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BUBBLE TREE FOR APPROXIMATE HARMONIC MAPS

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Abstract. In this paper, we set up the complete bubble tree theory for approximate harmonic maps from a Riemann surface with tension fields bounded in Zygmund class $L \ln^+ L$. Some special cases of this theory have previously been used in a number of papers.

On the other hand, one can see that this bubble tree theory is not true for the general target manifold if we only assume that the tension fields are bounded in L^1 uniformly.

1. Introduction and main results

Let (M, g) be a closed Riemannian manifold and (N, h) be a Riemannian manifold without boundary. For a map u from M to N in $W^{1,2}(M, N)$, the energy density of u is defined by

$$e(u) = \frac{1}{2}|du|^2 = \mathrm{Trace}_g u^* h,$$

where u^*h is the pull-back of the metric tensor h.

The energy of the map u is defined as

$$E(u) = \int_M e(u) dV,$$

where dV is the volume element of (M, g).

A map $u \in C^1(M, N)$ is called a harmonic map if it is a critical point of the energy E.

By the Nash embedding theorem we know that (N, h) can be isometrically embedded into a Euclidean space R^K with some positive integer K. Then (N, h) may be considered as a submanifold of R^K with the metric induced from the Euclidean metric. Thus a map $u \in C^1(M, N)$ can be considered as a map of $C^1(M, R^K)$ whose image lies on N. In this sense we can get the following Euler-Lagrange equation:

$$\triangle u = A(u)(du, du).$$

The tension field $\tau(u)$ is defined by

$$\tau(u) = \triangle_M u - A(u)(du, du),$$

where $A(u)(du, du)$ is the second fundamental form of N in R^K. So u is harmonic means that $\tau(u) = 0$.

The harmonic maps are of special interest when M is a Riemann surface, as the energy is conformal invariant in this case.

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Consider a sequence of maps u_n from Riemann surface M to N with bounded energies. It is clear that u_n converges weakly to u in $W^{1,2}(M,N)$ for some $u \in W^{1,2}(M,N)$. But in general, it may not converge to u strongly in $W^{1,2}(M,N)$.

The space $L\ln^+ L$ (Zygmund class) is defined by

$$\{f : \|f\|_{L\ln^+ L} = \int |f(x)| \ln(2 + |f(x)|)dx < \infty\}.$$

Note that $\|.\|_{L\ln^+ L}$ isn't a standard norm, but we can find a norm in the space $L\ln^+ L$.

We may use the following quantity in the space $L\ln^+ L$:

$$\|f\|^*_{L\ln^+ L} = \int_0^\infty f^*(t)\ln(2 + \frac{1}{t})dt,$$

where $f^*(t) = \inf\{s \geq 0 : |\{x : |f(x)| > s\}| \leq t\}$ is the nonincreasing rearrangement function of f.

We know that $\|\cdot\|^*_{L\ln^+ L}$ and $\|\cdot\|_{L\ln^+ L}$ can be controlled by each other through suitable functions.

We will show how u_n tends to u weakly in $W^{1,2}(M,N)$. Our main result is the following bubble tree theorem. Without loss of generality, let M be the unit disk B_1.

Theorem 1. *Let $\{u_n\}$ be a sequence of maps from B_1 to N in $W^{1,2}(B_1,N)$ with tension field $\tau(u_n)$. If u_n tends to u strongly in $W^{1,2}(B_1 \setminus B_{\frac{1}{2}}, N)$, weakly in $W^{1,2}(B_1,N)$ and*

$$\|u_n\|_{W^{1,2}(B_1)} + \|\tau(u_n)\|_{L\ln^+ L(B_1)} \leq \Lambda,$$

then there exist a subsequence of $\{u_n\}$ (we still denote it by $\{u_n\}$) and some nonnegative integer m. For any $i = 1, ..., m$, there exist points x^i_n, positive numbers r^i_n and nonconstant harmonic spheres ψ_i (which we view as a map from $R^2 \cup \{\infty\} \to N$) such that

(1) $x^i_n \to x^i \in B_{\frac{1}{2}}, r^i_n \to 0$ as $n \to \infty$.

(2) $\lim_{n\to\infty}(\frac{r^i_n}{r^j_n} + \frac{r^j_n}{r^i_n} + \frac{|x^i_n - x^j_n|}{r^i_n + r^j_n}) = \infty$ for any $i \neq j$.

