

ON COHOMOLOGICAL DECOMPOSABILITY OF ALMOST-KÄHLER STRUCTURES

DANIELE ANGELLA, ADRIANO TOMASSINI, AND WEIYI ZHANG

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ABSTRACT. We study the J -invariant and J -anti-invariant cohomological subgroups of the de Rham cohomology of a compact manifold M endowed with an almost-Kähler structure (J, ω, g) . In particular, almost-Kähler manifolds satisfying a Lefschetz type property and solvmanifolds endowed with left-invariant almost-complex structures are investigated.

INTRODUCTION

Cohomological properties of compact complex, and, more generally, almost-complex, manifolds have been recently studied by many authors; see, e.g., [3], respectively [11, 12], and the references therein. The study of the cohomology of almost-complex manifolds is motivated, in particular, by a question of Donaldson, [10, Question 2], relating the tamed and compatible symplectic cones of a compact 4-dimensional almost-complex manifold (see, e.g., [20]), and by the analogous question arising for compact higher-dimensional complex manifolds (see [20, page 678] and [26, Question 1.7]). (We recall that a symplectic structure ω on a manifold M is said to *tame* an almost-complex structure J if $\omega_x(u_x, J_x u_x) > 0$ for any $x \in M$ and for any $u \in T_x M \setminus \{0\}$, and it is said to be *compatible* with J if $g := \omega(\cdot, J\cdot)$ is a J -Hermitian metric; in the latter case, the triple (J, ω, g) is called an *almost-Kähler structure* on M .)

Following T.-J. Li and the third author, [20], an almost-complex structure J on a $2n$ -dimensional manifold M is called \mathcal{C}^∞ -*pure-and-full* if

$$H_{dR}^2(M; \mathbb{R}) = H_J^{(1,1)}(M)_{\mathbb{R}} \oplus H_J^{(2,0),(0,2)}(M)_{\mathbb{R}},$$

where $H_J^{(1,1)}(M)_{\mathbb{R}}$ and $H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$ denote the subgroups of $H_{dR}^2(M; \mathbb{R})$ whose elements can be represented by forms of types $(1, 1)$ and $(2, 0) + (0, 2)$ respectively. In the notation of T. Drăghici, T.-J. Li, and the third author, [11], $H_J^{(1,1)}(M)_{\mathbb{R}} =: H_J^+(M)$ and $H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} =: H_J^-(M)$ are the J -invariant and J -anti-invariant cohomology subgroups respectively.

In [11, Theorem 2.3], Drăghici, Li, and the third author proved that every almost-complex structure on a compact 4-dimensional manifold is \mathcal{C}^∞ -pure-and-full. This

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is no longer true in dimensions greater than four; see, e.g., [15, Example 3.3]; see also [1, 2].

The groups $H_J^{(1,1)}(M)_{\mathbb{R}}$ and $H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$ appear as a natural generalization of the Dolbeault cohomology groups to the non-integrable case; see, e.g., [20, Proposition 2.1]. In fact, compact Kähler manifolds are \mathcal{C}^∞ -pure-and-full, and, in this case, $H_J^{(1,1)}(M)_{\mathbb{R}} \simeq H_{\bar{\partial}}^{1,1}(M) \cap H_{dR}^2(M; \mathbb{R})$ and $H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} \simeq \left(H_{\bar{\partial}}^{2,0}(M) \oplus H_{\bar{\partial}}^{0,2}(M) \right) \cap H_{dR}^2(M; \mathbb{R})$.

We remark that on a compact complex manifold, other cohomologies can be defined, namely, the Bott-Chern and Aeppli cohomologies. In [3], the problem of cohomology decomposition in terms of the Bott-Chern cohomology groups is investigated, providing in particular a characterization of compact complex manifolds satisfying the $\partial\bar{\partial}$ -Lemma.

With compact Kähler manifolds being \mathcal{C}^∞ -pure-and-full, in this paper we are interested in the study of the cohomological subgroups $H_J^{(1,1)}(M)_{\mathbb{R}}$ and $H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$ for almost-Kähler manifolds.

On the one hand, A. Fino and the second author, [15, Proposition 3.2], as well as Drăghici, Li, and the third author, [11, Proposition 2.8], proved that the almost-complex structure of a compact almost-Kähler manifold is \mathcal{C}^∞ -pure. On the other hand, we prove the following result, therefore showing a difference between the integrable and the non-integrable cases.

Proposition 4.1. *Let $X := \mathbb{Z}[i]^3 \setminus (\mathbb{C}^3, *)$ be the real manifold underlying the Iwasawa manifold. Then there exists an almost-Kähler structure (J, ω, g) on X which is \mathcal{C}^∞ -pure and non- \mathcal{C}^∞ -full. Furthermore, the Lefschetz type operator $\mathcal{L}_\omega := \omega \wedge \cdot : \wedge^2 M \rightarrow \wedge^4 M$ of the almost-Kähler structure (J, ω, g) does not take g -harmonic 2-forms to g -harmonic 4-forms.*

In studying cohomological decomposition of the de Rham cohomology of almost-Kähler manifolds, the third author introduced a *Lefschetz type property* for 2-forms; see Definition 2.2. Such a property is stronger than the Hard Lefschetz Condition on 2-classes, namely, the property that $[\omega]^{n-2} \smile \cdot : H_{dR}^2(M; \mathbb{R}) \rightarrow H_{dR}^{2n-2}(M; \mathbb{R})$ is an isomorphism, where $2n := \dim M$.

We study such a Lefschetz type property on almost-Kähler manifolds (M, J, ω, g) in relation to the existence of a cohomological decomposition of $H_{dR}^2(M; \mathbb{R})$. More precisely, we prove the following result.

Theorem 2.3. *Let (M, J, ω, g) be a compact almost-Kähler manifold. Suppose that there exists a basis of $H_{dR}^2(X; \mathbb{R})$ represented by g -harmonic 2-forms which are of pure type with respect to J . Then the Lefschetz type property on 2-forms is satisfied.*

Note that by the hypothesis, it follows, in particular, that J is \mathcal{C}^∞ -pure-and-full and pure-and-full, [15, Theorem 3.7]. Note also that Fino and the second author provided in [15] several examples of compact non-Kähler solvmanifolds admitting a basis of harmonic representatives of pure type with respect to the almost-complex structure. In [13, §2], Drăghici, Li, and the third author asked whether such a Lefschetz type property on 2-forms is actually equivalent to \mathcal{C}^∞ -fullness for every almost-Kähler nilmanifold and solvmanifold, without any further assumption. Theorem 2.3 and Proposition 4.1 provide results and examples in favour of a possibly positive answer to their question.

