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WEIGHTED NORM INEQUALITIES FOR k-PLANE TRANSFORMS

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Abstract. We obtain sharp inequalities for the k-plane transform, the "j-plane to k-plane" transform, and the corresponding dual transforms, acting on L^p spaces with a radial power weight. The operator norms are explicitly evaluated. Some generalizations and open problems are discussed.

1. Introduction

Mapping properties of Radon-like transforms were studied by many authors, e.g., [3]-[7], [9, 10, 13, 14, 17, 22, 24, 27, 33, 36, 37], to mention a few. Most of the publications deal with L^p-L^q estimates or mixed norm inequalities when the problem is to minimize a gap between necessary and sufficient conditions and find the best possible bounds. We also mention a series of works devoted to weighted norm estimates for Radon-like transforms of radial functions; see, e.g., [8, 19, 20].

In the present article we show that for the k-plane transform in $\mathbb{R}^n$ [16, 28], the more general "j-plane to k-plane" transform [36, p. 701], [6, 12, 29], and the corresponding dual transforms, sharp estimates can be obtained if the action of these operators is considered in the L^p-L^q setting with $p = q$ and radial power weights. In this case the proofs are elementary and self-contained. Our approach was inspired by a series of publications on operators with homogeneous kernel dating back, probably, to Schur [31]; see, e.g., [15, 30, 35]. It is not surprising that the same ideas are applicable to some operators of integral geometry, because the common point is invariance under rotations and dilations.

The weighted L^2 estimate for the hyperplane Radon transform in $\mathbb{R}^n$, $n \geq 3$, was obtained by Quinto [26, p. 410] via spherical harmonics and Hardy's inequality. This estimate was generalized by Kumar and Ray [18] to all $n \geq 2$ and all p by making use of the similar spherical harmonics techniques combined with interpolation. Our approach is completely different, covers these results, and provides the best possible constants, which coincide with explicitly evaluated operator norms.

Let us proceed to the details. We denote by $\Pi_{n,k}$ the manifold of all nonoriented k-planes τ in $\mathbb{R}^n$; $G_{n,k}$ is the Grassmann manifold of k-dimensional linear subspaces ξ of $\mathbb{R}^n$; $1 \leq k \leq n-1$. Each k-plane τ is parameterized by the pair (ξ, u), where $\xi \in G_{n,k}$ and $u \in \xi^\perp$ (the orthogonal complement of ξ in $\mathbb{R}^n$). Thus, $\Pi_{n,k}$ is a bundle over $G_{n,k}$ with an $(n-k)$-dimensional fiber. The manifold $\Pi_{n,k}$ is endowed with the product measure $d\tau = d\xi du$, where $d\xi$ is the $O(n)$-invariant probability

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measure on $G_{n,k}$ and du denotes the volume element on $\xi^\perp$. We define

$$L^p(\Pi_{n,k};w) = \{f : ||f||_{p,w} \equiv ||wf||_p < \infty\}, \qquad 1 \le p \le \infty,$$

where $||\cdot||_p$ is the norm in $L^p(\Pi_{n,k})$. If $w(\tau) = |\tau|^\nu$, where $|\tau|$ denotes the Euclidean distance from the plane $\tau \in \Pi_{n,k}$ to the origin, we also write $L^p(\Pi_{n,k};w) = L^p_\nu(\Pi_{n,k})$ and $||f||_{p,w} = ||f||_{p,\nu}$.

The k-plane transform of a sufficiently good function f on $\mathbb{R}^n$ is a function $R_k f$ on $\Pi_{n,k}$ defined by

$$(R_k f)(\tau) \equiv (R_k f)(\xi, u) = \int\limits_{x\in\tau} f(x)\, d_\tau x \equiv \int\limits_{\xi} f(y+u)\, dy, \tag{1.1}$$

where dy is the volume element in ξ. The more general "j-plane to k-plane" transform takes a function f on $\Pi_{n,j}$ to a function $R_{j,k}f$ on $\Pi_{n,k}$, $0 \le j < k < n$, by the formula

$$(R_{j,k} f)(\tau) \equiv (R_{j,k} f)(\xi, u) = \int\limits_{\zeta\subset\tau} f(\zeta)\, d_\tau \zeta \equiv \int\limits_{\eta\subset\xi} d_\xi \eta \int\limits_{\eta^\perp\cap\xi} f(\eta, y+u)\, dy. \tag{1.2}$$

Here $d_\xi\eta$ denotes the probability measure on the manifold of all j-dimensional linear subspaces η of ξ.

Theorem 1.1. *Let $1 \le p \le \infty$, $1/p + 1/p' = 1$, $\nu = \mu - k/p'$, $\mu > k - n/p$. Then R_k is a linear bounded operator from $L^p_\mu(\mathbb{R}^n)$ to $L^p_\nu(\Pi_{n,k})$ with the norm*

$$||R_k|| = \pi^{k/2} \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}}\right)^{1/p} \frac{\Gamma((\mu + n/p - k)/2)}{\Gamma((\mu + n/p)/2)}, \tag{1.3}$$

where σ_{n-1} (σ_{n-k-1}) is the area of the unit sphere in $\mathbb{R}^n$ (resp., in $\mathbb{R}^{n-k}$).

