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# PRONORMAL SUBGROUPS AND ZEROS OF CHARACTERS

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ABSTRACT. We give a characterization of when a pronormal subgroup of a solvable group is normal by using character theory.

## 1. INTRODUCTION

There seems to be a certain connexion between non-zero values of characters and normality in finite groups. For instance, if $G$ is a finite group, then a *non-vanishing* $x \in G$ (that is, an element on which no irreducible character takes the value zero) lies in $\mathbf{F}(G)$ whenever the order of $x$ is coprime to 6 ([DNPST]). Dolfi, Pacifici, Sanus and Spiga have proved in [DPSS] that if all the elements of a Sylow $p$-subgroup $P \in \mathrm{Syl}_p(G)$ are non-vanishing, then $P \triangleleft G$. (Other related results can be found in [MT] or [NT].)

In fact, we have very recently proved in [MN] that if $P \in \mathrm{Syl}_p(G)$, then $P \triangleleft G$ if and only if $\chi(x) \neq 0$ for all $x \in P$ and all irreducible constituents $\chi$ of the permutation character $(1_P)^G$. This is the starting point of this note: If $H \triangleleft G$, then it is clear that $\chi(h) \neq 0$ for all $h \in H$ and all irreducible constituents $\chi$ of $(1_H)^G$, but the converse is not in general true. Are there other types of subgroups $H$, aside from Sylow subgroups, for which the converse holds?

Recall that a subgroup $H$ of $G$ is **pronormal** in $G$ if for every $g \in G$ we have that $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$. For instance, Sylow subgroups and Sylow normalizers are pronormal in every finite group, while Carter or Hall subgroups, among many others classes of subgroups, are pronormal in solvable groups.

If $H$ is a subgroup of $G$, let us denote by $\mathrm{Irr}((1_H)^G)$ the set of the irreducible constituents of the permutation character $(1_H)^G$.

**Theorem A.** *Suppose that $G$ is solvable and that $H$ is pronormal in $G$. Then $H \triangleleft G$ if and only if $\chi(h) \neq 0$ for all $\chi \in \mathrm{Irr}((1_H)^G)$ and all $h \in H$.*

Unfortunately, Theorem A is not true outside solvable groups: If $G = A_7$ and $H$ is any subgroup of $G$ isomorphic to $C_2 \times C_2$, then it can be checked that $H$ is pronormal in $G$ and that $\chi(h) \neq 0$ for all $\chi \in \mathrm{Irr}((1_H)^G)$ and all $h \in H$. As a matter of fact, we remark that any theorem implying the main result of [MN] or [DPSS] is not likely to have an elementary proof such as the one we give in the next section: all these results need the Classification of Finite Simple Groups.

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## 2. Proofs

*Proof of Theorem A.* If $H \lhd G$ and $\chi \in \mathrm{Irr}(G)$ is an irreducible constituent of $(1_H)^G$, then $\chi_H = \chi(1)1_H$, and therefore $\chi(h) \neq 0$ for all $h \in H$.

To prove the converse, we argue by induction on $|G|$. So we assume that $\chi(h) \neq 0$ for all $\chi \in \mathrm{Irr}((1_H)^G)$ and all $h \in H$, and we prove that $H \lhd G$. We may assume that $H > 1$. Let $N \lhd G$ and $g \in G$. Then $\langle H, H^g \rangle \subseteq \langle NH, NH^g \rangle$, and we see that $NH/N$ is pronormal in $G/N$. If $\chi \in \mathrm{Irr}(G)$ lies over $1_{HN}$, then $\chi$ lies over $1_H$, and therefore $\chi(h) \neq 0$ for all $h \in H$. If also $N \subseteq \ker(\chi)$, then $\chi(nh) = \chi(h) \neq 0$ for all $n \in N, h \in H$, and therefore $NH/N \lhd G/N$ if $N > 1$ by the inductive hypothesis. Thus, if $1 < N \lhd G$, then we have that $NH \lhd G$. In particular, we may assume that $\mathrm{core}_G(H) = 1$.

Now let $N$ be a minimal normal subgroup of $G$. Let $g \in G$. Since $NH \lhd G$, we have that $\langle H, H^g \rangle \subseteq NH$, and therefore $H^g = H^n$ for some $n \in N$. Thus $G = N\mathbf{N}_G(H)$. Since $N \cap H \lhd \mathbf{N}_G(H)$ and $N \cap H \lhd N$ because $N$ is abelian, then we deduce that $N \cap H \lhd G$. Thus $N \cap H = 1$ because $H$ has a trivial core. Also $\mathbf{C}_H(N) \lhd \mathbf{N}_G(H)$, and since $[\mathbf{C}_H(N), N] = 1$, we deduce that $\mathbf{C}_H(N) = 1$, so that $H$ acts faithfully on $N$.

Now, let $K/N$ be a chief factor of $G$ where $K \subseteq NH$, so that $K/N$ is an abelian $p$-group for some prime $p$. Since the abelian $p$-group $K \cap H$ acts faithfully on the necessarily $p'$-group $\mathrm{Irr}(N)$, we can conclude then that there is a regular orbit; that is, there exists $\lambda \in \mathrm{Irr}(N)$ such that no element of $K \cap H$ stabilizes $\lambda$. (See for instance Corollary 3.4 of [IsB].) Thus $\theta = \lambda^K \in \mathrm{Irr}(K)$ by the Clifford correspondence. Since

$$((1_H)^{NH})_N = (1_1)^N = \rho_N$$

is the regular character of $N$, it follows that some irreducible constituent $\mu \in \mathrm{Irr}((1_H)^{NH})$ lies over $\lambda$. Hence, if $\chi \in \mathrm{Irr}(G)$ lies over $\mu$, then $\chi$ lies over $1_H$ and over $\lambda$. In particular, $\chi$ lies over $\theta = \lambda^K$. However, $\theta^g = (\lambda^g)^K$ vanishes on every $1 \neq x \in K \cap H$ for all $g \in G$. Hence so does $\chi$ by Clifford's theorem. This is a contradiction that concludes the proof of the theorem. $\square$

To end this note, we change our question: are there groups for which the non-vanishing and the normality conditions are equivalent? Even for nilpotent groups this is not a complete triviality.