In [12, Theorem 1.1], starting with a compact complex surface (M, J) , it is shown that the dimension $h_{\tilde{J}}^-$ of the \tilde{J} -anti-invariant cohomology subgroup $H_{\tilde{J}}^-(M)$ of any *metric related* almost-complex structure \tilde{J} on M (namely, an almost-complex structure \tilde{J} on M inducing the same orientation as the one induced by J and with a common compatible metric), such that $\tilde{J} \neq \pm J$, can be 0, 1, or 2, and a description of such almost-complex structures \tilde{J} having $h_{\tilde{J}}^- \in \{1, 2\}$ is provided. Furthermore, it is conjectured that $h_{\tilde{J}}^- = 0$ for a generic almost-complex structure J on a compact 4-dimensional manifold, and that if $h_{\tilde{J}}^- \geq 3$, then J is integrable, [12, Conjecture 2.4, Conjecture 2.5]. One could set a similar question for higher-dimensional manifolds, asking Question 5.2: *are there examples of non-integrable almost-complex structures J on a compact $2n$ -dimensional manifold with $h_{\tilde{J}}^- > n(n - 1)$?*

Finally, we prove a Nomizu type result for the subgroups $H_J^\pm(M)$ of completely-solvable solvmanifolds $M = \Gamma \backslash G$ endowed with left-invariant almost-complex structures J . More precisely, denote the Lie algebra associated to G by \mathfrak{g} , and consider

$$H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} := \left\{ \mathfrak{a} = [\alpha \in H^\bullet(\wedge^\bullet \mathfrak{g}^*, d) : \alpha \in \wedge_J^{(p,q),(q,p)} \mathfrak{g}^*] \right\} \subseteq H_{dR}^\bullet(M; \mathbb{R})$$

the subgroup of $H_{dR}^\bullet(M; \mathbb{R})$ that consists of classes admitting a left-invariant representative of type $(p, q) + (q, p)$, where

$$\wedge_J^{(p,q),(q,p)} \mathfrak{g}^* := (\wedge^{p,q}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^* \oplus \wedge^{q,p}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^*) \cap \wedge^\bullet \mathfrak{g}^*.$$

Then the following result holds.

Theorem 5.4. *Let $M = \Gamma \backslash G$ be a solvmanifold endowed with a left-invariant almost-complex structure J , and denote the Lie algebra naturally associated to G by \mathfrak{g} . For any $p, q \in \mathbb{N}$, the map $j : H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \rightarrow H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}$ induced by left-translations is injective, and if $H_{dR}^\bullet(\wedge^\bullet \mathfrak{g}^*, d) \simeq H_{dR}^\bullet(M; \mathbb{R})$ (for instance, if M is a completely-solvable solvmanifold), then $j : H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \rightarrow H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}$ is in fact an isomorphism.*

In particular, it follows that $\dim_{\mathbb{R}} H_J^-(M) \leq n(n - 1)$ for every left-invariant almost-complex structure on a completely-solvable solvmanifold.

1. C^∞ -PURE-AND-FULL ALMOST-COMPLEX STRUCTURES

1.1. Subgroups of the de Rham cohomology of almost-complex manifolds.

We start by fixing some notation and recalling some recent results on cohomological properties of almost-complex manifolds; for more details see, e.g., [1, 2, 11–13, 15, 20] and the references therein.

Let J be a smooth almost-complex structure on a compact $2n$ -dimensional manifold M . Denote by $\wedge^r M$ the bundle of r -forms on M ; we denote with the same symbol $\wedge^r M := \Gamma(M, \wedge^r M)$ the space of smooth global sections of the bundle $\wedge^r M$. Then J extends to a complex automorphism of $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$ such that $T^{\mathbb{C}}M = T_J^{1,0}M \oplus T_J^{0,1}M$, where $T_J^{1,0}M$ and $T_J^{0,1}M$ are the $(\pm i)$ -eigenbundles. The action of J can be extended to the space $\wedge^r(M; \mathbb{C})$ of smooth global sections of the bundle $\wedge^r(M; \mathbb{C}) := \wedge^r M \otimes \mathbb{C}$, getting the following decomposition:

$$\wedge^r(M; \mathbb{C}) = \bigoplus_{p+q=r} \wedge_J^{p,q} M.$$

Then the space $\wedge^r M$ of real smooth differential r -forms decomposes as

$$\wedge^r M = \bigoplus_{p+q=r, p \leq q} \wedge_J^{(p,q),(q,p)}(M)_{\mathbb{R}},$$

where, for $p < q$ (later on, we do not distinguish the cases $p < q$ and $p = q$),

$$\begin{aligned} \wedge_J^{(p,q),(q,p)}(M)_{\mathbb{R}} &:= \{ \alpha \in \wedge_J^{p,q} M \oplus \wedge_J^{q,p} M : \alpha = \bar{\alpha} \}, \\ \wedge_J^{(p,p)}(M)_{\mathbb{R}} &:= \{ \alpha \in \wedge_J^{p,p} M : \alpha = \bar{\alpha} \}. \end{aligned}$$

In particular, for $r = 2$, we will adopt the following notation:

$$\wedge_J^{1,1}(M)_{\mathbb{R}} =: \wedge_J^{\perp} M, \quad \wedge_J^{(2,0),(0,2)}(M)_{\mathbb{R}} =: \wedge_J^{-} M;$$

this is consistent with the decomposition in invariant and anti-invariant parts of $\wedge^2 M$ under the natural action of J on $\wedge^2 M$, given by $J\alpha(\cdot, \cdot) := \alpha(J\cdot, J\cdot)$.

We will refer to forms in $\wedge_J^{1,1}(M)_{\mathbb{R}}$, respectively $\wedge_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$, as forms of *pure type with respect to J* .

