ is a basis of $H^2_{S^1}(M; \mathbb{Z})$, we may assume

$$[\widetilde{\omega}] = m_0 x + m_1 a_1 + \cdots + m_n a_n$$

for some real numbers m_i. (See Section 2, the definition of $x, a_1, \cdots, a_n$.) Therefore, for any fixed point $p_i^j \in M^{S^1}$, we have

$$[\widetilde{\omega}]|_{p_i^j} = (m_0 x + m_1 a_1 + \cdots + m_n a_n)|_{p_i^j}.$$

When $i = 0$ and $j = 1$, Lemma 3.4 implies that

$$[\widetilde{\omega}]|_{p_0^1} = m_0 u.$$

Since every a_i vanishes on p_0^1, the right hand side is

$$(m_0 x + m_1 a_1 + \cdots + m_n a_n)|_{p_0^1} = m_0 u.$$

Hence $m_0 = \mu(p_0^1)$. Similarly,

$$[\widetilde{\omega}]|_{p_1^i} = \mu(p_1^i) u$$

and

$$(m_0 x + m_1 a_1 + \cdots + m_n a_n)|_{p_1^i} = m_0 u + m_i u.$$

Hence we have $m_i = \mu(p_1^i) - m_0 = \mu(p_1^i) - \mu(p_0^1)$ for each $i = 1, \cdots, n$. Therefore $\{\mu(p_0^1), \mu(p_1^1), \cdots, \mu(p_1^n)\}$ determines the coefficients m_i of $[\widetilde{\omega}]$.

For p_k^j with $k > 1$, the relation $[\widetilde{\omega}]|_{p_k^j} = (m_0 x + m_1 a_1 + \cdots + m_n a_n)|_{p_k^j}$ implies

- $[\widetilde{\omega}]|_{p_k^j} = \mu(p_k^j) u$ and
- $(m_0 x + m_1 a_1 + \cdots + m_n a_n)|_{p_k^j} = m_0 u + \sum_{i \in \phi(p_k^j)} m_i u.$

Therefore, for fixed k, the set $\{(\mu(p_k^j), m_{p_k^j})_j \mid j = 1, \cdots, \binom{n}{k}\}$ is just

$$\{((m_0 + m_{i_1} + \cdots + m_{i_k}), (-1)^k)_{\{i_1, \cdots, i_k\}} \mid \{i_1, \cdots, i_k\} \subset \{1, 2, \cdots, n\}\},$$

and this set does not depend on the ordering of p_k^j. Hence $\{(\mu(F), m_F)_F \mid F \in M^{S^1}\} = \bigcup_{k=0}^{k=n} \{(\mu(p_k^j), m_{p_k^j})_j \mid j = 1, \cdots, \binom{n}{k}\}$ is completely determined by $\mu(p_0^1)$, $\mu(p_1^1), \cdots, \mu(p_1^n)$. $\square$

Now we are ready to prove Theorem 1.1.

Theorem 4.2 (Theorem 1.1). *Let (M, ω) be a closed symplectic manifold with a semifree Hamiltonian S^1-action whose fixed point set M^{S^1} consists of isolated points. Then the Duistermaat- Heckman measure is log-concave.*

Proof. Let $\mu : M \to \mathbb{R}$ be a moment map and let $M^{S^1} = \{p_k^j | k = 0, \cdots, n, j = 1, \cdots, \binom{n}{k}\}$ be the fixed point set, where p_k^j is a fixed point of index $2k$ labeled by $j = 1, \cdots, \binom{n}{k}$. Note that $\mu(p_0^1)$ is the minimum value of μ. Let $\phi : M^{S^1} \to 2^{[n]}$ be the identification map between the fixed point set M^{S^1} and the power set $2^{[n]}$ defined in Theorem 2.2. Then we may assume (by re-labeling, if necessary) that $\phi(p_1^j) = \{j\}$ for $j = 1, \cdots, n$. We will show that there exists a semifree holomorphic Hamiltonian S^1-manifold $(S^2 \times \cdots \times S^2), \sigma)$ with the moment map μ' such that $\psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}$ preserves their indices, weights, and the values of the moment map, where $\psi : (S^2 \times \cdots \times S^2)^{S^1} \to 2^{[n]}$ is the identification map described in Theorem 2.2.

