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A SUBADDITIVE PROPERTY OF THE ERROR FUNCTION

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Abstract. We prove the following subadditive property of the error function:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \quad (x \in \mathbf{R}).$$

Let a and b be real numbers. The inequality

$$\operatorname{erf}\big((x+y)^a\big)^b < \operatorname{erf}(x^a)^b + \operatorname{erf}(y^a)^b$$

holds for all positive real numbers x and y if and only if $ab \le 1$.

1. Introduction

The Gauss error function, for $x \in \mathbf{R}$, is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

It has remarkable applications in probability theory, statistics, financial mathematics (e.g. the Black-Scholes-Merton theory of stock option), and also in the theory of partial differential equations (e.g. the heat equation). Because of its relevance and wide applications, the error function has been studied thoroughly by many researchers. A collection of the most important properties can be found, for instance, in the well-known Handbook by Abramowitz and Stegun [AS, chapter 7]. Inequalities involving the error function have been given in [Alz03], [Alz09], [Alz10], [Bar08a], [Bar08b], [Chu55], [Mit70, p. 291].

It is the aim of this paper to present a subadditive property of the error function. We recall that a function $F : (0, \infty) \to \mathbf{R}$ is said to be subadditive if

$$F(x+y) \le F(x) + F(y) \quad \text{for all} \quad x, y > 0. \tag{1.1}$$

If strict inequality holds for all positive x and y, then F is called strictly subadditive. Furthermore, if (1.1) is valid with "$\ge$" instead of "$\le$", then we say that F is superadditive.

Sub- and superadditive functions play an important role in various branches including, for example, probability, semi-groups, convex bodies, and differential equations. For more information on this subject we refer to [Bec64], [Bru60], [Bru62], [Bru64], [BO62], [HP57], [RS99], [Ros50], [TWW89].

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In 2010, it was proved by the first author (Theorem 4.1 in [Alz10]) that the functions $x \mapsto \operatorname{erf}(x^\alpha)$ ($\alpha \in \mathbf{R}$) and $x \mapsto \operatorname{erf}(x \operatorname{erf}(x)^\beta)$ ($\beta \in \mathbf{R}$) are subadditive if and only if $\alpha \le 1$ and $\beta \le 0$, respectively.

We present, in Section 2, some lemmas which are needed in proving our main result. In Section 3, we present an extension of the above-mentioned result, Theorem 4.1 of [Alz10]. We determine all real parameters a, b such that $x \mapsto \operatorname{erf}(x^a)^b$ is subadditive. In Section 4, we conclude our paper with remarks on the complementary error function $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$.

2. Lemmas

The first lemma is known as l'Hôpital's rule for monotonicity. A weaker version (requiring that the functions F and G are increasing) is contained in the classical monograph by Hardy, Littlewood, and Pólya [HLP52, Proposition 148]. The same result is also cited in Mitrinović [Mit70, Proposition 3.9.49]. This result, without the additional monotone assumption, has been rediscovered in recent years and applied with success in establishing various new inequalities. The same proof given in [HLP52] actually works for the stronger version.

Lemma 1. *Let F and G be continuous on $[0,\infty)$ and differentiable on $(0,\infty)$. Furthermore, let $F(0) = G(0) = 0$ and $G' \ne 0$ on $(0,\infty)$. If F'/G' is strictly decreasing on $(0,\infty)$, then F/G is strictly decreasing on $(0,\infty)$.*

Proof. Since $G' \ne 0$, it follows that G' does not change its sign. If $G' > 0$, then $G(x) > G(0) = 0$ for $x > 0$; and, if $G' < 0$, then $G(x) < 0$ for $x > 0$. Thus, G'/G is positive on $(0,\infty)$. Applying the Mean Value Theorem gives

$$\frac{F(x)}{G(x)} = \frac{F(x) - F(0)}{G(x) - G(0)} = \frac{F'(\xi)}{G'(\xi)} \quad (0 < \xi < x),$$

so that the strict monotonicity of F'/G' leads to

$$\frac{d}{dx}\frac{F(x)}{G(x)} = \frac{G'(x)}{G(x)}\Big[\frac{F'(x)}{G'(x)} - \frac{F(x)}{G(x)}\Big] = \frac{G'(x)}{G(x)}\Big[\frac{F'(x)}{G'(x)} - \frac{F'(\xi)}{G'(\xi)}\Big] < 0.$$

This implies that F/G is strictly decreasing on $(0,\infty)$. □

Throughout, we use the notation

$$f = \operatorname{erf} \quad \text{and} \quad z_a(x) = f(x^a)^{1/a} = \operatorname{erf}(x^a)^{1/a}.$$

The following four lemmas provide some elementary properties of these functions.