For a finite set S of pairs of integers, let

$$\mathcal{Z}_J^S := \bigoplus_{(p,q) \in S, p \leq q} \mathcal{Z}_J^{(p,q),(q,p)}, \quad \mathcal{B}_J^S := \bigoplus_{(p,q) \in S, p \leq q} \mathcal{B}_J^{(p,q),(q,p)},$$

where

$$\begin{aligned} \mathcal{Z}_J^{(p,q),(q,p)} &:= \left\{ \alpha \in \wedge_J^{(p,q),(q,p)}(M)_{\mathbb{R}} : d\alpha = 0 \right\}, \\ \mathcal{B}_J^{(p,q),(q,p)} &:= \left\{ \beta \in \wedge_J^{(p,q),(q,p)}(M)_{\mathbb{R}} : \text{there exists } \gamma \text{ such that } d\gamma = \beta \right\}. \end{aligned}$$

Define

$$H_J^S(M)_{\mathbb{R}} := \frac{\mathcal{Z}_J^S}{\mathcal{B}_J^S}.$$

Let \mathcal{B} be the space of d-exact forms. Since $\frac{\mathcal{Z}_J^S}{\mathcal{B}_J^S} = \frac{\mathcal{Z}_J^S}{\mathcal{B} \cap \mathcal{Z}_J^S}$, a natural inclusion $\rho_S: \frac{\mathcal{Z}_J^S}{\mathcal{B}_J^S} \rightarrow \frac{\mathcal{Z}_J^S}{\mathcal{B}}$ is defined. As in [20], we will write $\rho_S \left(\frac{\mathcal{Z}_J^S}{\mathcal{B}_J^S} \right)$ simply as $\frac{\mathcal{Z}_J^S}{\mathcal{B}}$, and consequently the cohomology spaces $H_J^S(M)_{\mathbb{R}}$ can be identified as

$$H_J^S(M)_{\mathbb{R}} = \{ [\alpha] \in H_{dR}^{\bullet}(M; \mathbb{R}) : \alpha \in \mathcal{Z}_J^S \} = \frac{\mathcal{Z}_J^S}{\mathcal{B}}.$$

Therefore, there is a natural inclusion

$$H_J^{(1,1)}(M)_{\mathbb{R}} + H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} \subseteq H_{dR}^2(M; \mathbb{R}).$$

1.2. \mathcal{C}^∞ -pure-and-full and pure-and-full almost-complex structures. As in [20], we set the following definition.

Definition 1.1 ([20, Definition 2.2, Definition 2.3, Lemma 2.2]). An almost-complex structure J on a manifold M is said to be

- \mathcal{C}^∞ -pure if $H_J^{(1,1)}(M)_{\mathbb{R}} \cap H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} = \{0\}$,
- \mathcal{C}^∞ -full if $H_{dR}^2(M; \mathbb{R}) = H_J^{(1,1)}(M)_{\mathbb{R}} + H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$,
- \mathcal{C}^∞ -pure-and-full if

$$H_{dR}^2(M; \mathbb{R}) = H_J^{(1,1)}(M)_{\mathbb{R}} \oplus H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}.$$

According to the previous notation, we will write

$$H_J^+(M) := H_J^{(1,1)}(M)_{\mathbb{R}}, \quad H_J^-(M) := H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}.$$

Similar definitions in terms of currents can be given, introducing the notion of *pure-and-full* almost-complex structure; we refer to [20, §2.2] for further details and results. More precisely, on an almost complex manifold (M, J) , the space $\mathcal{E}_k(M)_{\mathbb{R}}$ of real k -currents has a decomposition $\mathcal{E}_k(M)_{\mathbb{R}} = \bigoplus_{\substack{p+q=k \\ p \leq q}} \mathcal{E}_{(p,q),(q,p)}^J(M)_{\mathbb{R}}$, where $\mathcal{E}_{(p,q),(q,p)}^J(M)_{\mathbb{R}}$ denotes the space of real k -currents of bi-dimension $(p, q) + (q, p)$.

Let $\mathcal{Z}_{(2,0),(0,2)}^J$ and $\mathcal{Z}_{(1,1)}^J$ denote the spaces of real d-closed currents of bi-dimension $(2, 0) + (0, 2)$, respectively $(1, 1)$, and $\mathcal{B}_{(2,0),(0,2)}^J$ and $\mathcal{B}_{(1,1)}^J$ denote the spaces of real d-exact currents of bi-dimension $(2, 0) + (0, 2)$, respectively $(1, 1)$. Denote by \mathcal{B} the space of boundaries. Let, as in [20],

$$H_{(1,1)}^J(M)_{\mathbb{R}} := \left\{ [\alpha] \in H_2(M; \mathbb{R}) : \alpha \in \mathcal{Z}_{(1,1)}^J \right\} = \frac{\mathcal{Z}_{(1,1)}^J}{\mathcal{B}},$$

$$H_{(2,0),(0,2)}^J(M)_{\mathbb{R}} := \left\{ [\alpha] \in H_2(M; \mathbb{R}) : \alpha \in \mathcal{Z}_{(2,0),(0,2)}^J \right\} = \frac{\mathcal{Z}_{(2,0),(0,2)}^J}{\mathcal{B}}.$$

We recall the following definition.

Definition 1.2 ([20, Definition 2.15, Definition 2.16]). An almost complex structure J on a manifold M is said to be *pure* if $H_{(1,1)}^J(M)_{\mathbb{R}} \cap H_{(2,0),(0,2)}^J(M)_{\mathbb{R}} = \{0\}$. It is said to be *full* if $H_2(M; \mathbb{R}) = H_{(1,1)}^J(M)_{\mathbb{R}} + H_{(2,0),(0,2)}^J(M)_{\mathbb{R}}$. Therefore, an almost complex structure J is *pure-and-full* if and only if

$$H_2(M, \mathbb{R}) = H_{(1,1)}^J(M)_{\mathbb{R}} \oplus H_{(2,0),(0,2)}^J(M)_{\mathbb{R}}.$$

In [20, Proposition 2.1] it is shown that, given a compact complex manifold (M, J) of complex dimension n , if $n = 2$ or J is Kähler, then J is \mathcal{C}^∞ -*pure-and-full*, and

$$H_J^{(1,1)}(M)_{\mathbb{R}} \simeq H_{\bar{\partial}}^{1,1}(M) \cap H_{dR}^2(M; \mathbb{R})$$

and

$$H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} \simeq \left(H_{\bar{\partial}}^{2,0}(M) \oplus H_{\bar{\partial}}^{0,2}(M) \right) \cap H_{dR}^2(M; \mathbb{R}).$$

In view of this result the subgroups $H_J^{(1,1)}(M)_{\mathbb{R}}$ and $H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$ of the de Rham cohomology can be viewed as an analogue of the Dolbeault cohomology groups for non-integrable almost-complex structures.

In [11, Theorem 2.3] the following result is proven.

Theorem 1.3 ([11, Theorem 2.3]). *If M is a compact manifold of dimension 4, then any almost-complex structure J on M is \mathcal{C}^∞ -pure-and-full.*

This is no longer true in dimensions higher than 4: in [15, Example 3.3], a compact non- \mathcal{C}^∞ -pure almost-complex structure on a 6-dimensional nilmanifold is constructed. Therefore, the previous theorem can be considered as a sort of Hodge decomposition theorem in the non-Kähler case.