Theorem 1.2. *Let $1 \le p \le \infty$, $1/p+1/p' = 1$, $\nu = \mu-(k-j)/p'$, $\mu > k-n/p-j/p'$. Then $R_{j,k}$ is a linear bounded operator from $L^p_\mu(\Pi_{n,j})$ to $L^p_\nu(\Pi_{n,k})$ with the norm*

$$||R_{j,k}|| = \pi^{(k-j)/2} \left(\frac{\sigma_{n-k-1}}{\sigma_{n-j-1}}\right)^{1/p} \frac{\Gamma((\mu + n/p - k + j/p')/2)}{\Gamma((\mu + n/p - j/p)/2)}. \tag{1.4}$$

Apart from these theorems, we obtain similar statements for the corresponding dual transforms; see Sections 3.2, 3.3, 4.2, 4.3. The assumptions for μ and ν are best possible. Generalizations and open problems are discussed in Section 5.

2. Preliminaries

Notation. In the following $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere S^{n-1} in $\mathbb{R}^n$; $d\sigma(\theta)$ stands for the surface element of S^{n-1}; $S^{n-1}_+ = \{x = (x_1, \cdots, x_n) \in S^{n-1} : x_n > 0\}$ is the "upper" hemisphere of S^{n-1}; $e_1, \ldots, e_n$ are coordinate unit vectors; $G = O(n)$ is the group of orthogonal transformations of $\mathbb{R}^n$ endowed with the invariant probability measure. This group acts on $G_{n,k}$ transitively. For $g, \gamma \in G$, we denote

$$f_g(x) = f(gx), \qquad \varphi_\gamma(\xi, u) = \varphi(\gamma\xi, \gamma u).$$

We will need the following simple statements.

Lemma 2.1. *The norm of a function $\varphi(\tau) \equiv \varphi(\xi, u) \in L^p_\nu(\Pi_{n,k})$ can be computed by the formula*

$$||\varphi||_{p,\nu} = \left(\sigma_{n-k-1} \int_0^\infty r^{n-k-1+\nu p}\, dr \int_G |\varphi_\gamma(\mathbb{R}^k, re_{k+1})|^p\, d\gamma\right)^{1/p} \tag{2.1}$$

if $1 \le p < \infty$, *and*

$$||\varphi||_{\infty,\nu} = \operatorname*{ess\,sup}_{\gamma\in G,\ r>0} \{r^\nu |\varphi_\gamma(\mathbb{R}^k, re_{k+1})|\} \tag{2.2}$$

if $p = \infty$.

Proof. The proof is straightforward and based on the definition

$$||\varphi||_{p,\nu} = \begin{cases} \left(\displaystyle\int_{G_{n,k}} d\xi \int_{\xi^\perp} \big||u|^\nu \varphi(\xi, u)\big|^p\, du\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{\xi,u} \{|u|^\nu |\varphi(\xi, u)|\} & \text{if } p = \infty. \end{cases}$$

□

Lemma 2.2. *Let $f \in L^1(\mathbb{R}^{n-1})$, $\theta = (\theta_1, \cdots, \theta_n) \in S^{n-1}$. Then*

$$\int_{\mathbb{R}^{n-1}} f(x)\, dx = \int_{S^{n-1}_+} f\Big(\frac{\theta'}{\theta_n}\Big) \frac{d\sigma(\theta)}{\theta_n^n}, \qquad \theta' = (\theta_1, \cdots, \theta_{n-1}). \tag{2.3}$$

Proof.

$$\int_{\mathbb{R}^{n-1}} f(x)\, dx = \int_0^\infty s^{n-2}\, ds \int_{S^{n-2}} f(s\omega)\, d\sigma(\omega) \qquad (s = \tan\varphi)$$

$$= \int_0^{\pi/2} \frac{\sin^{n-2}\varphi}{\cos^n\varphi}\, d\varphi \int_{S^{n-2}} f\Big(\frac{\omega \sin\varphi}{\cos\varphi}\Big)\, d\sigma(\omega) = \int_{S^{n-1}_+} f\Big(\frac{\theta'}{\theta_n}\Big) \frac{d\sigma(\theta)}{\theta_n^n}.$$

□

3. Mapping properties of the k-plane transform

3.1. **Preparations.** The following explicit equalities, reflecting action of R_k on weighted L^1 spaces, were obtained in [28, Theorem 2.3].

Lemma 3.1. *Let*

$$\lambda_\mu = \frac{\Gamma((n-k+\mu)/2)\,\Gamma(n/2)}{\Gamma((n+\mu)/2)\,\Gamma((n-k)/2)}, \qquad \mu > k - n. \tag{3.1}$$

Then

$$\int_{\Pi_{n,k}} (R_k f)(\tau)|\tau|^\mu d\tau = \lambda_\mu \int_{\mathbb{R}^n} f(x)|x|^\mu dx, \tag{3.2}$$

$$\int_{\Pi_{n,k}} (R_k f)(\tau) \frac{|\tau|^\mu\, d\tau}{(1+|\tau|^2)^{(n+\mu)/2}} = \lambda_\mu \int_{\mathbb{R}^n} f(x) \frac{|x|^\mu\, dx}{(1+|x|^2)^{(n-k+\mu)/2}}, \tag{3.3}$$

provided that either side of the corresponding equality exists in the Lebesgue sense.