Lemma 2. *Let*

$$k_1(t) = \pi(2t^2 - 1)f(t) + 2\sqrt{\pi}te^{-t^2} \quad \text{and} \quad k_2(t) = \pi f(t) - 2\sqrt{\pi}te^{-t^2}. \tag{2.1}$$

(i) *k_1 and k_2 are positive on $(0,\infty)$.*
(ii) *The function $Q = k_2/k_1$ is strictly decreasing on $(0,\infty)$ with $Q(0) = 1/2$ and $\lim_{t\to\infty} Q(t) = 0$.*

Proof. (i) Let $t > 0$. Since

$$k_1'(t) = 4\pi t f(t) > 0, \quad k_1(0) = 0, \quad k_2'(t) = 4\sqrt{\pi}t^2 e^{-t^2} > 0, \quad k_2(0) = 0,$$

we conclude that $k_1(t) > 0$ and $k_2(t) > 0$.

(ii) We define for $t > 0$:

$$u_1(t) = f(t) \quad \text{and} \quad u_2(t) = \frac{1}{\sqrt{\pi}} t e^{-t^2}.$$

Then we have

$$u_1(0) = u_2(0) = 0, \quad u_1'(t) = \frac{2}{\sqrt{\pi}} e^{-t^2} > 0, \quad \frac{u_2'(t)}{u_1'(t)} = \frac{1}{2} - t^2.$$

Applying Lemma 1 reveals that u_2/u_1 is strictly decreasing on $(0, \infty)$. Since

$$\frac{k_2'(t)}{k_1'(t)} = \frac{u_2(t)}{u_1(t)},$$

we obtain from Lemma 1 that Q is strictly decreasing on $(0, \infty)$.

Applying l'Hôpital's rule leads to

$$\lim_{t \to 0} Q(t) = \lim_{t \to 0} \frac{k_2'(t)}{k_1'(t)} = \frac{1}{\sqrt{\pi}} \lim_{t \to 0} \frac{e^{-t^2}}{f(t)/t} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2/\sqrt{\pi}} = \frac{1}{2}$$

and, since $\lim_{t \to \infty} f(t) = 1$, we obtain $\lim_{t \to \infty} Q(t) = 0$. □

Lemma 3. *Let $a \in (-1/2, 0)$. There exists a number $\tau > 0$ such that z_a'' is positive on $(0, \tau)$ and negative on (τ, ∞).*

Proof. Let $x > 0$. We get

$$z_a''(x) = -\frac{2}{\pi^{3/2}} x^{a-2} e^{-x^{2a}} f(x^a)^{-2+1/a} K(x^a), \tag{2.2}$$

with

$$K(t) = a k_1(t) + k_2(t), \tag{2.3}$$

where k_1 and k_2 are defined in (2.1). Since $-a \in (0, 1/2)$, we conclude from Lemma 2 that there exists a number $\tau_0 > 0$ such that

$$\frac{k_2(t)}{k_1(t)} > -a \quad \text{for} \quad t \in (0, \tau_0) \quad \text{and} \quad \frac{k_2(t)}{k_1(t)} < -a \quad \text{for} \quad t > \tau_0.$$

Thus, we obtain that K is positive on $(0, \tau_0)$ and negative on (τ_0, ∞). We set $\tau = \tau_0^{1/a}$. Applying identity (2.2) reveals that $z_a''(x) > 0$ for $x \in (0, \tau)$ and $z_a''(x) < 0$ for $x > \tau$. □

Lemma 4. *If $a < 0$, then we have for $x > 0$:*

$$z_a(2x) < 2 z_a(x). \tag{2.4}$$

Proof. Let $\delta \in (0, 1)$. We define for $t > 0$:

$$h(t) = f(\delta t) - \delta f(t).$$

Since

$$\frac{\partial}{\partial t} h(t) = \frac{2\delta}{\sqrt{\pi}} \left(e^{-\delta^2 t^2} - e^{-t^2} \right) > 0,$$

we get

$$h(t) > h(0) = 0.$$

Hence, with $\delta = 2^a$,

$$h(x^a) = z_a(2x)^a - \left(2 z_a(x) \right)^a > 0.$$

This yields (2.4). □

Lemma 5.

(i) *If $a > 0$, then $z_a''(x) < 0$ for $x > 0$.*
(ii) *If $a \le -1/2$, then $z_a''(x) > 0$ for $x > 0$.*

Proof. (i) Using (2.2), (2.3), and Lemma 2(i) we conclude that $z_a''(x) < 0$ if $x > 0$ and $a > 0$.

(ii) Let $a \le -1/2$. Applying Lemma 2 and (2.3) gives for $t > 0$:

$$K(t) \le -\frac{1}{2}k_1(t) + k_2(t) < 0,$$

so that (2.2) implies that $z_a''(x) > 0$ for $x > 0$. □

Moreover, we need a monotonicity property of power sums; see [BB65, p. 18].