2. COHOMOLOGICAL PROPERTIES OF ALMOST-KÄHLER MANIFOLDS

2.1. Lefschetz type property on almost-Kähler manifolds with pure type harmonic representatives. Given a compact $2n$ -dimensional almost-Kähler manifold (M, J, ω, g) , we are interested in studying the property of being C^∞ -pure-and-full.

First we recall the following result.

Proposition 2.1 ([11, Proposition 2.8], [15, Proposition 3.2]). *If J is an almost-complex structure on a compact manifold M and J admits a compatible symplectic structure, then J is C^∞ -pure.*

Furthermore, Fino and the second author proved that an almost-Kähler manifold admitting a basis of harmonic 2-forms whose elements are of pure type with respect to the almost-complex structure is C^∞ -pure-and-full and pure-and-full, [15, Theorem 3.7]. They also provided several examples of compact non-Kähler solvmanifolds satisfying the above assumption in [15].

For the purpose of studying the property of being C^∞ -pure-and-full on almost-Kähler manifolds, we recall the following definition.

Definition 2.2. Given a compact $2n$ -dimensional symplectic manifold (M, ω) , denote by

$$\mathcal{L}_\omega : \wedge^2 M \rightarrow \wedge^{2n-2} M, \quad \mathcal{L}_\omega(\alpha) := \omega^{n-2} \wedge \alpha$$

the *Lefschetz type operator* (on 2-forms) associated with ω .

Then one says that the compact $2n$ -dimensional almost-Kähler manifold (M, J, ω, g) satisfies the *Lefschetz type property (on 2-forms)* if \mathcal{L}_ω takes g -harmonic 2-forms to g -harmonic $(2n - 2)$ -forms.

Furthermore, we recall some notions and results from [6, 22, 27]; see also [7, 23].

Let (M, ω) be a compact $2n$ -dimensional symplectic manifold. Extend $\omega^{-1} : T^*M \rightarrow TM$ to the whole exterior algebra of T^*M . For any $k \in \mathbb{N}$, the *symplectic \star_ω operator* is defined as

$$\star_\omega : \wedge^k M \rightarrow \wedge^{2n-k} M, \quad \beta \wedge \star_\omega \alpha = \omega^{-1}(\alpha, \beta) \frac{\omega^n}{n!}, \quad \forall \alpha, \beta \in \wedge^k M.$$

One can prove that $\star_\omega^2 = \text{id}_{\wedge^\bullet M}$ [6, Lemma 2.1.2].

For any $k \in \mathbb{N}$, define the *symplectic co-differential operator*

$$\delta_\omega : \wedge^k M \rightarrow \wedge^{k-1} M, \quad \delta_\omega \lrcorner_{\wedge^k M} := (-1)^{k+1} \star_\omega d \star_\omega;$$

this operator has been studied by J.-L. Brylinski in [6] for Poisson manifolds. In the context of generalized complex geometry (see, e.g., [16]), it can be interpreted as the symplectic counterpart of the operator $d^c := -i(\partial - \bar{\partial})$ in complex geometry; see [7].

By definition, (M, ω) satisfies the *Hard Lefschetz Condition* if, for each $k \in \mathbb{N}$, the map

$$[\omega]^k \smile \cdot : H_{dR}^{n-k}(M; \mathbb{R}) \rightarrow H_{dR}^{n+k}(M; \mathbb{R})$$

is an isomorphism. O. Mathieu [22, Corollary 2] and, independently, D. Yan [27, Theorem 0.1] proved that, given a compact symplectic manifold (M, ω) , any de Rham cohomology class has a (possibly non-unique) ω -symplectically harmonic representative (that is, a d -closed δ_ω -closed representative) if and only if the Hard Lefschetz Condition holds.

We can now prove the following result.

Theorem 2.3. *Let (M, J, ω, g) be a compact almost-Kähler manifold. Suppose that there exists a basis of $H_{dR}^2(X; \mathbb{R})$ represented by g -harmonic 2-forms which are of pure type with respect to J . Then the Lefschetz type property on 2-forms is satisfied.*

Proof. Recall that on a $2n$ -dimensional almost-Kähler manifold (M, J, ω, g) the Hodge $*_g$ operator and the symplectic \star_ω operator are related by $\star_\omega = *_g J$ [6, Theorem 2.4.1, Remark 2.4.4]. Therefore, for forms of pure type with respect to J , the properties of being g -harmonic and of being ω -symplectically harmonic are equivalent. The theorem follows noting that [27, Lemma 1.2] $[\mathcal{L}_\omega, d] = 0$ and $[\mathcal{L}_\omega, \delta_\omega] = d$; hence \mathcal{L}_ω sends ω -symplectically harmonic 2-forms (of pure type with respect to J) to ω -symplectically harmonic $(2n - 2)$ -forms (of pure type with respect to J). □

Remark 2.4. We note that if (M, J, ω, g) is a compact $2n$ -dimensional almost-Kähler manifold satisfying the Lefschetz type property on 2-forms and J is \mathcal{C}^∞ -full, then J is \mathcal{C}^∞ -pure-and-full and pure-and-full.

Indeed, we have already remarked that J is \mathcal{C}^∞ -pure; see Proposition 2.1. Moreover, since J is \mathcal{C}^∞ -full, J is also pure by [20, Proposition 2.5]. We now recall the argument in [15] to prove that J is also full.

Firstly, note that if the Lefschetz type property on 2-forms holds, then

$$[\omega^{n-2}] \smile \cdot : H_{dR}^2(M; \mathbb{R}) \rightarrow H_{dR}^{2n-2}(M; \mathbb{R})$$

is an isomorphism. Therefore, we get that

$$H_{dR}^{2n-2}(M; \mathbb{R}) = H_J^{(n,n-2),(n-2,n)}(M)_{\mathbb{R}} + H_J^{(n-1,n-1)}(M)_{\mathbb{R}} ;$$

indeed (following the argument in [15, Theorem 4.1]), since $[\omega^{n-2}] \smile \cdot : H_{dR}^2(M; \mathbb{R}) \rightarrow H_{dR}^{2n-2}(M; \mathbb{R})$ is in particular surjective, we have

$$\begin{aligned} H_{dR}^{2n-2}(M; \mathbb{R}) &= [\omega^{n-2}] \smile H_{dR}^2(M; \mathbb{R}) \\ &= [\omega^{n-2}] \smile \left(H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} \oplus H_J^{(1,1)}(M)_{\mathbb{R}} \right) \\ &\subseteq H_J^{(n,n-2),(n-2,n)}(M)_{\mathbb{R}} + H_J^{(n-1,n-1)}(M)_{\mathbb{R}} , \end{aligned}$$

yielding the above decomposition of $H_{dR}^{2n-2}(M; \mathbb{R})$. Then, it follows that J is also full; see, for example, [1, Theorem 2.1].