Lemma 3.2. *Let $f \in L^p_\mu(\mathbb{R}^n)$, $1 \le p \le \infty$, and suppose that*

$$\mu \begin{cases} > k - n/p & \text{if } 1 < p \le \infty, \\ \ge k - n & \text{if } p = 1. \end{cases} \tag{3.4}$$

Then $(R_k f)(\tau)$ is finite for almost all $\tau \in \Pi_{n,k}$. If (3.4) fails, then there is a function $f_0 \in L^p_\mu(\mathbb{R}^n)$ such that $(R_k f)(\tau) \equiv \infty$. For instance,

$$f_0(x) = \frac{|x|^{-\mu}\,(2+|x|)^{-n/p}}{\log^{1/p+\delta}(2+|x|)}, \tag{3.5}$$

where $0 < \delta < 1/p'$ if $1 < p \le \infty$, and any $\delta > 0$ if $p = 1$.

Proof. If $\mu > k - n/p$, the first statement follows from (3.3) by Hölder's inequality. If $\mu = k - n$ (for $p = 1$), we set $\tilde\mu = k - n + \varepsilon$, $\varepsilon > 0$. Since

$$\int\limits_{\mathbb{R}^n} \frac{|f(x)|\,|x|^{\tilde\mu}\,dx}{(1+|x|^2)^{(n-k+\tilde\mu)/2}} = \int\limits_{\mathbb{R}^n} \frac{|f(x)|}{|x|^{n-k}}\,\frac{|x|^{\varepsilon}\,dx}{(1+|x|^2)^{\varepsilon/2}} \le \int\limits_{\mathbb{R}^n} \frac{|f(x)|}{|x|^{n-k}}\,dx < \infty,$$

then (3.3) holds with the new parameter $\tilde\mu$ and, therefore, $(R_k f)(\tau)$ is finite for almost all $\tau \in \Pi_{n,k}$. The second statement of the lemma follows from the Abel type representation [28, p. 98]:

$$(R_k f)(\tau) = \sigma_{k-1} \int\limits_{|\tau|}^{\infty} f_0(r)(r^2 - |\tau|^2)^{k/2-1} r\,dr, \qquad f(x) \equiv f_0(|x|). \tag{3.6}$$

□

The scaling argument (cf. [34, p. 118]) yields the following.

Lemma 3.3. *Let $1 \le p \le \infty$, $1/p + 1/p' = 1$. If $||R_k f||_{p,\nu} \le c\,||f||_{p,\mu}$ for a nonnegative function $f \not\equiv 0$ and a constant $c > 0$ independent of f, then $\nu = \mu - k/p'$.*

Lemmas 3.2 and 3.3 contain necessary conditions for the operator R_k to be bounded from $L^p_\mu(\mathbb{R}^n)$ to $L^p_\nu(\Pi_{n,k})$. It will be shown that these conditions, except $\mu = k - n$ when $p = 1$, are also sufficient.

The next statement is obvious.

Lemma 3.4. *Let $1 \le p \le \infty$, $1/p + 1/p' = 1$. If the operator $R_k : L^p_\mu(\mathbb{R}^n) \to L^p_\nu(\Pi_{n,k})$ is bounded, then*

$$\Big| \int\limits_{\Pi_{n,k}} (R_k f)(\tau)\,g(\tau)\,d\tau \Big| \le ||R_k||_{L^p_\mu(\mathbb{R}^n) \to L^p_\nu(\Pi_{n,k})}\,||f||_{p,\mu}\,||g||_{p',-\nu} \tag{3.7}$$

for any $f \in L^p_\mu(\mathbb{R}^n)$ and $g \in L^{p'}_{-\nu}(\Pi_{n,k})$.

The following "hemispherical" representation of the k-plane transform plays the crucial role in our consideration; cf. [23, p. 188] for $k = n - 1$, where it was used for different purposes. We denote

$$\mathbb{R}^k = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_k, \qquad \mathbb{R}^{k+1} = \mathbb{R}^k \oplus \mathbb{R}e_{k+1},$$

$$S^k = S^{n-1} \cap \mathbb{R}^{k+1}, \quad S^k_+ = \{\theta = (\theta_1, \cdots, \theta_{k+1}) \in S^k : \theta_{k+1} > 0\}.$$

Lemma 3.5. *Suppose that $u \neq 0$ and let $f_g(x) = f(gx)$, where $g \in G$ satisfies $g\,\mathbb{R}^k = \xi$ and $g\,e_{k+1} = u/|u|$. Then*

$$(R_k f)(\xi, u) = (R_k f_g)(\mathbb{R}^k, re_{k+1}) = r^k \int\limits_{S^k_+} f_g\Big(\frac{r\theta}{\theta_{k+1}}\Big)\,\frac{d\sigma(\theta)}{\theta_{k+1}^{k+1}}, \quad r = |u|. \tag{3.8}$$

Proof. Changing variables and using (2.3) (with $n = k+1$), we obtain

$$\begin{aligned}(R_k f)(\xi, u) &= \int\limits_{\mathbb{R}^k} f_g(z + re_{k+1})\,dz = (R_k f_g)(\mathbb{R}^k, re_{k+1}) \\ &= r^k \int\limits_{\mathbb{R}^k} f_g(r(z + e_{k+1}))\,dz = r^k \int\limits_{S^k_+} f_g\Big(\frac{r\theta}{\theta_{k+1}}\Big)\,\frac{d\sigma(\theta)}{\theta_{k+1}^{k+1}}.\end{aligned}$$

□

3.2. **Proof of Theorem 1.1.** Denote by c_k the constant on the right-hand side of (1.3).

