

MORSE THEORY AND GEODESICS IN THE SPACE OF KÄHLER METRICS

TAMÁS DARVAS

(Communicated by Lei Ni)

To my parents

ABSTRACT. Given a compact Kähler manifold (X, ω_0) let \mathcal{H}_0 be the set of Kähler forms cohomologous to ω_0 . As observed by Mabuchi, this space has the structure of an infinite dimensional Riemannian manifold if one identifies it with a totally geodesic subspace of \mathcal{H} , the set of Kähler potentials of ω_0 . Following Donaldson's research program, existence and regularity of geodesics in this space is of fundamental interest. In this paper, supposing enough regularity of a geodesic $u : [0, 1] \rightarrow \mathcal{H}$, connecting $u_0 \in \mathcal{H}$ with $u_1 \in \mathcal{H}$, we establish a Morse theoretic result relating the critical points of $u_1 - u_0$ to the critical points of $\dot{u}_0 = du/dt|_{t=0}$. As an application of this result, we prove that on all Kähler manifolds, connecting Kähler potentials with smooth geodesics is not possible in general. In particular, in the case $X \neq \mathbb{C}P^1$, we will also prove that the set of pairs of potentials that cannot be connected with smooth geodesics has nonempty interior. This is an improvement upon the findings of Lempert and Vivas and of the author and Lempert.

1. INTRODUCTION

Let (X^n, ω_0) be a connected compact Kähler manifold. The space of smooth Kähler potentials of ω_0 is the set

$$\mathcal{H} := \{v \in C^\infty(X) : \omega_0 + i\partial\bar{\partial}v > 0\}.$$

It is clear that \mathcal{H} is a Fréchet manifold, as it is an open subset of $C^\infty(X)$. From this, it also follows that for each $v \in \mathcal{H}$ we can identify $T_v\mathcal{H}$ with $C^\infty(X)$. Following Mabuchi, we define a Riemannian metric on $T_v\mathcal{H}$:

$$\langle \xi, \eta \rangle := \int_X \xi \eta (\omega_0 + i\partial\bar{\partial}v)^n, \quad \xi, \eta \in T_v\mathcal{H}.$$

If $\phi : [0, 1] \rightarrow \mathcal{H}$ is a smooth curve, one can think of ϕ as an element of $C^\infty([0, 1] \times X)$. Also, suppose that ψ is a tangent vector field along ϕ , which again can be treated as an element of $C^\infty([0, 1] \times X)$. The Levi-Civita connection of the metric introduced above is as follows:

$$\nabla_{\dot{\phi}}\psi := \dot{\psi} - \frac{1}{2}\langle \nabla\psi, \nabla\dot{\phi} \rangle.$$

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Hence a smooth curve $g : [0, 1] \rightarrow \mathcal{H}$ is a geodesic if we have

$$0 = \nabla_{\dot{g}}\dot{g} = \ddot{g} - \frac{1}{2}\langle \nabla\dot{g}, \nabla\dot{g} \rangle.$$

Let us now relate the above discussion to the space of Kähler metrics over X . Hodge theory implies that the correspondence $u \rightarrow \omega_0 + i\partial\bar{\partial}u$ gives an onto map from \mathcal{H} to \mathcal{H}_0 . As found by Mabuchi, \mathcal{H}_0 can be identified with a totally geodesic subspace of \mathcal{H} . This correspondence will automatically put a Riemannian structure on \mathcal{H}_0 . Moreover, one can prove that with this structure, \mathcal{H} is isometric to the Riemannian product $\mathcal{H}_0 \times \mathbb{R}$. Hence to study the question of existence of geodesics in \mathcal{H}_0 one can study the same question in \mathcal{H} .