2.2. A family of almost-Kähler manifolds satisfying the Lefschetz type property on 2-forms. Let \mathfrak{n} be the 6-dimensional nilpotent Lie algebra whose structure equations, with respect to a basis $\{e^j\}_{j \in \{1, \dots, 6\}}$ of \mathfrak{n}^* , are given by

$$de^1 = de^2 = de^3 = 0, \quad de^4 = e^{23}, \quad de^5 = e^{13}, \quad de^6 = e^{12}$$

(where we write e^{jk} instead of $e^j \wedge e^k$). Using a result of Mal'tsev [21, Theorem 7], the connected simply-connected Lie group G associated with \mathfrak{n} admits a discrete co-compact subgroup Γ : let $N := \Gamma \backslash G$ be the (compact) nilmanifold obtained as a quotient of G by Γ . Note that N is not formal by a theorem of K. Hasegawa [17, Theorem 1, Corollary].

Fix $\alpha > 1$ and take

$$\omega_\alpha := e^{14} + \alpha \cdot e^{25} + (\alpha - 1) \cdot e^{36} ;$$

since $d\omega_\alpha = 0$ and $\omega_\alpha^3 \neq 0$, we get that ω_α is a left-invariant symplectic form on N . Set

$$\begin{aligned} J_\alpha e_1 &:= e_4, & J_\alpha e_2 &:= \alpha e_5, & J_\alpha e_3 &:= (\alpha - 1) e_6, \\ J_\alpha e_4 &:= -e_1, & J_\alpha e_5 &:= -\frac{1}{\alpha} e_2, & J_\alpha e_6 &:= -\frac{1}{\alpha-1} e_3, \end{aligned}$$

where $\{e_1, \dots, e_6\}$ denotes the global dual frame of $\{e^1, \dots, e^6\}$ on N . It is immediate to check that

- setting $g_\alpha(\cdot, \cdot) := \omega_\alpha(\cdot, J_\alpha \cdot)$, the triple $(J_\alpha, \omega_\alpha, g_\alpha)$ gives rise to a family of left-invariant almost-Kähler structures on N ;
- denoting by

$$\begin{aligned} E_\alpha^1 &:= e^1, & E_\alpha^2 &:= \alpha e^2, & E_\alpha^3 &:= (\alpha - 1) e^3, \\ E_\alpha^4 &:= e^4, & E_\alpha^5 &:= e^5, & E_\alpha^6 &:= e^6, \end{aligned}$$

then $\{E_\alpha^1, \dots, E_\alpha^6\}$ is a g_α -orthonormal co-frame on N ; with respect to this new co-frame, we easily obtain the following structure equations:

$$\begin{aligned} dE_\alpha^1 = dE_\alpha^2 = dE_\alpha^3 = 0, & \quad dE_\alpha^4 = \frac{1}{\alpha(\alpha-1)} E_\alpha^{23}, \\ dE_\alpha^5 = \frac{1}{\alpha-1} E_\alpha^{13}, & \quad dE_\alpha^6 = \frac{1}{\alpha} E_\alpha^{12}. \end{aligned}$$

Then

$$\varphi_\alpha^1 := E_\alpha^1 + i E_\alpha^4, \quad \varphi_\alpha^2 := E_\alpha^2 + i E_\alpha^5, \quad \varphi_\alpha^3 := E_\alpha^3 + i E_\alpha^6$$

are $(1,0)$ -forms with respect to the almost-complex structure J_α , and

$$\omega_\alpha = E_\alpha^{14} + E_\alpha^{25} + E_\alpha^{36} .$$

By a result of K. Nomizu [25, Theorem 1] (see Theorem 5.3), the de Rham cohomology of N is straightforwardly computed:

$$H_{dR}^2(N; \mathbb{R}) \simeq \mathbb{R} \left\langle E_\alpha^{15}, E_\alpha^{16}, E_\alpha^{24}, E_\alpha^{26}, E_\alpha^{34}, E_\alpha^{35}, E_\alpha^{14} + \frac{1}{\alpha} E_\alpha^{25}, \frac{1}{\alpha} E_\alpha^{25} + \frac{1}{\alpha-1} E_\alpha^{36} \right\rangle$$

(where we have listed the g_α -harmonic representatives instead of their classes).

Note that the listed g_α -harmonic representatives of $H_{dR}^2(N; \mathbb{R})$ are of pure type with respect to J_α : hence, the almost-complex structure J_α is C^∞ -pure-and-full by [15, Theorem 3.7]. In particular, note that

$$\begin{aligned} H_{dR}^2(N; \mathbb{R}) \simeq \mathbb{R} \left\langle i \alpha \varphi_\alpha^{1\bar{1}} + i \varphi_\alpha^{2\bar{2}}, i(\alpha-1) \varphi_\alpha^{2\bar{2}} + i \alpha \varphi_\alpha^{3\bar{3}}, \Im \varphi_\alpha^{1\bar{2}}, \Im \varphi_\alpha^{1\bar{3}}, \Im \varphi_\alpha^{3\bar{2}} \right\rangle \\ \oplus \left\langle \Im \varphi_\alpha^{1\bar{2}}, \Im \varphi_\alpha^{1\bar{3}}, \Im \varphi_\alpha^{2\bar{3}} \right\rangle ; \end{aligned}$$

hence $h_{J_\alpha}^+(N) = 5$ and $h_{J_\alpha}^-(N) = 3$.

Moreover, one explicitly notes that

$$\begin{aligned} \mathcal{L}_{\omega_\alpha} E_\alpha^{15} &= E_\alpha^{1536} = *_{g_\alpha} E_\alpha^{24}, & \mathcal{L}_{\omega_\alpha} E_\alpha^{16} &= E_\alpha^{1625} = *_{g_\alpha} E_\alpha^{34}, \\ \mathcal{L}_{\omega_\alpha} E_\alpha^{24} &= E_\alpha^{2436} = *_{g_\alpha} E_\alpha^{15}, & \mathcal{L}_{\omega_\alpha} E_\alpha^{26} &= E_\alpha^{2614} = *_{g_\alpha} E_\alpha^{35}, \\ \mathcal{L}_{\omega_\alpha} E_\alpha^{34} &= E_\alpha^{3425} = *_{g_\alpha} E_\alpha^{16}, & \mathcal{L}_{\omega_\alpha} E_\alpha^{35} &= E_\alpha^{3514} = *_{g_\alpha} E_\alpha^{26}, \end{aligned}$$

while

$$\mathcal{L}_{\omega_\alpha} \left(E_\alpha^{14} + \frac{1}{\alpha} E_\alpha^{25} \right) = -\frac{\alpha + 1}{\alpha} E_\alpha^{1245} - \frac{1}{\alpha} E_\alpha^{2356} - E_\alpha^{1346}$$

where

$$d *_{g_\alpha} \mathcal{L}_{\omega_\alpha} \left(E_\alpha^{14} + \frac{1}{\alpha} E_\alpha^{25} \right) = d \left(-\frac{\alpha + 1}{\alpha} E_\alpha^{36} - E_\alpha^{25} - \frac{1}{\alpha} E_\alpha^{14} \right) = 0,$$

and, by a similar computation, $d *_{g_\alpha} \mathcal{L}_{\omega_\alpha} (e^{25} + e^{36}) = 0$. This proves explicitly that ω_α satisfies the Lefschetz type property on 2-forms.