Let ω_i be the Fubini-Study form on S^2 such that the symplectic volume is $\mu(p_1^i) - \mu(p_0^1)$. Let S be the south pole and N be the north pole of S^2 so that S is the minimum (N is the maximum, respectively) of the moment map on (S^2, ω_i) with the standard semifree circle action on S^2. Then $(S^2 \times \cdots \times S^2, \omega_1 \oplus \cdots \oplus \omega_n)$ is a symplectic manifold with the diagonal semifree Hamiltonian circle action. Let $\mu' : S^2 \times \cdots \times S^2 \to \mathbb{R}$ be the moment map whose minimum is $\mu(p_0^1)$. Let $\psi : (S^2 \times \cdots \times S^2)^{S^1} \to 2^{[n]}$ be the identification map between the fixed point set $(S^2 \times \cdots \times S^2)^{S^1}$ and the power set $2^{[n]}$ such that

$$\psi^{-1}(\{i\}) := q_1^i = (S, \cdots, S, N, S, \cdots, S)$$

for all $i = 1, \cdots, n$, where $q_1^i = (S, \cdots, S, N, S, \cdots, S)$ is a fixed point on $(S^2 \times \cdots \times S^2)$ of index 2 such that the i-th coordinate is N and the other coordinates are S. Then we can easily see that $\psi^{-1} \circ \phi(p_1^j) = q_1^j$ and $\mu(p_1^j) = \mu'(q_1^j) = \mu(p_1^j) - \mu(p_0^1)$ for all $j = 1, \cdots, n$. By Lemma 4.1, we have

$$\{(\mu(F), m_F)_F | F \in M^{S^1}\} = \{(\mu'(F), m_F)_F | F \in (S^2 \times \cdots \times S^2)^{S^1}\},$$

and $\psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}$ satisfies the condition in Corollary 3.10. Therefore the Duistermaat-Heckman measure is log-concave on $\mu(M)$. □

Remark 4.3 (Summary). Let (M, ω) be a $2n$-dimensional compact semifree Hamiltonian S^1-manifold with isolated fixed points. Let $\mu : M \to \mathbb{R}$ be a moment map. Let $\phi : M^{S^1} \to 2^{[n]}$ be the identification described in Theorem 2.2. In the proof of Theorem 1.1, we proved that there exists a Kähler form $\omega_1 \oplus \cdots \oplus \omega_n$ on $S^2 \times \cdots \times S^2$ with the diagonal semifree Hamiltonian action with the moment map $\mu' : S^2 \times \cdots \times S^2 \to \mathbb{R}$ satisfying the following:

- There is an identification $\psi : (S^2 \times \cdots \times S^2)^{S^1} \to 2^{[n]}$ such that $\psi^{-1}(\{i\}) = q_1^i = (S, \cdots, S, N, S, \cdots, S)$, where $q_1^i = (S, \cdots, S, N, S, \cdots, S)$ is a fixed point on $S^2 \times \cdots \times S^2$ of index 2 such that the i-th coordinate is N and the other coordinates are S.
- The composition map $\psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}$ preserves their indices, weights, and the values of the moment map.
- By Corollary 2.4, $\psi^{-1} \cdots \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}$ induces an isomorphism $f : H^*_{S^1}(M; \mathbb{Z}) \to H^*_{S^1}(S^2 \times \cdots \times S^2; \mathbb{Z})$. Moreover, by the proof of Lemma 4.1, f sends the equivariant symplectic class $[\widetilde{\omega}]$ in $H^2_{S^1}(M; \mathbb{R})$ to the one in $H^2_{S^1}(S^2 \times \cdots \times S^2; \mathbb{R})$.