(3) ψ_i is the weak limit or strong limit of $u_n(x^i_n + r^i_n x)$ in $W^{1,2}_{Loc}(R^2,N)$. Moreover, if

$$\lim_{n\to\infty} \frac{|x^i_n - x^j_n|}{r^i_n} = \infty, \forall j \neq i,$$

then $u_n(x^i_n + r^i_n x) \to \psi_i$ strongly in $W^{1,2}_{Loc}(R^2,N)$.

(4) For any $x_n \to x_0 \in B_1, r_n \to 0$, if for any $i = 1, 2, ..., m$, we have

$$\lim_{n\to\infty}(\frac{r^i_n}{r_n} + \frac{r_n}{r^i_n} + \frac{|x^i_n - x_n|}{r^i_n + r_n}) = \infty,$$

then $u_n(x_n + r_n x) \to 0$ weakly in $W^{1,2}_{Loc}(R^2,N)$.

Remark. Property (2) means that the sequences $u_n(x^i_n + r^i_n x)$ and $u_n(x^j_n + r^j_n x)$ do not tend to the same bubble weakly in $W^{1,2}_{Loc}(R^2,N)$. Property (3) shows the picture about the bubbles. Property (4) implies that there are no other bubbles for this subsequence.

If there is no energy on the neck domain during blowing up, i.e.

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty} E(u_n, \bigcup_i^m B(x^i,\delta)\setminus\bigcup_i^m B(x_n^i, r_n^i R)) = 0,$$

which is equivalent to

$$\lim_{n\to\infty} E(u_n) = E(u) + \sum_{i=1}^m E(\psi_i),$$

we say that the energy identity holds.

If there is no oscillation on the neck domain, i.e.

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty}\sup_{x,y\in\bigcup_i^m B(x^i,\delta)\setminus\bigcup_i^m B(x_n^i,r_n^i R)} |u_n(x)-u_n(y)| = 0,$$

we say that there is no neck or necklessness during blowing up. The geometrical meaning is that the image $u(B_1)\cup\bigcup_{i=1}^m \psi_j(R^2)$ is a connected set if there is no neck during blowing up.

When $\tau(u_n)=0$ or $\tau(u_n)$ are bounded in L^2, the bubble tree, energy identity and necklessness have been proved in [1,4–8]. Though we have not found the complete proof of bubble tree theorem, it seems that it is not difficult to extend the bubble tree theorem to the case where $\tau(u_n)$ are bounded in L^p for some $p>1$. In [3] we extended the energy identity and necklessness for maps with tension fields bounded in $L^{\frac{6}{5}}$.

In [2], when the target manifold is a standard sphere, we proved the bubble tree theorem and the energy identity for a sequence of u_n with bounded energies and tension fields bounded in $L\ln^+ L$. It is worth pointing out that there may exist a positive neck in this case.

Our proof of the bubble tree theorem in [2] depends on a convergence result and a new compactness theorem. The convergence result was proved via a div-curl lemma and it is not valid for the general target manifold.

Here we use a special induction on the quantity $\sup_n E(u_n, B_1)$ to prove the main theorem. Also our proof depends on the compactness theorem for approximate harmonic maps proved in [2] and a convergence lemma for the general target manifold.

Throughout this paper, without illustration the letter C denotes a positive constant which may depend on Λ, N and may vary in different cases. Furthermore we do not distinguish the sequence and its subsequence.

2. Proof of the main theorem

We recall the following compactness theorem ([2], Theorem 1.2).

Theorem 2. *Let N be a compact Riemannian manifold. Assume that a sequence of maps $u_n\in W^{1,2}(B_1,N)$ satisfies*

$$\|u_n\|_{W^{1,2}(B_1)} + \|\tau(u_n)\|_{L\ln^+ L(B_1)} \le \Lambda.$$

There exists a positive constant ϵ_N which depends only on the target manifold such that if $E(u_n,B_1)\le\epsilon_N$, then there exists a subsequence of u_n (still denoted by u_n) and u such that

$$\lim_{n\to\infty}\|u_n-u\|_{W^{1,2}(B_{\frac{1}{2}})} = 0.$$

It is easy to see that this theorem is not true if we only assume that $\tau(u_n)$ are bounded in L^1.