Step 1. Let us show that $||R_k|| \le c_k$. By Lemmas 2.1 and 3.5, owing to rotation invariance and Minkowski's inequality for integrals, for $1 \le p < \infty$ we have

$$\begin{aligned}||R_k f||_{p,\nu} &= \left(\sigma_{n-k-1} \int\limits_0^\infty r^{n-k-1+\nu p}\,dr \int\limits_G |(R_k f)_\gamma(\mathbb{R}^k, re_{k+1})|^p\,d\gamma\right)^{1/p} \\ &= \left(\sigma_{n-k-1} \int\limits_0^\infty r^{n-k-1+\nu p}\,dr \int\limits_G \left| r^k \int\limits_{S^k_+} f_\gamma\left(\frac{r\theta}{\theta_{k+1}}\right)\frac{d\sigma(\theta)}{\theta_{k+1}^{k+1}}\right|^p d\gamma\right)^{1/p} \\ &\le \sigma_{n-k-1}^{1/p} \int\limits_{S^k_+} \frac{A(\theta)}{\theta_{k+1}^{k+1}}\,d\sigma(\theta).\end{aligned}$$

Here, for $\nu = \mu - k/p'$,

$$A(\theta) = \left(\int\limits_0^\infty r^{n-1+\mu p}\,dr \int\limits_G \left| f_\gamma\left(\frac{r\theta}{\theta_{k+1}}\right)\right|^p d\gamma\right)^{1/p} = \frac{\theta_{k+1}^{\mu+n/p}}{\sigma_{n-1}^{1/p}}\,||f||_{p,\mu}.$$

If $p = \infty$, we similarly get

$$||R_k f||_{\infty,\nu} \le \int\limits_{S^k_+} \frac{A(\theta)}{\theta_{k+1}^{k+1}}\,d\sigma(\theta)$$

with

$$A(\theta) = \operatorname*{ess\,sup}_{r,\gamma} r^\mu \left| f_\gamma\left(\frac{r\theta}{\theta_{k+1}}\right)\right| \le \theta_{k+1}^\mu\,||f||_{\infty,\mu}.$$

This gives $||R_k f||_{p,\nu} \le c_k\,||f||_{p,\mu}$, where

$$c_k = \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}}\right)^{1/p} \int\limits_{S^k_+} \frac{d\sigma(\theta)}{\theta_{k+1}^{k+1-\mu-n/p}}, \qquad 1 \le p \le \infty. \tag{3.9}$$

The last integral equals

$$\sigma_{k-1}\int_0^1 (1-t^2)^{k/2-1}\, t^{\mu+n/p-k-1} dt = \frac{\pi^{k/2}\,\Gamma((\mu+n/p-k)/2)}{\Gamma((\mu+n/p)/2)},$$

as desired. Thus, $||R_k|| \le c_k$.

Step 2. Let us show that $||R_k|| \ge c_k$. To this end, we transform (3.7) by choosing f and g in a proper way. Suppose that both f and g are nonnegative, $f(x) \equiv f_0(|x|)$, $g(\tau) \equiv g_0(|\tau|)$, and denote by I the integral on the left-hand side of (3.7). By Lemma 3.5,

$$\begin{aligned} I &= \sigma_{n-k-1}\int_0^\infty g_0(r)\, r^{n-1}\, dr \int_{S_+^k} f_0\left(\frac{r}{\theta_{k+1}}\right)\frac{d\sigma(\theta)}{\theta_{k+1}^{k+1}} \\ &= \sigma_{n-k-1}\int_{S_+^k} \frac{d\sigma(\theta)}{\theta_{k+1}^{k+1}} \int_0^\infty g_0(r)\, f_0\left(\frac{r}{\theta_{k+1}}\right) r^{n-1}\, dr. \end{aligned} \tag{3.10}$$

Let $1 < p < \infty$. Then $||f||_{p,\mu}$ and $||g||_{p',-\nu}$ in (3.7) have the form

$$\begin{aligned} ||f||_{p,\mu} &= \left(\sigma_{n-1}\int_0^\infty r^{n-1+\mu p}\, f_0^p(r)\, dr\right)^{1/p}, \\ ||g||_{p',-\nu} &= \left(\sigma_{n-k-1}\int_0^\infty r^{n-k-1-\nu p'}\, g_0^{p'}(r)\, dr\right)^{1/p'}. \end{aligned}$$

Choose g_0 so that $g_0(r) = r^{\mu p}\, f_0^{p-1}(r)$. Then

$$||g||_{p',-\nu} = \tilde{c}\, ||f||_{p,\mu}^{p-1}, \qquad \tilde{c} = \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}}\right)^{1/p'}.$$