The nilmanifold N is not formal by a theorem of Hasegawa [17, Theorem 1, Corollary]. The non-formality of M can also be proved by giving a non-zero triple Massey product on N ; see [9]. Since

$$[E_\alpha^1] \smile [E_\alpha^3] = (\alpha - 1) [d E_\alpha^5] = 0, \quad [E_\alpha^3] \smile [E_\alpha^2] = -\alpha (\alpha - 1) [d E_\alpha^4] = 0,$$

we get that the triple Massey product

$$\langle [E_\alpha^1], [E_\alpha^3], [E_\alpha^2] \rangle = -(\alpha - 1) [E_\alpha^{25} + \alpha E_\alpha^{14}]$$

does not vanish, and hence N is not formal.

In summary, we have proven the following result.

Proposition 2.5. *There is a non-formal 6-dimensional nilmanifold N endowed with a 1-parameter family $\{(J_\alpha, \omega_\alpha, g_\alpha)\}_{\alpha > 1}$ of left-invariant almost-Kähler structures being \mathcal{C}^∞ -pure-and-full and pure-and-full and satisfying the Lefschetz type property on 2-forms.*

Remark 2.6. It has to be noted that $\omega_\alpha \wedge \cdot : \wedge^2 N^6 \rightarrow \wedge^4 N^6$ induces an isomorphism in cohomology $[\omega_\alpha] \smile \cdot : H_{dR}^2(N, \mathbb{R}) \rightarrow H_{dR}^4(N, \mathbb{R})$, while according to [5, Theorem A], $[\omega_\alpha]^2 \smile \cdot : H_{dR}^1(N, \mathbb{R}) \rightarrow H_{dR}^5(N, \mathbb{R})$ is not an isomorphism.

3. ALMOST-KÄHLER \mathcal{C}^∞ -PURE-AND-FULL STRUCTURES

3.1. The Nakamura manifold of completely solvable type. Take $A \in \text{SL}(2; \mathbb{Z})$ with two different real eigenvalues e^λ and $e^{-\lambda}$ with $\lambda > 0$, and fix $P \in \text{GL}(2; \mathbb{R})$ such that $PAP^{-1} = \text{diag}(e^\lambda, e^{-\lambda})$. For example, take

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad P := \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{\sqrt{5}-1}{2} \end{pmatrix},$$

and consequently $\lambda = \log \frac{3+\sqrt{5}}{2}$. Let $M^6 :=: M^6(\lambda)$ be the compact complex manifold

$$M^6 := \mathbb{S}_{x^2}^1 \times \frac{\mathbb{R}_{x^1} \times \mathbb{T}_{\mathbb{C}}^2(x^3, x^4, x^5, x^6)}{\langle T_1 \rangle},$$

where $\mathbb{T}_{\mathbb{C}}^2$ is the 2-dimensional complex torus $\mathbb{T}_{\mathbb{C}}^2 := \frac{\mathbb{C}^2}{P\mathbb{Z}[i]^2}$ and T_1 acts on $\mathbb{R} \times \mathbb{T}_{\mathbb{C}}^2$ as $T_1(x^1, x^3, x^4, x^5, x^6) := (x^1 + \lambda, e^{-\lambda} x^3, e^\lambda x^4, e^{-\lambda} x^5, e^\lambda x^6)$. The manifold M^6 can be seen as a compact quotient of a completely-solvable Lie group by a discrete co-compact subgroup [14, Example 3.1] (denote the Lie algebra naturally associated to the completely-solvable Lie group of M^6 by \mathfrak{g}). Using coordinates x^2 on \mathbb{S}^1 , x^1 on \mathbb{R} and (x^3, x^4, x^5, x^6) on $\mathbb{T}_{\mathbb{C}}^2$, we set

$$\begin{aligned} e^1 &:= dx^1, & e^2 &:= dx^2, & e^3 &:= e^{x^1} dx^3, \\ e^4 &:= e^{-x^1} dx^4, & e^5 &:= e^{x^1} dx^5, & e^6 &:= e^{-x^1} dx^6 \end{aligned}$$

as a basis for \mathfrak{g}^* ; therefore, with respect to $\{e^i\}_{i \in \{1, \dots, 6\}}$, the structure equations are the following:

$$de^1 = de^2 = 0, \quad de^3 = e^{13}, \quad de^4 = -e^{14}, \quad de^5 = e^{15}, \quad de^6 = -e^{16}.$$

3.2. The de Rham cohomology of the Nakamura manifold. Let J be the almost-complex structure on M^6 defined by the complex $(1, 0)$ -forms given by

$$\varphi^1 := \frac{1}{2}(e^1 + ie^2), \quad \varphi^2 := e^3 + ie^5, \quad \varphi^3 := e^4 + ie^6.$$

It is straightforward to check that J is integrable.