The point $x \in B_1$ is called a concentration point of u_n if for any $r, 0 < r < 1-|x|$,

$$\sup_n E(u_n, B(x,r)) > \epsilon_N.$$

By Theorem 2, if x_0 is not a concentration point of u_n, then u_n has a subsequence converging strongly in $W^{1,2}$-norm near x_0.

We show that there exists a subsequence of u_n (we still denote it by u_n) that has only finite concentration points. If it is not true, take a positive integer $J > \frac{\Lambda}{\epsilon_N}$. Then we can find a subsequence of u_n that has at least J concentration points $x_1, x_2, \ldots, x_J$. So we have

$$\lim_{n\to\infty} E(u_n, B_1) \geq \sum_1^J \lim_{r\to 0} \lim_{n\to\infty} E(u_n, B(x_i, r)) > J\epsilon_N > \Lambda,$$

which contradicts the assumption.

As a corollary, we get that immediately

Lemma 3. *Suppose M is a closed Riemannian surface without boundary. Then for any nontrivial harmonic map u from M to N, there holds*

$$E(u) \geq \epsilon_N.$$

We also need a convergence lemma.

Lemma 4. *Suppose that u_n are maps from a Riemann surface M to N with energies bounded and tension fields $\tau(u_n)$ bounded in $L\ln^+ L$, i.e.*

$$\|u_n\|_{W^{1,2}(M)} + \|\tau(u_n)\|_{L\ln^+ L(M)} \leq \Lambda.$$

Then there exists $u \in W^{1,2}(M,N)$ and a subsequence of u_n (still denoted by u_n) such that $u_n \to u$ weakly in $W^{1,2}(M,N)$ and

$$\triangle u_n + A(u_n)(du_n, du_n) \to \triangle u + A(u)(du, du) \text{ in } L^1(M).$$

Remark. When the target manifold is a standard sphere or the Euclidean space, this lemma is true without any conditions on $\tau(u_n)$.

Proof. Passing to a subsequence, it is easy to assume that $u_n \rightharpoonup u$ weakly in $W^{1,2}$ and $u_n \to u$ in L^1.

Without loss of generality suppose that there is only one concentration point $\{0\} \in M$ for the sequence u_n; in this case

$$|\nabla u_n|^2 \rightharpoonup |\nabla u|^2 + a\delta_0,$$

where $a > 0$. By Theorem 2 we know that $u_n \to u$ strongly in $W^{1,2}(M \setminus B_\delta)$ for any $\delta > 0$.

For any positive integer k, set $F_{n,k} = \{x \in M : |u_n(x) - u(x)| > \frac{1}{k}\}$. Obviously there holds $|F_{n,k}| \to 0$ as $n \to \infty$. So for any k, there exists n_k such that $|F_{n_k,k}| < 2^{-k}$. Choose the subsequence u_{n_k} and rewrite it as u_n. Then for any n we have

$$|\{x \in M : |u_n(x) - u(x)| > \frac{1}{n}\}| < 2^{-n}.$$

Take n_δ such that $2^{-n_\delta} < \delta$ and set $F_0 = \bigcup_{n > n_\delta} \{x \in M : |u_n(x) - u(x)| > \frac{1}{n}\}$. Then $|F_0| < 2^{-n_0} < \delta$ and $u_n \to u$ strongly in $L^\infty(M \setminus F_0)$.

Now we show that $A(u_n)(du_n, du_n) \to A(u)(du, du)$ in $L^1(M \setminus (F_0 \cup B_\delta))$. As the coefficient of the second fundamental form is smooth in N, it can be shown that

$$\begin{aligned}
& |A(u_n)(du_n, du_n) - A(u)(du, du)| \\
= \;& |(A(u_n)(du_n, du_n) - A(u_n)(du_n, du)) + (A(u_n)(du_n, du) - A(u_n)(du, du)) \\
& +(A(u_n)(du, du) - A(u)(du, du))| \\
\leq \;& |A(u_n)|(|\nabla u_n||\nabla u_n - \nabla u| + |\nabla u||\nabla u_n - \nabla u|) + |A(u_n) - A(u)||\nabla u|^2 \\
\leq \;& C((|\nabla u_n| + |\nabla u|)|\nabla(u_n - u)| + |u_n - u||\nabla u|^2).
\end{aligned}$$