As discovered by Semmes [S], the geodesic equation can be rewritten as a complex Monge-Ampère equation. Let $S = \{s \in \mathbb{C} : 0 < \text{Im}s < 1\}$, and let ω be the pullback of the Kähler form ω_0 to $\bar{S} \times X$ by the projection $\pi_X : S \times X \rightarrow X$. If $\phi : [0, 1] \rightarrow \mathcal{H}$ is a smooth curve, let $u \in C^\infty(\bar{S} \times X)$ be defined by $u(s, x) := \phi(\text{Im}s, x)$. Semmes observed that ϕ is a geodesic if and only if the following equation is satisfied for u :

$$(\omega + i\partial\bar{\partial}u)^{n+1} = 0.$$

Since the form $\omega + \partial\bar{\partial}u$ is nondegenerate on each X -fiber, the above equation is the same as writing $(\omega + i\partial\bar{\partial}u)^{n+1} = 0$ but $(\omega + i\partial\bar{\partial}u)^n \neq 0$. Summing up, we can say that if there exists a geodesic connecting $u_0 \in \mathcal{H}$ with $u_1 \in \mathcal{H}$, then there exists a smooth function $u \in C^\infty(\bar{S} \times X)$ that satisfies the following conditions:

$$\begin{aligned} &(\omega + i\partial\bar{\partial}u)^{n+1} = 0, \\ &(\omega + i\partial\bar{\partial}u)^n \neq 0, \\ (1.1) \quad &u(r + it, x) = u(it, x) \quad \forall x \in X, t \in [0, 1], r \in \mathbb{R}, \\ &u(0, x) = u_0(x), \quad u(1, x) = u_1(x) \quad \forall x \in X. \end{aligned}$$

Solutions to the above problem have been intensely studied. Uniqueness has already been found by Donaldson [D1]. Chen [C] proved that one always has a ω -plurisubharmonic solution to (1.1) in the sense of Bedford and Taylor [BT] for which the current $\partial\bar{\partial}u$ is represented by a bounded form. Of course in this setting the condition $(\omega + i\partial\bar{\partial}u)^n \neq 0$ does not make any sense, and hence it is omitted from the problem.

In [LV] the authors proved that on certain Kähler manifolds admitting special symmetries there exist boundary data for which (1.1) has no C^3 solutions. This provided the first example showing that, using smooth geodesics, connecting points in \mathcal{H} is not possible in general. In [DL], under similar circumstances, it was shown that the regularity result of Chen we just mentioned is sharp.

In this paper we only investigate C^3 solutions of problem (1.1). More specifically, with the above conventions in mind, we prove the following result:

Theorem 1.1. *Suppose u is a C^3 solution of (1.1). Then all the critical points of $\dot{u}_0 := du(it, \cdot)/dt|_{t=0} \in C^2(X)$ are among the critical points of $u_1 - u_0$. Moreover, if $u_1 - u_0$ is a Morse function, then \dot{u}_0 is a Morse function as well.*

The above result establishes a connection between the boundary data and initial tangent vector of C^3 solutions of the complex Monge-Ampère equation. This

theorem is proved by a careful analysis of the first order variation of the Monge-Ampère foliation of u at each critical point of the initial tangent vector \dot{u}_0 . A similar method has been applied with success in [LV] as well. As an application we prove the following theorem:

Theorem 1.2. *For each connected compact Kähler manifold (X, ω_0) one can find a pair of potentials $u_0, u_1 \in \mathcal{H}$ that cannot be connected by a smooth geodesic. If $X \neq \mathbb{C}P^1$, then the set of such pairs of potentials has nonempty interior in $C^\infty(X) \times C^\infty(X)$.*

This result is an improvement upon the similar nonexistence result of [LV]. The examples constructed there only work on manifolds admitting holomorphic isometries $A : X \rightarrow X$ having an isolated fixed point and satisfying the identity $A^2 = Id$.

In Kähler geometry the special role of symmetries, especially that of one-parameter group actions, has a long history. The first result of this kind was that of Matsushima [M], who proved that if the Lie algebra of holomorphic vector fields on a complex manifold is not reductive, then the manifold admits no Kähler-Einstein metrics. By contrast, in the absence of holomorphic vector fields, the search for canonical Kähler metrics becomes more promising. Calabi [Ca] observed that in this case, all extremal metrics will have constant scalar curvature. The Futaki invariant, that serves as an obstruction to finding constant scalar curvature metrics, vanishes in this situation as well [F].