With M^6 being a compact quotient of a completely-solvable Lie group, one computes the de Rham cohomology of M^6 easily by A. Hattori’s theorem [19, Corollary 4.2]; see Theorem 5.3:

$$H_{dR}^1(M^6; \mathbb{C}) \simeq \mathbb{C}\langle \varphi^1, \bar{\varphi}^1 \rangle, \quad H_{dR}^2(M^6; \mathbb{C}) \simeq \mathbb{C}\langle \varphi^{1\bar{1}}, \varphi^{2\bar{3}}, \varphi^{3\bar{2}}, \varphi^{2\bar{3}}, \varphi^{\bar{2}\bar{3}} \rangle,$$

$$H_{dR}^3(M^6; \mathbb{C}) \simeq \mathbb{C}\langle \varphi^{12\bar{3}}, \varphi^{13\bar{2}}, \varphi^{12\bar{3}}, \varphi^{1\bar{2}\bar{3}}, \varphi^{2\bar{1}\bar{3}}, \varphi^{3\bar{1}\bar{2}}, \varphi^{23\bar{1}}, \varphi^{\bar{1}\bar{2}\bar{3}} \rangle$$

(for the sake of clearness, we write, for example, $\varphi^{A\bar{B}}$ in place of $\varphi^A \wedge \bar{\varphi}^B$ and we list the harmonic representatives with respect to the metric $g := \sum_{j=1}^3 \varphi^j \odot \bar{\varphi}^j$ instead of their classes). Therefore, M^6 is *geometrically formal*; i.e., the product of g -harmonic forms is still g -harmonic, and therefore it is *formal*. That is, the de Rham complex of M is formal as a differential graded algebra; see, e.g., [9]. Furthermore, it can be easily checked that

$$\omega := e^{12} + e^{34} + e^{56}$$

gives rise to a symplectic structure on M^6 satisfying the Hard Lefschetz Condition. We obtain the following result.

Proposition 3.1 ([14, Proposition 3.2]). *The manifold M^6 is formal and admits a symplectic form ω satisfying the Hard Lefschetz Condition.*

Note also that $\tilde{\omega} := \frac{i}{2}(\varphi^{1\bar{1}} + \varphi^{2\bar{2}} + \varphi^{3\bar{3}})$ is not d-closed but $d\tilde{\omega}^2 = 0$, from which it follows that the manifold M^6 admits a balanced metric.

Moreover, since M^6 is a compact quotient of a completely-solvable Lie group, by the Hasegawa theorem [18, Main Theorem] we have the following result; see also [14, Theorem 3.3]. (We recall that a compact complex manifold is said to belong to *class \mathcal{C} of Fujiki* if it admits a proper modification from a Kähler manifold.)

Theorem 3.2 ([18, Main Theorem]). *The manifold M^6 admits no Kähler structure and it is not in class \mathcal{C} of Fujiki.*

3.3. An almost-Kähler structure on the Nakamura manifold. According to Hasegawa’s theorem [18, Main Theorem], any integrable complex structure on M^6 (for example, the J defined in §3.2) does not admit any symplectic structure compatible with it. Therefore, we consider the almost-complex structure J' defined by

$$J'e^1 := -e^2, \quad J'e^3 := -e^4, \quad J'e^5 := -e^6.$$

Considering

$$\psi^1 := \frac{1}{2}(e^1 + ie^2), \quad \psi^2 := e^3 + ie^4, \quad \psi^3 := e^5 + ie^6$$

as a co-frame for the space of $(1, 0)$ -forms on (M^6, J') , one can compute

$$d\psi^1 = 0, \quad d\psi^2 = \psi^{1\bar{2}} + \psi^{\bar{1}2}, \quad d\psi^3 = \psi^{1\bar{3}} + \psi^{\bar{1}3},$$

from which it is clear that J' is not integrable. Note that the J' -compatible 2-form

$$\omega' := e^{12} + e^{34} + e^{56}$$

is d-closed. Hence, (M^6, J', ω') is an almost-Kähler manifold.

Moreover, recall that

$$H_{dR}^2(M^6; \mathbb{R}) \simeq \underbrace{\mathbb{R} \langle i\psi^{1\bar{1}}, i\psi^{2\bar{2}}, i\psi^{3\bar{3}}, i(\psi^{2\bar{3}} + \psi^{3\bar{2}}) \rangle}_{\subseteq H_{J'}^+(M^6)_{\mathbb{R}}} \oplus \underbrace{\mathbb{R} \langle i(\psi^{2\bar{3}} - \psi^{\bar{2}3}) \rangle}_{\subseteq H_{J'}^-(M^6)_{\mathbb{R}}},$$

where we have listed the harmonic representatives with respect to the metric $g' := \sum_{j=1}^6 e^j \odot e^j$ instead of their classes; note that the listed g' -harmonic representatives are of pure type with respect to J' . Therefore, J' is obviously \mathcal{C}^∞ -full; it is also \mathcal{C}^∞ -pure by [15, Proposition 3.2] or [11, Proposition 2.8] (see Proposition 2.1). Moreover, since any cohomology class in $H_{J'}^+(M^6)_{\mathbb{R}}$ (respectively, in $H_{J'}^-(M^6)_{\mathbb{R}}$) has a g' -harmonic representative in $\mathcal{Z}_{J'}^{(1,1)}$ (respectively, in $\mathcal{Z}_{J'}^{(2,0),(0,2)}$), by [15, Theorem 3.7] we have that J' is also pure-and-full. One can explicitly check that the Lefschetz type operator $\mathcal{L}_{\omega'} : \wedge^2 M^6 \rightarrow \wedge^4 M^6$ introduced in §2 takes g' -harmonic 2-forms to g' -harmonic 4-forms, since

$$\begin{aligned} \mathcal{L}_{\omega'} e^{12} &= e^{1234} + e^{1256} = *_{g'}(e^{34} + e^{56}), & \mathcal{L}_{\omega'} e^{36} &= e^{1236} = *_{g'} e^{45}, \\ \mathcal{L}_{\omega'} e^{34} &= e^{1234} + e^{3456} = *_{g'}(e^{12} + e^{56}), & \mathcal{L}_{\omega'} e^{45} &= e^{1245} = *_{g'} e^{36}, \\ \mathcal{L}_{\omega'} e^{56} &= e^{1256} + e^{3456} = *_{g'}(e^{12} + e^{34}). \end{aligned}$$

Resuming, we have shown the following result.

Proposition 3.3. *Let M^6 be the Nakamura manifold. Then there exist a complex structure J and an almost-Kähler structure (J', ω', g') , both of which are \mathcal{C}^∞ -pure-and-full and pure-and-full.*

Furthermore, the Lefschetz type operator of the almost-Kähler structure (J', ω', g') takes g' -harmonic 2-forms to g' -harmonic 4-forms.

Inspired by the argument of the proof of [11, Theorem 2.3] (see Theorem 1.3), one can ask the following question (compare also [13, §2]). We provide in Proposition 4.1 an example of a non- \mathcal{C}^∞ -full almost-Kähler structure for which the Lefschetz type property on 2-forms does not hold.