This equality, together with (3.10) and (3.7), yields

$$\sigma_{n-k-1}\int_{S_+^k} \frac{d\sigma(\theta)}{\theta_{k+1}^{k+1}} \int_0^\infty f_0^{p-1}(r)\, f_0\left(\frac{r}{\theta_{k+1}}\right) r^{n-1+\mu p}\, dr \le \tilde{c}\, ||R_k||\, ||f||_{p,\mu}^p.$$

Now we assume $f_0(r) = 0$ if $r < 1$ and $f_0(r) = r^{-\mu-n/p-\varepsilon}$, $\varepsilon > 0$, if $r > 1$. Then $||f||_{p,\mu}^p = \sigma_{n-1}/\varepsilon p$ and we have

$$\sigma_{n-k-1}\int_{S_+^k} \frac{d\sigma(\theta)}{\theta_{k+1}^{k+1-\mu-n/p-\varepsilon}} \le \tilde{c}\, \sigma_{n-1}\, ||R_k||.$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$||R_k|| \ge \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}}\right)^{1/p} \int_{S_+^k} \frac{d\sigma(\theta)}{\theta_{k+1}^{k+1-\mu-n/p}},$$

as desired; cf. (3.9).

If $p=1$, then $\nu=\mu$. We choose $g_0(r)=r^{\mu}$ and proceed as above. If $p=\infty$, we choose $f_0(r)=r^{-\mu}$. Then $||f||_{\infty,\mu}=1$ and, by (3.7),

$$\int\limits_{S^k_+} \frac{d\sigma(\theta)}{\theta_{k+1}^{k+1-\mu}} \int\limits_0^\infty g_0(r)\, r^{n-1-\mu}\, dr \le ||R_k|| \int\limits_0^\infty g_0(r)\, r^{n-1-\nu}\, dr.$$

Suppose $g_0(r)=0$ if $r<1$ and $g_0(r)=r^{-\delta}$ if $r>1$, where δ is big enough. Then

$$\frac{\nu-n+\delta}{\mu-n+\delta} \int\limits_{S^k_+} \frac{d\sigma(\theta)}{\theta_{k+1}^{k+1-\mu}} \le ||R_k||.$$

Letting $\delta\to\infty$, we obtain the result. □

Some comments are in order. In the case $p=1$, the constant (1.3) coincides with (3.1). In the case $p=2$ it differs from that in [26, p. 410]. The case $p=1$, $\mu=\nu=k-n$, is skipped in Theorem 1.1, though it is included in Lemma 3.2. The reason is that the boundedness of R_k from $L^1_{k-n}(\mathbb{R}^n)$ to $L^1_{k-n}(\Pi_{n,k})$ fails to hold. Take, for instance, $f(x)=f_0(|x|)$ with $f_0(r)\equiv 0$ if $r<10$ and $f_0(r)=r^{k-\delta}$, $\delta>0$, otherwise. Clearly, $f\in L^1_{k-n}(\mathbb{R}^n)$; however, by (3.6),

$$\int_{\Pi_{n,k}} (R_kf)(\tau)|\tau|^{k-n}\, d\tau = c\int\limits_0^\infty \frac{dt}{t} \int\limits_t^\infty f_0(r)(r^2-|\tau|^2)^{k/2-1} r\, dr = \infty.$$

3.3. **The dual k-plane transform.** The dual k-plane transform of a function φ on $\Pi_{n,k}$ is defined by the formula

$$(R_k^*\varphi)(x) = \int\limits_G \varphi(\gamma\mathbb{R}^k + x)\, d\gamma, \quad x\in\mathbb{R}^n, \tag{3.11}$$

and satisfies the duality relation [16, 28]

$$\int\limits_{\mathbb{R}^n} (R_k^*\varphi)(x)\, f(x)\, dx = \int\limits_{\Pi_{n,k}} (R_kf)(\tau)\, \varphi(\tau)\, d\tau. \tag{3.12}$$

The following lemma gives precise information about the L^1 case.

Lemma 3.6 ([28, Theorem 2.3])**.** *Let*

$$\tilde\lambda_\mu = \frac{\pi^{k/2}\,\Gamma(-\mu/2)}{\Gamma((k-\mu)/2)}, \qquad \mu<0, \qquad \varkappa\in\{0,1\}. \tag{3.13}$$

Then

$$\int\limits_{\mathbb{R}^n} (R_k^*\varphi)(x)\, (\varkappa+|x|^2)^{(\mu-k)/2}\, dx = \tilde\lambda_\mu \int\limits_{\Pi_{n,k}} \varphi(\tau)\, (\varkappa+|\tau|^2)^{\mu/2}\, d\tau, \tag{3.14}$$

provided that either side of the equality exists in the Lebesgue sense.

In the L^p case, the scaling argument yields:

Lemma 3.7. *Let $1\le p\le\infty$, $1/p+1/p'=1$. If $||R_k^*\varphi||_{p,\nu}\le c\,||\varphi||_{p,\mu}$ for a nonnegative function $\varphi\not\equiv 0$ and a constant $c>0$ independent of φ, then $\nu=\mu-k/p$.*

The following statement is dual to Theorem 1.1.