With the above picture and the examples of [LV] in mind, one might perhaps think that constructing pairs of Kähler potentials that cannot be connected by a smooth geodesic requires the presence of holomorphic symmetries. The interest of Theorem 1.2 is showing that this is not the case.

It would be interesting to see if one can prove the last statement of the above result for $\mathbb{C}P^1$ as well. We summarize the proof of Theorem 1.2 here. In the case $X = \mathbb{C}P^1$ every Kähler metric ω_0 is in the Kähler class of some multiple of the Fubini-Study metric ω_{FS} . However, each metric $c\omega_{FS}$ satisfies the symmetry conditions of Theorem 1.1 in [LV], and so this theorem provides the $u_0, u_1 \in \mathcal{H}$ that we seek. Hence, to prove the result, we only have to consider the case $X \neq \mathbb{C}P^1$. The rest of the argument will be a consequence of the following more precise statement:

Theorem 1.3. *Suppose $X \neq \mathbb{C}P^1$. For each connected compact Kähler manifold (X, ω_0) one can find boundary data $u_0, u_1 \in \mathcal{H}$ for which problem (1.1) has no C^3 solutions. In particular, the set of such pairs has nonempty interior in \mathcal{H} .*

2. THE MONGE-AMPÈRE FOLIATION

In this section we summarize known facts about the Monge-Ampère foliation, first discussed in [BK]. For a more complete picture, the reader is referred to [D2] or Section 2.1 in [RZ].

The reason C^3 solutions of the complex Monge-Ampère equation are special is that this is the minimal amount of regularity that is needed for the existence of the Monge-Ampère foliation. This is defined as follows. Let $u \in C^\infty(\bar{S} \times X)$ be a solution of (1.1). This means that for each $(s, x) = (r + it, x) \in \bar{S} \times X$, $\text{Ker}(\omega + i\partial\bar{\partial}u)|_{(s,x)}$ is a one dimensional complex subspace of $T_{(s,x)}(\bar{S} \times X)$, called the Monge-Ampère distribution. Since the form $\omega + i\partial\bar{\partial}u$ is closed, this complex

subbundle of $T(\bar{S} \times X)$ is integrable, and hence gives rise to a foliation in which each leaf is a one dimensional complex submanifold of $\bar{S} \times X$.

We will denote by \mathcal{F}_x the leaf of this foliation passing through the point $(0, x)$. Also, $u_t(\cdot)$ and $\dot{u}_t(\cdot)$ will stand for $u(it, \cdot)$ and $du(it, \cdot)/dt$ for $t \in [0, 1]$. We should note that our formulation of (1.1) allows solutions u for which some of the forms $\omega_0 + i\partial_X\bar{\partial}_X u_t$ are degenerate on X . Because of this, in general, the leaves \mathcal{F}_x cannot be assumed to be diffeomorphic to the strip \bar{S} . This will only be a minor inconvenience, as we will see in a moment.

As observed by Semmes [S], the vector field $\partial/\partial r + J\nabla_{g_0}\dot{u}_0/2$ generates $\text{Ker}(\omega + i\partial\bar{\partial}u)|_{(r,x)}$, where g_0 is the Riemannian metric corresponding to the Kähler metric $\omega_0 + i\partial_X\bar{\partial}_X u_0$ and $x \in X, r \in \mathbb{R}$. Since $J\nabla_{g_0}\dot{u}_0/2$ is independent of r , this has a very interesting consequence. If $r \rightarrow f_x(r), r \in \mathbb{R}$, is the trajectory of the vector field $J\nabla_{g_0}\dot{u}_0/2$ with initial data $f_x(0) = x$, then the leaf \mathcal{F}_x contains the image of the arc $r \rightarrow \Gamma_x(r) = (r, f_x(r))$. Hence, with a slight abuse of precision, one should think of each leaf \mathcal{F}_x as the unique analytic extension of the curve Γ_x . This fact will be used in the next section.

3. PROOF OF THEOREM 1.1

By replacing ω_0 with $\omega_0 + i\partial\bar{\partial}u_0$ and $u(s, x)$ with $u(s, x) - u_0(x)$, we can assume that $u_0 = 0$. We prove Theorem 1.1 in two steps.