Because $u_n \to u$ in $L^\infty(M \setminus F_0)$ and $u_n \to u$ in $W^{1,2}(M \setminus B_\delta)$, we can get that

$$\begin{aligned}
& \lim_{n\to\infty} \|A(u_n)(du_n, du_n) - A(u)(du, du)\|_{L^1(M\setminus(F_0\cup B_\delta))} \\
\leq \;& C \lim_{n\to\infty} (\|\nabla(u_n + u)\|_2 \|\nabla(u_n - u)\|_{L^2(M\setminus B_\delta)} + \|u_n - u\|_{L^\infty(M\setminus F_0)} \|\nabla u\|_2^2) \\
= \;& 0. \qquad (1)
\end{aligned}$$

As $\triangle u_n = -A(u_n)(du_n, du_n) + \tau(u_n)$ are bounded in L^1, we can assume that

$$|\triangle u_n| \rightharpoonup f + b\delta_0,$$

where $b \geq 0$. It is worth pointing out that here b may be equal to 0. So there exists $f_0 \in L^1(M \setminus B_\delta)$ such that $\triangle u_n \to f_0$ in $L^1(M \setminus B_\delta)$. On the other hand, as $\nabla u_n \to \nabla u$ weakly in $L^2(M)$, there holds $\triangle u_n \to \triangle u$ in the sense of distribution. So it must be $\triangle u = f_0$ on $M \setminus B_\delta$, i.e.

$$\lim_{n\to\infty} \|\triangle u_n - \triangle u\|_{L^1(M\setminus B_\delta)} = 0. \qquad (2)$$

Take a cut-off function $\varphi \in C_0^\infty(B_2)$ with $\varphi = 1$ on B_1 and $\varphi_\delta(x) = \varphi(\frac{x}{\delta})$. It can be shown that

$$\begin{aligned}
& (\triangle u_n + A(u_n)(du_n, du_n)) - (\triangle u + A(u)(du, du)) \\
= \;& (1 - \varphi_\delta)(1 - \chi_{F_0})[(\triangle u_n - \triangle u) + (A(u_n)(du_n, du_n) - A(u)(du, du))] \\
& +(\varphi_\delta + \chi_{F_0} - \varphi_\delta\chi_{F_0})(\triangle u_n + A(u_n)(du_n, du_n)) \\
& -(\varphi_\delta + \chi_{F_0} - \varphi_\delta\chi_{F_0})(\triangle u + A(u)(du, du)) \\
= \;& -(1 - \varphi_\delta)\chi_{F_0}\triangle u - (\varphi_\delta + \chi_{F_0} - \varphi_\delta\chi_{F_0})A(u)(du, du) \\
& +(1 - \varphi_\delta)(1 - \chi_{F_0})[(\triangle u_n - \triangle u) + (A(u_n)(du_n, du_n) - A(u)(du, du))] \\
& +(\varphi_\delta + \chi_{F_0} - \varphi_\delta\chi_{F_0})\tau(u_n) \\
& -\varphi_\delta\triangle u.
\end{aligned}$$

At first, as $\triangle u \in L^1(M \setminus B_\delta)$ and $|\nabla u|^2 \in L^1(M)$, by the absolute continuity, for any $\epsilon > 0$, let δ be small enough such that

$$\int_M (1 - \varphi_\delta)\chi_{F_0}|\triangle u|dx + \int_M (\varphi_\delta + \chi_{F_0} - \varphi_\delta\chi_{F_0})|A(u)(du, du)|dx < \epsilon. \qquad (3)$$

From (1) and (2), there exists $n_\epsilon > n_\delta$, such that for any $n > n_\epsilon$,

$$\int_M (1 - \varphi_\delta)(1 - \chi_{F_0})|(\triangle u_n - \triangle u) + (A(u_n)(du_n, du_n) - A(u)(du, du))|dx < \epsilon. \qquad (4)$$