Theorem 3.8. *Let $1 \le p \le \infty$, $1/p + 1/p' = 1$, $\nu = \mu - k/p$, $\mu < (n-k)/p'$. Then the dual k-plane transform R_k^* is a linear bounded operator from $L^p_\mu(\Pi_{n,k})$ to $L^p_\nu(\mathbb{R}^n)$ with the norm*

$$||R_k^*|| = \pi^{k/2} \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}}\right)^{1/p'} \frac{\Gamma(((n-k)/p' - \mu)/2)}{\Gamma((n/p' + k/p - \mu)/2)}. \tag{3.15}$$

Proof. By the duality (3.12), operators $R_k : L^p_\mu(\mathbb{R}^n) \to L^p_\nu(\Pi_{n,k})$ and $R_k^* : L^{p'}_{-\nu}(\Pi_{n,k}) \to L^{p'}_{-\mu}(\mathbb{R}^n)$ are bounded simultaneously and their norms coincide. Hence, replacing p by p' and making obvious changes in the statement of Theorem 1.1, we obtain the result. □

4. The j-plane to k-plane transform

This transform is defined by (1.2). The reader is referred to [29] for additional information.

4.1. **Preparations.**

Lemma 4.1 (cf. Theorem 2.5 in [29])**.** *Let $0 \le j < k < n$,*

$$\lambda_{j,\mu} = \frac{\Gamma((n-k+\mu)/2)\,\Gamma((n-j)/2)}{\Gamma((n+\mu-j)/2)\,\Gamma((n-k)/2)}, \qquad \mu > k-n. \tag{4.1}$$

Then

$$\int\limits_{\Pi_{n,k}} (R_{j,k}f)(\tau)\,|\tau|^\mu\,d\tau = \lambda_{j,\mu} \int\limits_{\Pi_{n,j}} f(\zeta)\,|\zeta|^\mu\,d\zeta, \tag{4.2}$$

$$\int\limits_{\Pi_{n,k}} \frac{(R_{j,k}f)(\tau)\,|\tau|^\mu}{(1+|\tau|^2)^{(n+\mu-j)/2}}\,d\tau = \lambda_{j,\mu} \int\limits_{\Pi_{n,j}} \frac{f(\zeta)\,|\zeta|^\mu}{(1+|\zeta|^2)^{(n+\mu-k)/2}}\,d\zeta, \tag{4.3}$$

provided that either side of the corresponding equality exists in the Lebesgue sense.

Lemma 4.2. *Let $f \in L^p_\mu(\Pi_{n,j})$, $1 \le p \le \infty$, and suppose that*

$$\mu \begin{cases} > k - n/p - j/p' & \text{if } 1 < p \le \infty, \\ \ge k - n & \text{if } p = 1. \end{cases} \tag{4.4}$$

Then $(R_{j,k}f)(\tau)$ is finite for almost all $\tau \in \Pi_{n,k}$. If (4.4) fails, then there is a function $f_0 \in L^p_\mu(\Pi_{n,j})$ such that $(R_{j,k}f)(\tau) \equiv \infty$. For instance,

$$f_0(\zeta) = \frac{|\zeta|^{-\mu}\,(2+|\zeta|)^{(j-n)/p}}{\log^{1/p+\delta}(2+|\zeta|)}, \tag{4.5}$$

where $0 < \delta < 1/p'$ if $1 < p \le \infty$, and any $\delta > 0$ if $p = 1$.

Proof. The proof mimics that of Lemma 3.2, using Hölder's inequality in (4.3) and the known formula for radial functions [29, p. 5051]:

$$(R_{j,k}f)(\tau) = \sigma_{k-j-1} \int\limits_{|\tau|}^{\infty} f_1(r)(r^2 - |\tau|^2)^{(k-j)/2-1} r\,dr, \quad f(\zeta) \equiv f_1(|\zeta|).$$

□

The scaling argument (in the fibers) yields the following statement.

Lemma 4.3. *Let $1 \le p \le \infty$, $1/p + 1/p' = 1$. If $||R_{j,k}f||_{p,\nu} \le c\,||f||_{p,\mu}$ for a nonnegative function $f \not\equiv 0$ and a constant $c > 0$ independent of f, then $\nu = \mu - (k-j)/p'$.*

To obtain an analogue of (3.8) for $R_{j,k}f$, we denote

$$\mathbb{R}^j = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_j, \quad \mathbb{R}^{k-j} = \mathbb{R}e_{j+1} \oplus \cdots \oplus \mathbb{R}e_k, \quad \mathbb{R}^{k-j+1} = \mathbb{R}^{k-j} \oplus \mathbb{R}e_{k+1},$$
$$S^{k-j} = S^{n-1} \cap \mathbb{R}^{k-j+1}, \quad S^{k-j}_+ = \{\theta = (\theta_{j+1}, \cdots, \theta_{k+1}) \in S^{k-j} : \theta_{k+1} > 0\}.$$

Lemma 4.4. *Let $(R_{j,k}f)(\xi, u)$ be the transformation (1.2) with $u \ne 0$. If $g \in G$ satisfies $g\,\mathbb{R}^k = \xi$ and $g\,e_{k+1} = u/|u|$, then*

$$(R_{j,k}f)(\xi, u) = r^{k-j} \int\limits_{O(k)} d\gamma \int\limits_{S^{k-j}_+} f_{g\gamma}\Big(\mathbb{R}^j, \frac{r\theta}{\theta_{k+1}}\Big) \frac{d\sigma(\theta)}{\theta_{k+1}^{k-j+1}}, \quad r = |u|. \tag{4.6}$$