Lemma 3.1. *If x_0 is a critical point of \dot{u}_0 , then it is a critical point of \dot{u}_t for all $t \in [0, 1]$ and the leaf of the Monge-Ampère foliation through x_0 is $\bar{S} \times \{x_0\}$. In particular, $t \rightarrow u_t(x_0)$ is linear.*

Proof. Denote $\omega_t = \omega_0 + i\partial_X\bar{\partial}_X u_t$. We saw in the preceding section that the vector $\partial/\partial r + J\nabla_{g_0}\dot{u}_0/2$ generates the Monge-Ampère distribution at any point $(r, x) \in \mathbb{R} \times X$.

Since x_0 is a critical point of \dot{u}_0 , the vector field $J\nabla_{g_0}\dot{u}_0/2$ vanishes at x_0 , hence its trajectory $r \rightarrow f_{x_0}(r)$ is x_0 . Therefore, following the reasoning of the preceding section, we obtain that $\mathcal{F}_{x_0} = \bar{S} \times \{x_0\}$. Hence, $\partial/\partial r$ generates $\text{Ker}(\omega + i\partial\bar{\partial}u)|_{(s,x_0)}$ for any $s \in \bar{S}$. In any holomorphic chart of X around x_0 , this implies that $u_{s\bar{z}_j}(s, x_0) = u_{s\bar{z}_j}(s, x_0) = 0$ and $u_{s\bar{s}}(s, x_0) = 0$ for all $s \in \bar{S}$ and for all $j = 1, \dots, n$. As u is translation invariant, $\dot{u}_t = 2i\partial u/\partial s, \ddot{u}_t = 4\partial^2 u/\partial s\bar{\partial}s$ and we obtain the desired result. □

From this lemma, we obtain that if x_0 is a critical point of \dot{u}_0 , then it is a critical point of u_1 as well, since in any holomorphic chart around x_0 we have $u_{1z_j}(x_0) = \int_0^1 \dot{u}_{tz_j}(x_0)dt = 0$ and $u_{1\bar{z}_j}(x_0) = \int_0^1 \dot{u}_{t\bar{z}_j}(x_0)dt = 0$ for all $j = 1, \dots, n$. All that is left is to prove the nondegeneracy in Theorem 1.1. We will do this in the following lemma.

Lemma 3.2. *If x_0 is a degenerate critical point of \dot{u}_0 , then it is a degenerate critical point of u_1 as well. Moreover, for the Hessians of \dot{u}_0 and u_1 at x_0 , we have the inclusion $\text{Ker } H\dot{u}_0(x_0) \subseteq \text{Ker } Hu_1(x_0)$.*

Proof. Most of the computations to follow are taken over from [LV, Section 2]. We extend the Hessian forms of both u_1 and \dot{u}_0 to complex bilinear forms. Then we

have the following representation of these forms at any point $x \in X$ in any complex coordinate:

$$Hu_1(x) = \begin{bmatrix} u_{1zz}(x) & u_{1z\bar{z}}(x) \\ u_{1\bar{z}z}(x) & u_{1\bar{z}\bar{z}}(x) \end{bmatrix}, H\dot{u}_0(x) = \begin{bmatrix} \dot{u}_{0zz}(x) & \dot{u}_{0z\bar{z}}(x) \\ \dot{u}_{0\bar{z}z}(x) & \dot{u}_{0\bar{z}\bar{z}}(x) \end{bmatrix},$$

where $u_{1zz}(x) = \{u_{1z_j z_k}(x)\}_{j,k=1,n}$, and similarly for all the other terms.

We fix local coordinates $z_j, j = 1, \dots, n$, on a neighborhood $V \subset X$ of x_0 such that $\omega_0|_{x_0} = idz_j \wedge d\bar{z}_j|_{x_0}$, the local coordinates map V on a convex set in \mathbb{C}^n and there exist a potential w_0 on V such that $\omega_0 = i\partial\bar{\partial}w_0$ on V . We can clearly suppose that $w_{0z_j z_k}|_{x_0} = 0$. We denote by w the function $\pi_X^* w_0$. We identify V with its image in \mathbb{C}^n and x_0 with $0 \in \mathbb{C}^n$. Then $\bar{S} \times V$ is identified with a subset of $\bar{S} \times \mathbb{C}^n$.