On the other hand, some simple computations show that

$$
\begin{aligned}
& \int_M (\varphi_\delta + \chi_{F_0} - \varphi_\delta \chi_{F_0}) |\tau(u_n)| dx \\
\leq\ & \int_{F_0 \cup B_{2\delta}} |\tau(u_n)| dx \\
\leq\ & \int_0^{|F_0 \cup B_{2\delta}|} |\tau(u_n)^*| dt \\
\leq\ & \int_0^{C\delta^2} (|\tau(u_n)^*| \ln \frac{1}{t})(\ln \frac{1}{t})^{-1} dt \\
\leq\ & C(\ln \frac{1}{\delta})^{-1} \|\tau(u_n)\|^*_{L \ln^+ L} \\
\leq\ & C(\ln \frac{1}{\delta})^{-1}. \qquad (5)
\end{aligned}
$$

At last, for any $\phi \in C^\infty(M, N)$, we have

$$
\begin{aligned}
& |\int_M \varphi_\delta \phi \cdot \triangle u dx| \\
=\ & |\int_M \nabla(\varphi_\delta \phi) \nabla u dx| \\
\leq\ & \|\nabla(\varphi_\delta \phi)\|_2 \|\nabla u\|_{L^2(B_{2\delta})} \\
\leq\ & C(1 + \delta \|\nabla \phi\|_\infty) \epsilon^{\frac{1}{2}}. \qquad (6)
\end{aligned}
$$

Here we use the fact that $\int_{B_{2\delta}} |\nabla u|^2 dx < \epsilon$.

For any $\phi \in C^\infty(M, N)$ and $\epsilon > 0$, take δ small enough. Then from (3), (4), (5) and (6) there exists a positive integer n_ϵ such that for any $n > n_\epsilon$ there holds

$$
\begin{aligned}
& |\int_M [(\triangle u_n + A(u_n)(du_n, du_n)) - (\triangle u + A(u)(du, du))] \phi dx| \\
<\ & C((\ln \frac{1}{\delta})^{-1} + \epsilon^{\frac{1}{2}}) \|\nabla \phi\|_{C^1} \\
<\ & C\epsilon^{\frac{1}{2}}.
\end{aligned}
$$

So we get that $\triangle u_n + A(u_n)(du_n, du_n) \to \triangle u + A(u)(du, du)$ in the sense of distribution, i.e. $\tau(u_n) \to \tau(u)$ in the sense of distribution. On the other hand, as $\tau(u_n)$ are bounded in L^1, there exists a subsequence of $\tau \in L^1$ and $\tau \in L^1$ such that

$$\tau(u_n) \rightharpoonup \tau + \mu$$

with $|spt(\mu)| = 0$. Because $\tau(u_n)$ are bounded in $L \ln^+ L$, there is no concentration point for the sequence $\tau(u_n)$, i.e. there must be $\mu = 0$. So there must be $\tau(u) = \tau$ which implies that $\triangle u_n + A(u_n)(du_n, du_n) \to \triangle u + A(u)(du, du)$ in $L^1(M)$. □

As a simple application of this lemma, we show that

Lemma 5. *Suppose that u_n are maps from B_1 to N with bounded energies and tension fields $\tau(u_n)$ bounded in $L \ln^+ L$, i.e.*

$$\|u_n\|_{W^{1,2}(B_1)} + \|\tau(u_n)\|_{L \ln^+ L(B_1)} \leq \Lambda.$$

For any sequence $r_n \to 0$ and $x_n \in B_{\frac{1}{2}}$, there is a subsequence of $w_n(x) = u_n(x_n + r_n x)$ which converges to a harmonic map w weakly in $W^{1,2}_{Loc}(R^2, N)$.

Some simple computations show that $\tau(w_n)$ are bounded in $L\ln^+ L$ and we have $\int_{B_R} |\tau(w_n)|dx \to 0$. We omit the details.

Proof of the main theorem. Now we use the induction on the quantity $I = \lim_{n\to\infty} \frac{|E(u_n,B_1)-E(u,B_1)|}{\epsilon_N}$ to prove the main theorem. □

Step 1. If $\lim_{n\to\infty} \frac{|E(u_n,B_1)-E(u,B_1)|}{\epsilon_N} < 2$, it is easy to see that there is at most one bubble. In this case, if $E(u_n, B_1) < \frac{\epsilon_N}{2}$, take $r_n^1 = 1, x_n^1 = 0$. Otherwise by the continuity of integral, for any n, we can find $r_n^1, x_n^1 \in B_{\frac{1}{2}}$ such that

$$E(u_n, B(x_n^1, r_n^1)) = \sup_{B(y,r_n^1)\subset B_1} E(u_n, B(y, r_n^1)) = \frac{\epsilon_N}{2}.$$