Proof. Changing variables, we write $(R_{j,k}f)(\xi, u)$ as

$$\begin{aligned}(R_{j,k}f)(g\,\mathbb{R}^k, rg\,e_{k+1}) &= \int\limits_{G_{k,j}} d\tilde\eta \int\limits_{\tilde\eta^\perp \cap \mathbb{R}^k} f_g(\tilde\eta, re_{k+1} + z)\,dz \\ &= r^{k-j} \int\limits_{O(k)} d\gamma \int\limits_{\mathbb{R}^{k-j}} f_{g\gamma}(\mathbb{R}^j, r(e_{k+1} + \tilde z))\,d\tilde z.\end{aligned}$$

It remains to transform the inner integral using (2.3). □

4.2. **Proof of Theorem 1.2.** Denote by $c_{j,k}$ the constant on the right-hand side of (1.4).

Step 1. Let us show that $||R_{j,k}|| \le c_{j,k}$. By Lemmas 2.1 and 4.4, for $1 \le p < \infty$ we have

$$\begin{aligned}||R_{j,k}f||_{p,\nu} &= \Bigg(\sigma_{n-k-1} \int\limits_0^\infty r^{n-k-1+\nu p}\,dr \\ &\times \int\limits_G \Bigg| r^{k-j} \int\limits_{O(k)} d\gamma \int\limits_{S^{k-j}_+} f_{g\gamma}\left(\mathbb{R}^j, \frac{r\theta}{\theta_{k+1}}\right) \frac{d\sigma(\theta)}{\theta_{k+1}^{k-j+1}} \Bigg|^p dg\Bigg)^{1/p} \\ &\le \sigma_{n-k-1}^{1/p} \int\limits_{O(k)} d\gamma \int\limits_{S^{k-j}_+} \frac{A_\gamma(\theta)}{\theta_{k+1}^{k-j+1}}\,d\sigma(\theta), \quad \nu = \mu - (k-j)/p',\end{aligned}$$

where

$$A_\gamma^p(\theta) = \int\limits_0^\infty r^{n-j-1+\mu p}\,dr \int\limits_G \Big| f_{g\gamma}\left(\mathbb{R}^j, \frac{r\theta}{\theta_{k+1}}\right)\Big|^p dg = \frac{\theta_{k+1}^{n-j+\mu p}}{\sigma_{n-j-1}}\,||f||_{p,\mu}^p.$$

Hence, $||R_{j,k}f||_{p,\nu} \le c_{j,k}\,||f||_{p,\mu}$,

$$c_{j,k} = \left(\frac{\sigma_{n-k-1}}{\sigma_{n-j-1}}\right)^{1/p} \int\limits_{S^{k-j}_+} \frac{d\sigma(\theta)}{\theta_{k+1}^{k+1-\mu-n/p-j/p'}}. \tag{4.7}$$

This result also covers the case $p = \infty$, when the calculation is straightforward. The last integral gives the constant in (1.4).

Step 2. To prove that $||R_{j,k}|| \geq c_{j,k}$, we proceed as in the proof of Theorem 1.1 and use the relevant analogue of (3.7). Let f and g be nonnegative radial functions, $f(\zeta) \equiv f_0(|\zeta|)$, $g(\tau) \equiv g_0(|\tau|)$. Then

$$\int\limits_{\Pi_{n,k}} (R_{j,k}f)(\tau)\, g(\tau)\, d\tau = \sigma_{n-k-1} \int\limits_{S_+^{k-j}} \frac{d\sigma(\theta)}{\theta_{k+1}^{k-j+1}} \int\limits_0^\infty g_0(r)\, f_0\left(\frac{r}{\theta_{k+1}}\right) r^{n-j-1}\, dr,$$

If $1 < p < \infty$, then

$$\begin{aligned} ||f||_{p,\mu} &= \left(\sigma_{n-j-1} \int\limits_0^\infty r^{n-j-1+\mu p}\, f_0^p(r)\, dr\right)^{1/p}, \\ ||g||_{p',-\nu} &= \left(\sigma_{n-k-1} \int\limits_0^\infty r^{n-k-1-\nu p'}\, g_0^{p'}(r)\, dr\right)^{1/p'}. \end{aligned}$$

Choose g_0 so that $g_0(r) = r^\alpha\, f_0^{p/p'}$, $\alpha = (k-j)/p' + \nu + \mu(p-1)$, and set $f_0(r) = 0$ if $r < 1$ and $f_0(r) = r^{-\mu-(n-j)/p-\varepsilon}$, $\varepsilon > 0$, if $r > 1$. Then, as in the proof of Theorem 1.1, we get

$$\left(\frac{\sigma_{n-k-1}}{\sigma_{n-j-1}}\right)^{1/p} \int\limits_{S_+^{k-j}} \frac{d\sigma(\theta)}{\theta_{k+1}^{k+1-\mu-n/p-j/p'-\varepsilon}} \leq ||R_{j,k}||.$$