From Lemma 3.1 it follows that $f_{x_0} = f_0 \equiv 0$ and the leaf of the Monge Ampère foliation passing through $(0, x_0)$ is $\bar{S} \times \{x_0\}$. As discussed in Section 2 of [LV], this implies that for any $d > 0$ and for small enough x , it makes sense to holomorphically extend the trajectories $r \rightarrow f_x(r)$ from the segment $[-d, d] \subset \mathbb{R}$ to the compact set $\{z \in \bar{S} \mid -d \leq \text{Re } z \leq d\}$. Because of our regularity assumptions, we have that the leaves of the Monge-Ampère foliation change differentiably. Hence for any $a \in \mathbb{C}^n$ one can define the functions $\varphi_j := df_{(ta)_j}/dt|_{t=0}$, continuous on \bar{S} with values in \mathbb{C} , that are holomorphic on S for $j = 1, \dots, n$. It follows from basic properties of the Monge-Ampère foliation that whenever the maps below are defined, they are holomorphic:

$$s \rightarrow w_{z_j}(s, f_{ta}(s)) + u_{z_j}(s, f_{ta}(s)), \quad j = 1, \dots, n.$$

Differentiating the above maps with respect to t we obtain that the functions

$$s \rightarrow \sum_{k=1}^n \left(\{w_{z_j z_k}(s, 0) + u_{z_j z_k}(s, 0)\} \varphi_k(s) + \{w_{z_j \bar{z}_k}(s, 0) + u_{z_j \bar{z}_k}(s, 0)\} \overline{\varphi_k(s)} \right),$$

$j = 1, \dots, n, s \in S$, are holomorphic as well. From our choice of w_0 it follows that $w_{zz}(s, 0) \equiv 0$ and $w_{z\bar{z}}(s, 0) \equiv I$. After differentiating the above maps by $\partial/\partial\bar{s}$ as well, we find that

$$\sum_{k=1}^n \left(u_{z_j z_k \bar{s}}(s, 0) \varphi_k(s) + u_{z_j \bar{z}_k \bar{s}}(s, 0) \overline{\varphi_k(s)} + \{\delta_{jk} + u_{z_j \bar{z}_k}(s, 0)\} \overline{\varphi'_k(s)} \right) = 0,$$

$j = 1, \dots, n$. Summing up, with the notation introduced at the beginning, we obtain that the function $\psi : S \rightarrow \mathbb{C}^n$ defined by

$$\psi(s) := u_{zz}(s, 0) \varphi(s) + \{I + u_{z\bar{z}}(s, 0)\} \overline{\varphi(s)}$$

is holomorphic and

$$(3.1) \quad \{I + u_{z\bar{z}}(s, 0)\} \varphi'(s) = -u_{\bar{z}zs}(s, 0) \varphi(s) - u_{z\bar{z}s}(s, 0) \overline{\varphi(s)}.$$

With the notation $P = I + u_{z\bar{z}}(i, 0) = I + u_{1z\bar{z}}(0), Q = u_{zz}(i, 0) = u_{1zz}(0), A = -u_{\bar{z}zs}(0, 0)$ and $B = -u_{z\bar{z}s}(0, 0)$, we find that

$$(3.2) \quad \psi(z) = \begin{cases} \overline{\varphi(s)} & \text{if } \text{Im } s = 0, \\ P\overline{\varphi(s)} + Q\varphi(s) & \text{if } \text{Im } s = 1. \end{cases}$$

Restricting (3.1) to real s we have

$$\varphi'(s) = A\varphi(s) + B\overline{\varphi(s)}, \quad s \in \mathbb{R}.$$

We also observe that $\varphi(0) = a$. Hence, to find the values of φ , we need to solve the following initial value problem on the real line:

$$\begin{bmatrix} \varphi'(s) \\ \overline{\varphi}'(s) \end{bmatrix} = \begin{bmatrix} A & B \\ \overline{B} & \overline{A} \end{bmatrix} \begin{bmatrix} \varphi(s) \\ \overline{\varphi}(s) \end{bmatrix}, \quad \begin{bmatrix} \varphi(0) \\ \overline{\varphi}(0) \end{bmatrix} = \begin{bmatrix} a \\ \overline{a} \end{bmatrix}.$$