If $\lim_{n\to\infty} r_n^1 > 0$, there is no bubble, otherwise set $w_n^1(x) = u_n(x_n^1 + r_n^1 x)$. It is easy to check that

$$E(w_n^1, B_1) = \sup_y E(w_n^1, B(y,1)) = E(u_n, B(x_n^1, r_n^1)) = \frac{\epsilon_N}{2}.$$

For any $R > 0$, there holds $E(w_n^1, B_R) \le \Lambda$, so by Theorem 2 there exists $\psi^1 \in W^{1,2}(R^2)$ such that $w_n^1 \to \psi^1$ strongly in $W^{1,2}_{Loc}(R^2)$.

Obviously ψ^1 is not a constant map. From Lemma 5 we know ψ^1 is a weakly harmonic map. By Helein's theorem, ψ^1 is a nonconstant smooth harmonic map. Thus we prove the theorem in this case.

Step 2. Assume that when $I = \lim_{n\to\infty} \frac{|E(u_n,B_1)-E(u,B_1)|}{\epsilon_N} < k$, the theorem is true. Now suppose that

$$I = \lim_{n\to\infty} \frac{|E(u_n,B_1) - E(u,B_1)|}{\epsilon_N} < k+1.$$

By the same argument in Step 1 we can show that there is no bubble or obtain the first bubble ψ^1 which is the strong limit of $w_n^1(x) = u_n(x_n^1 + r_n^1 x)$ in $W^{1,2}_{Loc}(R^2)$. It does not matter if we assume that $x_n^1 = 0$ for any n.

We claim that there exist $\epsilon > 0$ and a sequence $t_n \to 0$ such that

(1) $\int_{B_{2t_n}\setminus B_{t_n}} |\nabla u_n|^2 dx > \epsilon, \forall n$;

(2) for any sequence $s_n \to 0$, if $\lim_{n\to\infty} \frac{s_n}{t_n} = \infty$, then there holds

$$\lim_{n\to\infty} \int_{B_{2s_n}\setminus B_{s_n}} |\nabla u_n|^2 dx = 0.$$

If it is not true, then we can get infinity sequences $t_n^j \to 0$ such that $\lim_{n\to\infty} \frac{t_n^{j+1}}{t_n^j} = \infty$ and

$$\lim_{n\to\infty} \int_{B_{2t_n^j}\setminus B_{t_n^j}} |\nabla u_n|^2 dx > 0.$$

Set $v_n^j(x) = u_n(t_n^j x)$ and suppose that $v_n^j \to v^j$ weakly in $W^{1,2}_{Loc}(R^2)$.

Case 1. If there exists a concentration point $x^j \in B_4 \setminus B_{\frac{1}{2}}$ for the sequence v_n^j, then from Theorem 2 we have

$$\begin{aligned}\lim_{R\to\infty}\lim_{n\to\infty} E(v_n^j, B_R) &\ge \lim_{n\to\infty} E(v_n^j, B_4 \setminus B_{\frac{1}{2}}) + \lim_{n\to\infty} E(v_n^j, B_{\frac{1}{2}}) \\ &> \epsilon_N + \lim_{R\to\infty}\lim_{n\to\infty} E(v_n^{j-1}, B_R).\end{aligned}$$

Case 2. If there is no concentration point on $B_4 \setminus B_{\frac{1}{2}}$ for the sequence v_n^j, then from Theorem 2 and Lemma 5 we know that v^j is a nonconstant harmonic map and the origin 0 is a concentration point for the sequence v_n^j. So there holds

$$\begin{aligned}\lim_{R\to\infty}\lim_{n\to\infty} E(v_n^j, B_R) &\geq E(v^j) + \lim_{\delta\to 0}\lim_{n\to\infty} E(v_n^j, B_\delta) \\ &> \epsilon_N + \lim_{R\to\infty}\lim_{n\to\infty} E(v_n^{j-1}, B_R).\end{aligned}$$

So we always have

$$\begin{aligned}\lim_{n\to\infty}(E(u_n, B_1) - E(u)) &\geq \lim_{R\to\infty}\lim_{n\to\infty} E(v_n^j, B_R) \\ &> \epsilon_N + \lim_{R\to\infty}\lim_{n\to\infty} E(v_n^{j-1}, B_R) \\ &> j\epsilon_N.\end{aligned}$$

Take $j > k+1$; it is a contradiction, so the claim is true.