It remains to pass to the limit as $\varepsilon \to 0$; cf. (4.7). The cases $p = 1$ and $p = \infty$ are treated as in the proof of Theorem 1.1. □

4.3. **The dual j-plane to k-plane transform.** For $0 \leq j < k < n$, the dual j-plane to k-plane transform takes functions $\varphi(\tau) \equiv \varphi(\xi, u)$ on $\Pi_{n,k}$ to functions $(R_{j,k}^*\varphi)(\zeta) \equiv (R_{j,k}^*\varphi)(\eta, v)$ on $\Pi_{n,j}$ by the formula

$$(R_{j,k}^*\varphi)(\zeta) = \int\limits_{\tau \supset \zeta} \varphi(\tau)\, d_\zeta\tau = \int\limits_{O(n-j)} \varphi(g_\eta \rho \mathbb{R}^k + v)\, d\rho. \tag{4.8}$$

Here $g_\eta \in G$ is an orthogonal transformation that sends $\mathbb{R}^j = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_j$ to η and $O(n-j)$ is the orthogonal group of the coordinate plane $\mathbb{R}^{n-j} = \mathbb{R}e_{j+1} \oplus \cdots \oplus \mathbb{R}e_n$. This transform averages $\varphi(\tau)$ over all k-planes τ containing the j-plane ζ. The case $j = 0$ gives the dual k-plane transform (3.11). The duality relation has the form

$$\int\limits_{\Pi_{n,k}} (R_{j,k}f)(\tau)\, \varphi(\tau)\, d\tau = \int\limits_{\Pi_{n,j}} f(\zeta)\, (R_{j,k}^*\varphi)(\zeta)\, d\zeta. \tag{4.9}$$

The following exact equalities generalize those in Lemma 3.6.

Lemma 4.5 ([29, Theorem 2.5])**.** *Let*

$$\tilde{\lambda}_{j,\mu} = \frac{\pi^{(k-j)/2}\, \Gamma(-\mu/2)}{\Gamma((k-j-\mu)/2)}, \qquad \mu < 0, \qquad \varkappa \in \{0, 1\}.$$

Then

$$\int\limits_{\Pi_{n,j}} (R^*_{j,k}\varphi)(x)\,(\varkappa+|\zeta|^2)^{(\mu-k+j)/2}\,d\zeta = \tilde{\lambda}_{j,\mu}\int\limits_{\Pi_{n,k}} \varphi(\tau)\,(\varkappa+|\tau|^2)^{\mu/2}\,d\tau,$$

provided that either side of the equality exists in the Lebesgue sense.

The dual of Theorem 1.2 is the following.

Theorem 4.6. *Let $1 \le p \le \infty$, $1/p+1/p'=1$, $\nu=\mu-(k-j)/p$, $\mu<(n-k)/p'$. Then $R^*_{j,k}$ is a linear bounded operator from $L^p_\mu(\Pi_{n,k})$ to $L^p_\nu(\mathbb{R}^n)$ with the norm*

$$||R^*_{j,k}|| = \pi^{(k-j)/2}\left(\frac{\sigma_{n-k-1}}{\sigma_{n-j-1}}\right)^{1/p}\frac{\Gamma((\mu+n/p-k+j/p')/2)}{\Gamma((\mu+n/p-j/p)/2)}. \tag{4.10}$$

The proof of this statement is similar to the proof of Theorem 3.8.

5. Some generalizations and open problems

1. Owing to projective invariance of the Radon transforms [11, p. xi], all theorems of the present article can be transferred to totally geodesic Radon transforms on the hyperbolic and elliptic spaces. Almost all formulas which are needed for this transition are available in the literature. Specifically, for the hyperplane Radon transforms and the k-plane transform see [1, 2, 21]. The correspondence between the affine j-plane to k-plane transform and the similar transform for planes through the origin was established in [29]. We believe that a similar transition holds to the hyperbolic space.

2. It might be challenging to establish a connection between weighted L^p inequalities of the present article and known L^p-L^q or mixed norm estimates.

3. The k-plane transform (1.1) is a member of the analytic family of the Semyanistyi type integrals

$$(R^\alpha_k f)(\tau)=\frac{1}{\gamma_{n-k}(\alpha)}\int\limits_{\mathbb{R}^n} f(x)|x-\tau|^{\alpha+k-n}dx, \quad \gamma_{n-k,\alpha}=\frac{2^\alpha\pi^{(n-k)/2}\Gamma(\alpha/2)}{\Gamma((n-k-\alpha)/2)},$$

where $Re\,\alpha>0$ and $|x-\tau|$ denotes the Euclidean distance between $x\in\mathbb{R}^n$ and $\tau\in\Pi_{n,k}$; see [32] for $k=n-1$ and [28, p. 104] for any $1\le k<n$. We conjecture that the method of our article extends to these operators and their duals. Since $\lim\limits_{\alpha\to 0} R^\alpha_k f = R_k f$ in a suitable sense, we expect that $\lim\limits_{\alpha\to 0}||R^\alpha_k|| = ||R_k||$ on the relevant weighted spaces.

4. To the best of our knowledge, Semyanistyi type integrals associated to the j-plane to k-plane transform $R_{j,k}$ are unknown. It might be interesting to properly introduce them and study their mapping properties; see [25] for similar operators on matrix spaces.

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