Since $A = iu_{0\overline{z}z}(0)/2$, $B = i\overline{u}_{0z\overline{z}}(0)/2$ and $H\dot{u}(0)$ is assumed to be degenerate, we obtain that the matrix of the above linear initial value problem is degenerate too. This means that there exists a nonzero $\alpha \in \mathbb{C}^n$ such that

$$\begin{bmatrix} A\alpha + B\overline{\alpha} \\ \overline{B}\alpha + \overline{A}\overline{\alpha} \end{bmatrix} = 0.$$

Thus, for $a = \alpha$, $\varphi(s) \equiv \alpha$ is a solution of the initial value problem. Using (3.2) and analytic continuation we obtain $\overline{\alpha} = P\overline{\alpha} + Q\alpha$. Hence $u_{1z\overline{z}}(0)\overline{\alpha} + u_{1\overline{z}z}(0)\alpha = 0$. So the vector $[\alpha, \overline{\alpha}]^T$ is in the kernel of the Hessian of u_1 at 0, and therefore x_0 is a degenerate critical point of u_1 . The inclusion $\text{Ker } H\dot{u}_0(x_0) \subseteq \text{Ker } Hu_1(x_0)$ clearly follows from our arguments. □

4. PROOF OF THEOREM 1.3

We start out with a Morse function $w : X \rightarrow \mathbb{R}$ with a minimal amount of critical points and distinct critical values. The condition $X \neq \mathbb{C}P^1$ implies that $b_i(X) > 0$ for some $0 < i < 2n$. From the Morse inequalities it follows that v has at least one saddle point x_0 . Let p be the index of w at x_0 .

We choose a holomorphic coordinate chart $(z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n)$ around the point x_0 such that $\omega_0|_{x_0} = \sum_{j=1}^n idz_j \wedge d\overline{z}_j$. It is easy to see that one can choose a smooth diffeomorphism φ of X such that $\varphi(x_0) = x_0$, and in our fixed chart, the Hessian of the Morse function $v = w \circ \varphi$ at x_0 is diagonal with the following entries on the diagonal:

$$\frac{\partial^2 v}{\partial x_j^2}(x_0) = \frac{\partial^2 v}{\partial y_k^2}(x_0) = 1 \text{ if } 1 \leq j \leq n, p+1 \leq k \leq n \text{ and } \frac{\partial^2 v}{\partial y_l^2}(x_0) = -1 \text{ if } 1 \leq l \leq p.$$

By multiplying v with a small constant, we can assume that $v \in \mathcal{H}$. We clearly have $v_{z_1\overline{z}_1}(x_0) = 0$ and $|v_{z_1z_1}(x_0)| \neq 0$. We can choose a number $\gamma > 0$ such that

$$|v_{z_1z_1}(x_0)| > 2\gamma + v_{z_1\overline{z}_1}(x_0) \text{ and } \gamma + v_{z_1\overline{z}_1}(x_0) > 0.$$

Using this and the fact that $v_{z\overline{z}}(x_0)$ is diagonal, by a lemma that will be provided below, one can find a smooth function ρ supported inside our coordinate patch that is identically equal to 1 in a neighborhood of x_0 such that

$$\omega_0 + i\partial\overline{\partial}(\gamma - 1)\rho(z)|z_1|^2 > 0 \text{ and } (\omega_0 + i\partial\overline{\partial}v) + i\partial\overline{\partial}(\gamma - 1)\rho(z)|z_1|^2 > 0.$$

Assume that problem (1.1) has a C^3 solution u with boundary data $u_0 = (\gamma - 1)\rho(z)|z_1|^2$ and $u_1 = v + (\gamma - 1)\rho(z)|z_1|^2$. By Theorem 1.1, \dot{u}_0 is a Morse function, and all its critical points are critical points of v . As v has the minimal amount of critical points, the critical sets of \dot{u}_0 and v coincide; in particular, x_0 is critical for \dot{u}_0 .