Lemma 6. *Let*

$$S(r) = (\alpha^r + \beta^r)^{1/r} \quad (\alpha, \beta > 0;\ r \ne 0).$$

S is decreasing on $(-\infty, 0)$ and $(0, \infty)$.

3. Main result

We determine now all real parameters a and b such that the function $x \mapsto \operatorname{erf}(x^a)^b$ is strictly subadditive. An analogous result for the complementary error function will be given in the next section.

Theorem 1. *Let a and b be real numbers. The inequality*

$$\operatorname{erf}\big((x+y)^a\big)^b < \operatorname{erf}(x^a)^b + \operatorname{erf}(y^a)^b \tag{3.1}$$

holds for all positive real numbers x and y if and only if $ab \le 1$.

Proof. First, we assume that (3.1) is valid for all $x, y > 0$. We set $x = y$ and $t = x^a$. Then we get

$$\left[2^a \cdot \frac{t}{f(t)} \cdot \frac{f(2^a t)}{2^a t}\right]^b < 2 \quad (t > 0). \tag{3.2}$$

We let t tend to 0 and make use of $\lim_{t\to 0} f(t)/t = 2/\sqrt{\pi}$. Then, (3.2) gives $2^{ab} \le 2$. Thus, $ab \le 1$.

Next, we prove: if $ab \le 1$, then (3.1) holds for all $x, y > 0$. We distinguish two cases.

Case 1. $ab = 1$. Define

$$\Delta(x, y) = z_a(x) + z_a(y) - z_a(x+y).$$

Note that (3.1) is equivalent to the assertion $\Delta(x, y) > 0$. We study three subcases.

Case 1.1. $a > 0$. Applying Lemma 5(i) gives

$$\frac{\partial}{\partial y}\Delta(x, y) = z_a'(y) - z_a'(x+y) > 0.$$

Since $z_a(0) = 0$, we get

$$\Delta(x, y) > \Delta(x, 0) = 0.$$

Case 1.2. $a \le -1/2$. Lemma 5(ii) implies that z_a is strictly convex on $(0, \infty)$. Using Jensen's inequality and (2.4) yields

$$\Delta(x, y) \ge 2z_a\Big(\frac{x+y}{2}\Big) - z_a(x+y) > 0.$$

Case 1.3. $-1/2 < a < 0$. Let us define for $s > 0$:

$$D(s) = \Delta(s, y).$$

In other words, we keep the second variable of the function Δ to be fixed at y while varying the first variable to obtain D. It is easy to see that $\lim_{s\to 0} D(s) = 1$; this fact can be used to extend D to be a continuous function on $[0, \infty)$.

Let $s > \tau$, where τ is given in Lemma 3. Using the Mean Value Theorem, we see that there exists a number $\theta \in (s, s + y)$ such that

$$D'(s) = z_a'(s) - z_a'(s + y) = -yz_a''(\theta).$$

By Lemma 3, $z_a''(\theta) < 0$. Hence, $D'(s) > 0$. This implies that D is strictly increasing on $[\tau, \infty)$. It follows that D must attain a global minimum at some point $x_1 \in [0, \tau]$.

If $x_1 = 0$, then $D(s) \geq D(0) = 1$ for all $x \in (0, \infty)$. In particular, $\Delta(x, y) = D(x) \geq 1$ and (3.1) holds.

Suppose now that $x_1 \in (0, \tau]$. Then

$$\Delta(x, y) = D(x) \geq D(x_1) = \Delta(x_1, y). \tag{3.3}$$

We next define

$$\bar{D}(t) = \Delta(x_1, t).$$

This time, we fix the first variable of Δ at x_1 while varying the second variable.

Using exactly the same argument as above reveals that we may assume that $\bar{D}$ attains a global minimum at some $y_1 \in (0, \tau]$. We thus have

$$\bar{D}(t) \geq \bar{D}(y_1), \qquad \text{for all } t > 0.$$

In particular, for $t = y$ we have

$$\Delta(x_1, y) = \bar{D}(y) \geq \bar{D}(y_1) = \Delta(x_1, y_1). \tag{3.4}$$

Since z_a is convex on $(0, \tau]$ and $x_1, y_1 \in (0, \tau]$, we conclude from Jensen's inequality and Lemma 4 that

$$\Delta(x_1, y_1) = z_a(x_1) + z_a(y_1) - z_a(x_1 + y_1) \geq 2z_a\Big(\frac{x_1 + y_1}{2}\Big) - z_a(x_1 + y_1) > 0. \tag{3.5}$$

Combining (3.3)-(3.5) yields $\Delta(x, y) > 0$ for $x, y > 0$.

Case 2. $ab < 1$. If $a = 0$ or $b = 0$, then (3.1) is valid. Next, we suppose that $ab \neq 0$. We consider four subcases.