We will refer to Remark 4.3 in the proof of Theorem 1.2 in Section 5.

5. The hard Lefschetz property of the reduced symplectic forms

For a Kähler manifold (N, σ) with a holomorphic circle action preserving σ, let $t \in R$ be any regular value of the moment map $H : N \to \mathbb{R}$. Since the reduced space $N_t := H^{-1}(t)/S^1$ with the reduced symplectic form σ_t is again Kähler, σ_t satisfies the hard Lefschetz property for every regular value $t \in \mathbb{R}$. In this section, we show that the same thing happens when (M, ω) is a closed semifree Hamiltonian S^1-manifold whose fixed points are all isolated. The following theorem is due to Tolman and Weitsman.

Theorem 5.1 ([20])**.** *Let (M, ω) be a closed Hamiltonian S^1-manifold with a moment map $\mu : M \to \mathbb{R}$. Assume that all fixed points are isolated and 0 is a regular value. Let M^{S^1} be the set of fixed points. Define $K_+^M := \{\alpha \in H^*_{S^1}(M; \mathbb{Z}) |\ \alpha|_{F_+} = 0\}$ where $F_+ := M^{S^1} \cap \mu^{-1}(0, \infty)$ and $K_-^M := \{\alpha \in H^*_{S^1}(M; \mathbb{Z}) |\ \alpha|_{F_-} = 0\}$ where $F_- := M^{S^1} \cap \mu^{-1}(-\infty, 0)$. Then there is a short exact sequence*

$$0 \longrightarrow K \longrightarrow H^*_{S^1}(M; \mathbb{Z}) \xrightarrow{\kappa} (M_{\mathrm{red}}; \mathbb{Z}) \longrightarrow 0$$

where κ is the Kirwan map.

Now we are ready to prove Theorem 1.2.

Theorem 5.2 (Theorem 1.2)**.** *Let (M, ω) be a closed semifree Hamiltonian S^1-manifold whose fixed points are all isolated, and let μ be the moment map. Then ω satisfies the hard Lefschetz property. Moreover, the reduced symplectic form ω_t satisfies the hard Lefschetz property for every regular value t.*

Proof. Let $\mu : M \to \mathbb{R}$ be a moment map such that $0 \in \mathbb{R}$ is a regular value of μ. For $M_{\mathrm{red}} \cong \mu^{-1}(0)/S^1$ with the reduced symplectic form ω_0, let $\kappa_M : H^*_{S^1}(M; \mathbb{R}) \to H^*(M_{\mathrm{red}}; \mathbb{R})$ be the Kirwan map for (M, ω) and let κ be the one for $(S^2 \times \cdots \times S^2, \sigma)$, where $\sigma := \omega_1 \oplus \cdots \oplus \omega_n$ is chosen in the proof of Theorem 1.1 in Section 4. (See also Remark 4.3.) As in Remark 4.3, we proved that there exists a semifree holomorphic Hamiltonian S^1-manifold $(S^2 \times \cdots \times S^2, \sigma)$ with the moment map μ' such that $\psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}$ preserves their indices, weights, and the values of the moment map. Also, the induced ring isomorphism $f : H^*_{S^1}(M; \mathbb{Z}) \to H^*_{S^1}(S^2 \times \cdots \times S^2; \mathbb{Z})$ given in Corollary 2.4 satisfies $\alpha|_x = f(\alpha)|_{\psi^{-1} \circ \phi(x)}$ for any $\alpha \in H^*_{S^1}(M; \mathbb{Z})$ and any fixed point $x \in M^{S^1}$. Hence $\psi^{-1} \circ \phi$ identifies K_+^M with $K_+^{S2 \times \cdots \times S^2}$ and K_-^M with $K_-^{S2 \times \cdots \times S^2}$. Hence if $\alpha \in K_+^M$, then $f(\alpha) \in K_+^{S^2 \times \cdots \times S^2}$. Similarly for any $\alpha \in K_-^M$, we have $f(\alpha) \in K_-^{S^2 \times \cdots \times S^2}$. Therefore $f(\alpha)$ is in $\ker \kappa$ if and only if $\alpha \in \ker \kappa_M$ by Theorem 5.1.