(2.1) **Theorem.** *Suppose that $G$ is nilpotent and let $H \leq G$. Then $H \lhd G$ if and only if $\chi(h) \neq 0$ for all $\chi \in \mathrm{Irr}((1_H)^G)$ and all $h \in H$.*

*Proof.* We assume that $\chi(h) \neq 0$ for all $\chi \in \mathrm{Irr}((1_H)^G)$ and all $h \in H$, and we prove that $H \lhd G$. We assume that $H > 1$. Let $Z$ be a minimal normal subgroup of $G$ so that $|Z| = p$ a prime and $Z$ is central. By induction, we have that $ZH \lhd G$. Thus we may assume that the core in $G$ of $H$ is trivial. Hence $Z \cap H = 1$. Now, let $K/Z$ be a chief factor of $G$, where $K$ is contained in $ZH$. Thus $|K/Z| = q$, a prime. If $p \neq q$, then $K \cap H$ is a normal Sylow $q$-subgroup of $K$, so it is normal in $G$. But the core of $H$ is trivial, and this is not possible. So we may assume that $|K| = p^2$. Let $\langle h \rangle = H \cap K = Q$, which we know is not normal in $G$. Let $1 \neq \lambda \in \mathrm{Irr}(Z)$ and let $\theta = \lambda \times 1_Q$. Suppose that $g \in G$ does not centralize $h$. Since $(Zh)^g = Zh$, then we have that $h^g = zh$ for some $z \neq 1$. Now $h^{g^i} = z^i h$ for $i \geq 0$, and we see that the conjugacy class of $h$ in $G$ has size $p$. Hence $|G : \mathbf{C}_G(h)| = p$ and

$$G = \mathbf{C}_G(h) \cup \mathbf{C}_G(h)g \cup \ldots \cup \mathbf{C}_G(h)g^{p-1}$$

is a disjoint union. Also, since $Q = \ker(\theta)$, it easily follows that the stabilizer of $\theta$ in $G$ is $\mathbf{N}_G(Q) = \mathbf{C}_G(Q) = \mathbf{C}_G(h)$. Now, let $\chi \in \mathrm{Irr}(G)$ be over $\theta$. By Clifford's theorem, we have that

$$\chi_K = e\sum_{j=0}^{p-1}\theta^{g^j}$$

for some integer $e$. Then

$$\chi(h) = e\sum_{j=0}^{p-1}\theta^{g^j}(h) = e(1+\lambda(z)+\cdots+\lambda(z^{p-1})) = 0$$

because $[\lambda, 1_Z] = 0$. This is a contradiction which proves the theorem. $\square$

If $G = C_3 \times S_3$, then every subgroup contained in the Fitting subgroup $F$ of $G$ is non-vanishing, while there is a subgroup inside $F$ which is not normal in $G$. So Theorem (2.1) does not hold even for supersolvable groups.

Also, we would like to point out that we do not know of any solvable group $G$ having a subgroup $H$ such that $\chi(h) \neq 0$ for all $\chi \in \mathrm{Irr}((1_H)^G)$ and all $h \in H$, and where $H$ is not subnormal in $G$. This would imply Theorem A and certainly be related to the main open problem in [INW].

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## References

[DNPST] Silvio Dolfi, Gabriel Navarro, Emanuele Pacifici, Lucia Sanus, and Pham Huu Tiep, *Non-vanishing elements of finite groups*, J. Algebra **323** (2010), no. 2, 540–545, DOI 10.1016/j.jalgebra.2009.08.014. MR2564856 (2010k:20015)

[DPSS] Silvio Dolfi, Emanuele Pacifici, Lucia Sanus, and Pablo Spiga, *On the orders of zeros of irreducible characters*, J. Algebra **321** (2009), no. 1, 345–352, DOI 10.1016/j.jalgebra.2008.10.004. MR2469367 (2009k:20013)

[Is] I. Martin Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423]. MR2270898

[IsB] I. Martin Isaacs, *Finite group theory*, Graduate Studies in Mathematics, vol. 92, American Mathematical Society, Providence, RI, 2008. MR2426855 (2009e:20029)

[INW] I. M. Isaacs, Gabriel Navarro, and Thomas R. Wolf, *Finite group elements where no irreducible character vanishes*, J. Algebra **222** (1999), no. 2, 413–423, DOI 10.1006/jabr.1999.8007. MR1733678 (2000k:20008)

[MN] Gunter Malle and Gabriel Navarro, *Characterizing normal Sylow p-subgroups by character degrees*, J. Algebra **370** (2012), 402–406, DOI 10.1016/j.jalgebra.2012.07.050. MR2966846

[MT] S. Marinelli and P. H. Tiep, *Zeros of real irreducible characters of finite groups*, J. Algebra and Number Theory, to appear.

[NT] Gabriel Navarro and Pham Huu Tiep, *Degrees and p-rational characters*, Bull. Lond. Math. Soc. **44** (2012), no. 6, 1246–1250, DOI 10.1112/blms/bds054. MR3007657

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