Case 2.1. $a < 0 < b$. Since f is strictly increasing, we conclude from $x^a > (x + y)^a$ that $f(x^a) > f((x + y)^a)$. Thus,

$$f(x^a)^b + f(y^a)^b > f(x^a)^b > f((x + y)^a)^b. \tag{3.6}$$

Case 2.2. $b < 0 < a$. We have $x^a < (x + y)^a$. This leads to

$$f(x^a) < f((x + y)^a) \quad \text{and} \quad f(x^a)^b > f((x + y)^a)^b.$$

It follows that (3.6) holds.

Case 2.3. $0 < a, b$. We apply Lemma 6 with $\alpha = f(x^a)$ and $\beta = f(y^a)$. Since $0 < b < 1/a$, we get

$$\big[f(x^a)^b + f(y^a)^b\big]^{1/b} \geq \big[f(x^a)^{1/a} + f(y^a)^{1/a}\big]^a > f((x + y)^a),$$

where the last inequality follows from (3.1) with $b = 1/a$. Since $b > 0$, we get (3.1).

Case 2.4. $a, b < 0$. Using $1/a < b < 0$ and (3.1) with $b = 1/a$ gives

$$\left[f(x^a)^b + f(y^a)^b\right]^{1/b} \le \left[f(x^a)^{1/a} + f(y^a)^{1/a}\right]^a < f((x+y)^a).$$

This leads to (3.1). The proof of Theorem 1 is complete. □

It is natural to ask whether there exist real parameters a and b such that $x \mapsto \operatorname{erf}(x^a)^b$ is superadditive. The answer in "no". Otherwise, we obtain for all $x, y > 0$:

$$R_{a,b}(x,y) = \frac{f(x^a)^b + f(y^a)^b}{f((x+y)^a)^b} \le 1.$$

However, this is impossible, since we have

$$R_{0,b}(x,y) = 2, \quad \lim_{x\to\infty} R_{a,b}(x,y) = 1 + f(y^a)^b > 1, \quad \text{if} \quad a > 0,$$

and

$$\lim_{x\to 0} R_{a,b}(x,y) = 1 + f(y^a)^{-b} > 1, \quad \text{if} \quad a < 0.$$

4. Concluding remarks

The complementary error function is given by

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \quad (x \in \mathbf{R}).$$

We have the following counterpart of Theorem 1.

Theorem 2. *Let a and b be real numbers. The inequality*

$$\operatorname{erfc}\left((x+y)^a\right)^b < \operatorname{erfc}(x^a)^b + \operatorname{erfc}(y^a)^b \tag{4.1}$$

holds for all positive real numbers x and y if and only if $ab \ge 0$.

Proof. The proof is short and simple. We set $x = y$ and $t = x^a$ so that (4.1) leads to

$$\left[\frac{1 - f(2^a t)}{1 - f(t)}\right]^b < 2.$$

We assume that $ab < 0$. Then,

$$\lim_{t\to\infty} \left[\frac{1 - f(2^a t)}{1 - f(t)}\right]^b = \lim_{t\to\infty} \left[2^a e^{(1-4^a)t^2}\right]^b = \infty,$$

a contradiction. Thus, if (4.1) holds for all $x, y > 0$, then $ab \ge 0$.

Conversely, if $ab = 0$, then (4.1) is valid. Next, let $ab > 0$. The monotonicity of erfc gives

$$\operatorname{erfc}((x+y)^a) < \operatorname{erfc}(x^a) \le \left(\operatorname{erfc}(x^a)^b + \operatorname{erfc}(y^a)^b\right)^{1/b}, \quad \text{if} \quad a, b > 0,$$

and

$$\left(\operatorname{erfc}(x^a)^b + \operatorname{erfc}(y^a)^b\right)^{1/b} < \operatorname{erfc}(x^a) \le \operatorname{erfc}((x+y)^a), \quad \text{if} \quad a, b < 0.$$

In both cases we conclude that (4.1) holds. □

Let us define for $x, y > 0$:

$$R^*_{a,b}(x,y) = \frac{\operatorname{erfc}(x^a)^b + \operatorname{erfc}(y^a)^b}{\operatorname{erfc}((x+y)^a)^b},$$

where a, b are real numbers. Then we have

$$R^*_{0,b}(x,y) = 2, \quad \lim_{x\to 0} R^*_{a,b}(x,y) = 1 + \operatorname{erfc}(y^a)^{-b} > 1, \quad \text{if} \quad a > 0,$$

$$\lim_{x\to\infty} R^*_{a,b}(x,y) = 1 + \operatorname{erfc}(y^a)^b > 1, \quad \text{if} \quad a < 0.$$

This implies that there are no real parameters a, b such that $x \mapsto \operatorname{erfc}(x^a)^b$ is superadditive.

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