Now, let $\widetilde{\omega}$ be the equivariant symplectic form with respect to the moment map μ. (See Section 3.) Note that $\kappa(f([\widetilde{\omega}]))$ is the cohomology class of the reduced symplectic form of $S^2 \times \cdots \times S^2$ at $\mu^{-1}(0)/S^1$. Since the Kähler quotient of the holomorphic action is again Kähler, $\kappa(f([\widetilde{\omega}]))$ satisfies the hard Lefschetz property. Now, assume that ω_0 does not satisfy the hard Lefschetz property. Then there exists a positive integer $k (< n)$ and some non-zero $\alpha \in H^k(M_{\mathrm{red}}; \mathbb{R})$ such that $\alpha \cdot [\omega_0]^{n-k} = 0$ in $H^{2n-k}(M_{\mathrm{red}})$. By the Kirwan surjectivity theorem [13], we can find $\widetilde{\alpha} \in H^k_{S^1}(M; \mathbb{R})$ with $\kappa(\widetilde{\alpha}) = \alpha$. Then $\widetilde{\alpha} \cdot [\widetilde{\omega}]^{n-k}$ is in $\ker \kappa_M$ and hence the image $f(\widetilde{\alpha} \cdot [\widetilde{\omega}]^{n-k})$ is in $\ker \kappa$. It implies that $f(\widetilde{\alpha}) = 0$ by the hard Lefschetz condition for $f([\widetilde{\omega}])$. Since $[\widetilde{\alpha}] \in \ker \kappa_M$ if and only if $f([\widetilde{\alpha}]) \in \ker \kappa$ and $[\widetilde{\alpha}]$ is not in $\ker \kappa_M$, it is a contradiction.

It remains to show that (M, ω) satisfies the hard Lefschetz property. Recall that $\psi^{-1} \circ \phi : M^{S^1} \to (S^2 \times \cdots \times S^2)^{S^1}$ induces an isomorphism

$$f : H^*_{S^1}(M; \mathbb{Z}) \to H^*_{S^1}(S^2 \times \cdots \times S^2; \mathbb{Z}),$$

which sends the equivariant symplectic class $[\widetilde{\omega}]$ to $[\widetilde{\sigma}]$ as we have seen in Section 4. (See Remark 4.3.) Here, $\widetilde{\sigma}$ is an equivariant symplectic form induced by $\sigma - d(\mu'\theta)$ in $H^*_{S^1}(S^2 \times \cdots \times S^2; \mathbb{R})$. Since f is an $H^*(BS^1; \mathbb{R})$-algebra isomorphism, it induces a ring isomorphism

$$f_u : \frac{H^*_{S^1}(M; \mathbb{R})}{u \cdot H^*_{S^1}(M; \mathbb{R})} \to \frac{H^*_{S^1}(S^2 \times \cdots \times S^2; \mathbb{R})}{u \cdot H^*_{S^1}(S^2 \times \cdots \times S^2; \mathbb{R})}.$$

Moreover, the quotient map $\pi_M : H^*_{S^1}(M; \mathbb{R}) \to \frac{H^*_{S^1}(M;\mathbb{R})}{u \cdot H^*_{S^1}(M;\mathbb{R})} \cong H^*(M; \mathbb{R})$ ($\pi_{S^2 \times \cdots \times S^2}$, resp.) is a ring homomorphism which comes from an inclusion $M \hookrightarrow M \times_{S^1} ES^1$ as a fiber. Therefore $\pi_M([\widetilde{\omega}]) = [\omega]$ and $\pi_{S^2 \times \cdots \times S^2}([\widetilde{\sigma}]) = [\sigma]$. Therefore the isomorphism f_u maps $[\omega]$ to $[\sigma]$. Since σ is a Kähler form, it satisfies the hard Lefschetz property. Hence so does ω. $\square$

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