Since $\text{rank}(\omega + i\partial\overline{\partial}u) \equiv n$ and $\omega + i\partial\overline{\partial}u$ is semi-positive on the boundary of $\overline{S} \times X$, we obtain that $\omega + i\partial\overline{\partial}u \geq 0$; in particular, u is ω -plurisubharmonic. It follows from this and Lemma 3.1 that all the conditions of Lemma 3.2 in [DL] are satisfied for the Kähler metric $\omega' = \omega + i\partial\overline{\partial}(\gamma - 1)\rho(z)|z_1|^2$ and the ω' -plurisubharmonic

function $u' = u - (\gamma - 1)\rho(z)|z_1|^2$. However, the conclusion of this result is violated since for $v' = u'_1$ and $\xi' = \partial/\partial z_1$ we have

$$(4.1) \quad \left| \sum_{j,k} v'_{z_j z_k}(x_0) \xi'_j \xi'_k \right| > \sum_{j,k} (2\omega'_{jk} \xi'_j \bar{\xi}'_k + v'_{z_j \bar{z}_k}(x_0) \xi'_j \bar{\xi}'_k).$$

To prove the last statement of the theorem, we refer to elements of Morse theory. If one perturbs slightly u_0 and u_1 in the C^∞ topology, it is a standard result that $u_1 - u_0$ is still a Morse function with distinct critical values. Also, by [GG, Theorem 2.2, p. 79], we know that for small perturbations, $u_1 - u_0$ will still have a minimal amount of critical points. Lastly, if the perturbation is small enough, then $u_0, u_1 \in \mathcal{H}$, and at one of the critical points (4.1) still holds. This completes the main argument of the proof.

As promised, we provide the following lemma initially found by Donaldson [D2, Lemma 8]; see also [LV, Lemma 3.3]. Since its original formulation is slightly inadequate for our purposes, we reformulate it here and give a proof for completeness.

Lemma 4.1. *Suppose ω^0 and ω^1 are two Kähler metrics on a complex manifold X . Let q^0 and q^1 be two real smooth functions on X with critical points at x_0 and $q^0(x_0) = q^1(x_0) = 0$. If in some holomorphic coordinate patch U around x_0 we have $q^l_{z\bar{z}}(x_0)(\xi, \bar{\xi}) + \omega^l(x_0)(\xi, \bar{\xi}) > 0$ for all nonzero $\xi \in T_{x_0}X$, $l = 0, 1$, then one can find a smooth function ρ supported in U that is identically 1 in a neighborhood of x_0 such that $\omega^l + i\partial\bar{\partial}(\rho q^l) > 0$, $l = 0, 1$.*

Proof. By shrinking U and possibly rescaling the coordinates, there exists $1 > \alpha > 0$ such that $\omega^l + i\partial\bar{\partial}q^l > \alpha\omega^l$ for $l = 0, 1$ on U and the coordinates map x_0 to 0 and U to a neighborhood of $\{|z| \leq 2\} \subset \mathbb{C}^n$.

Now take a smooth function $\beta : [-\infty, +\infty) \rightarrow [0, 1]$ such that $\beta(t) = 0$ if $t > 0$ and $\beta(t) = 1$ if $t < -1$. Then $\rho(t) = \beta(\varepsilon \log t)$, $\varepsilon > 0$, is supported in $[0, 1]$, $\rho(t) = 1$ if $t < e^{-1/\varepsilon}$ and $t|\rho'(t)|, t^2|\rho''(t)| < C\varepsilon$ for some constant C . From all of this it is easy to see that on U we have

$$\omega^l + i\partial\bar{\partial}(\rho(|z|^2)q^l) > \omega^l + i\rho(|z|^2)\partial\bar{\partial}q^l - D\varepsilon\omega^l > (\alpha - D\varepsilon)\omega^l$$

for some constant D independent of ε , $l = 0, 1$. Since $\rho(|z|^2)q^l$ is identically 0 outside U , for small enough ε , $\rho(|z|^2)$ has the required properties. \square

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907
E-mail address: tdarvas@math.purdue.edu