Set $\tilde{u}_n(x) = u_n(t_n x)$. Assume that $\tilde{u}_n \to \tilde{u}$ weakly in $W^{1,2}_{Loc}(R^2)$.

Case 1. Assume there exists a concentration point on $B_4 \setminus B_{\frac{1}{2}}$ for the sequence $\tilde{u}_n$. Without loss of generality, suppose that $x_0 = (1,0)$ is a concentration point of $\tilde{u}_n$. Then $\tilde{u}_n$ has at least two concentration points, $(0,0)$ and $(1,0)$. For $R > 2$, consider $\tilde{u}_n$ on the ball $B_R^1 = B_R((-R + \frac{2}{3}, 0))$. It is easy to see that

$$\begin{aligned}&\lim_{n\to\infty}(E(\tilde{u}_n, B_R^1) - E(\tilde{u}, B_R^1)) \\ \leq\ &\lim_{R_0\to\infty}\lim_{n\to\infty}(E(\tilde{u}_n, B_{R_0}) - E(\tilde{u}, B_{R_0})) - \lim_{\delta\to 0}\lim_{n\to\infty} E(\tilde{u}_n, B_\delta(x_0)) \\ \leq\ &\lim_{\delta\to 0}\lim_{n\to\infty} E(u_n, B_\delta) - \epsilon_N \\ <\ &k\epsilon_N.\end{aligned}$$

By the induction assumption we can construct the bubble tree of $\tilde{u}_n$ on the ball B_R^1. Similarly, we can obtain the bubble tree of $\tilde{u}_n$ on the ball $B_R^2 = B_R((R + \frac{2}{3}, 0))$. It is easy to see that for any x, if R is large enough, then $x \in B_R^1 \cup B_R^2$. So we obtain the bubble tree of $\tilde{u}_n$.

Case 2. If there is no concentration point on $B_4 \setminus B_{\frac{1}{2}}$ for the sequence $\tilde{u}_n$, then we know that $\tilde{u}$ is a nonconstant harmonic map and the origin 0 is a concentration point for the sequence $\tilde{u}_n$. So there holds

$$\begin{aligned}\lim_{R\to\infty}\lim_{n\to\infty}(E(\tilde{u}_n, B_R) - E(\tilde{u})) &\leq \lim_{R\to\infty}\lim_{n\to\infty} E(\tilde{u}_n, B_R) - \epsilon_N \\ &= \lim_{\delta\to 0}\lim_{n\to\infty} E(u_n, B_\delta) - \epsilon_N \\ &\leq \lim_{n\to\infty}(E(u_n) - E(u)) - \epsilon_N \\ &< k\epsilon_N.\end{aligned}$$

We can also get the bubble tree $\tilde{u}_n$ on R^2 by the induction assumption.

Now assume that all the bubbles of $\tilde{u}_n$ are produced by $\tilde{u}_n(y_n^i + s_n^i y)$. By the selection of t_n, for any x_n, r_n with $\frac{|x_n| + r_n}{t_n} \to \infty$, one can see that $u_n(x_n + r_n x) \to 0$ weakly in $W^{1,2}_{Loc}(R^2, N)$, i.e. any bubble of u_n is a bubble of $\tilde{u}_n$ or produced by $\tilde{u}_n$ itself. Set $x_n^i = t_n y_n^i, r_n^i = t_n s_n^i$. If $\tilde{u}$ is constant, we get that all the bubbles of u_n are produced by $u_n(x_n^i + r_n^i y)$. If $\tilde{u}$ is nonconstant, $\tilde{u}_n = u_n(t_n x)$ produces a new bubble $\tilde{u}$. So we prove the theorem.

Remark. Consider $\triangle u_n = f_n = n^2 f(nx)$ on B_1 where $f \in C_0^\infty(B_1) \geq 0$. It is easy to see that this theorem is false if we require only that $\tau(u_n)$ are bounded in L^1.

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