Question 3.4. *Let (M, J, ω, g) be a compact $2n$ -dimensional almost-Kähler manifold satisfying the Lefschetz type property on 2-forms. Is J \mathcal{C}^∞ -full?*

4. AN ALMOST-KÄHLER NON- \mathcal{C}^∞ -FULL STRUCTURE

Let $X := \mathbb{Z}[i]^3 \setminus (\mathbb{C}^3, *)$ be the *Iwasawa manifold*, where the group structure on \mathbb{C}^3 is defined by

$$(z_1, z_2, z_3) * (w_1, w_2, w_3) := (z_1 + w_1, z_2 + w_2, z_3 + z_1 w_2 + w_3).$$

Considering the standard complex structure induced by the one on \mathbb{C}^3 and setting $\{\varphi^1, \varphi^2, \varphi^3\}$ as a global co-frame for the $(1, 0)$ -forms on X , by Hattori's theorem

[19, Corollary 4.2] (see Theorem 5.3) one gets that

$$H_{dR}^2(X; \mathbb{C}) \simeq \mathbb{R} \left\langle \varphi^{13} + \varphi^{\bar{1}\bar{3}}, i \left(\varphi^{13} - \varphi^{\bar{1}\bar{3}} \right), \varphi^{23} + \varphi^{\bar{2}\bar{3}}, i \left(\varphi^{23} - \varphi^{\bar{2}\bar{3}} \right), \right. \\ \left. \varphi^{1\bar{2}} - \varphi^{2\bar{1}}, i \left(\varphi^{1\bar{2}} + \varphi^{2\bar{1}} \right), i \varphi^{1\bar{1}}, i \varphi^{2\bar{2}} \right\rangle \otimes_{\mathbb{R}} \mathbb{C},$$

where we have listed the harmonic representatives with respect to the metric $g := \sum_{h=1}^3 \varphi^h \odot \bar{\varphi}^h$ instead of their classes. Set

$$\varphi^1 := e^1 + i e^2, \quad \varphi^2 := e^3 + i e^4, \quad \varphi^3 := e^5 + i e^6;$$

then

$$d e^5 = -e^{13} + e^{24}, \quad d e^6 = -e^{14} - e^{23},$$

the other differentials being zero. Therefore,

$$H_{dR}^2(X; \mathbb{R}) \simeq \mathbb{R} \left\langle e^{15} - e^{26}, e^{16} + e^{25}, e^{35} - e^{46}, e^{36} + e^{45}, e^{13} + e^{24}, e^{23} - e^{14}, e^{12}, e^{34} \right\rangle.$$

Set

$$v_1 := e^{15} - e^{26}, \quad v_2 := e^{16} + e^{25}, \quad v_3 := e^{35} - e^{46}, \quad v_4 := e^{36} + e^{45}, \\ v_5 := e^{13} + e^{24}, \quad v_6 := e^{23} - e^{14}, \quad v_7 := e^{12}, \quad v_8 := e^{34}.$$

Consider the almost-Kähler structure (J, ω, g) on X defined by

$$J e^1 := -e^6, \quad J e^2 := -e^5, \quad J e^3 := -e^4, \quad \omega := e^{16} + e^{25} + e^{34}.$$

We easily get that

$$\mathbb{R} \langle v_2, v_3 + v_5, v_4 - v_6, v_8 \rangle \subseteq H_J^+(X), \quad \mathbb{R} \langle v_1, v_3 - v_5, v_4 + v_6 \rangle \subseteq H_J^-(X).$$

We claim that the previous inclusions are actually equalities, and in particular that J is a non- \mathcal{C}^∞ -full almost-Kähler structure on X . Indeed, we first note that by [15, Proposition 3.2] or [11, Proposition 2.8] (see Proposition 2.1), J is \mathcal{C}^∞ -pure, since it admits a symplectic structure compatible with it. Moreover, we recall that a \mathcal{C}^∞ -full almost-complex structure is also pure by [20, Proposition 2.30], and therefore it also satisfies that

$$(1) \quad H_J^{(3,1),(1,3)}(X)_{\mathbb{R}} \cap H_J^{(2,2)}(X)_{\mathbb{R}} = \{0\};$$

see [1, Theorem 2.4]. Therefore, our claim reduces to proving that J does not satisfy (1). Note that

$$\begin{aligned} [e^{3456}] &= [e^{3456} - d e^{135}] = [e^{3456} + e^{1234}] \\ &= [e^{3456} + d e^{135}] = [e^{3456} - e^{1234}] \end{aligned}$$

and that $e^{3456} + e^{1234} \in \wedge_J^{(3,1),(1,3)}(X)_{\mathbb{R}}$ while $e^{3456} - e^{1234} \in \wedge_J^{(2,2)}(X)_{\mathbb{R}}$, and so $H_J^{(3,1),(1,3)}(X)_{\mathbb{R}} \cap H_J^{(2,2)}(X)_{\mathbb{R}} \ni [e^{3456}]$; therefore (1) does not hold, and hence J is not \mathcal{C}^∞ -full.

Let \mathcal{L}_ω be the Lefschetz type operator of the almost-Kähler structure (J, ω, g) . Then, we have $\mathcal{L}_\omega(e^{12}) = e^{1234} = d(e^{245})$; i.e., \mathcal{L}_ω does not take g -harmonic 2-forms in g -harmonic 4-forms.

Hence, we have proved the following result.

Proposition 4.1. *Let $X := \mathbb{Z}[i]^3 \setminus (\mathbb{C}^3, *)$ be the real manifold underlying the Iwasawa manifold. Then there exists an almost-Kähler structure (J, ω, g) on X which is C^∞ -pure and non- C^∞ -full.*

Furthermore, the Lefschetz type operator of the almost-Kähler structure (J, ω, g) does not take g -harmonic 2-forms to g -harmonic 4-forms.

5. ALMOST-COMPLEX MANIFOLDS
WITH LARGE ANTI-INVARIANT COHOMOLOGY

Given an almost-complex structure J on a compact manifold M , it is natural to ask how large the cohomology subgroup $H_J^{(2,0),(0,2)}(M)_\mathbb{R}$ can be. In this direction, Drăghici, Li, and the third author raised the following question in [12].

Question 5.1 ([12, Conjecture 2.5]). *Are there compact 4-dimensional manifolds M endowed with non-integrable almost-complex structures J such that $\dim_\mathbb{R} H_J^-(M) \geq 3$?*

Here we present a 1-parameter family $\{J_t\}_t$ of (non-integrable) almost-complex structures on the 6-dimensional torus \mathbb{T}^6 having $h_{J_t}^- := \dim_\mathbb{R} H_{J_t}^-(\mathbb{T}^6)_\mathbb{R}$ greater than 3; see also [1, §4]. For t small enough, set $\alpha_t := \alpha_t(x^3) \in C^\infty(\mathbb{T}^6)$ such that $\alpha_0(x^3) \equiv 1$ and set

$$\varphi_t^1 := dx^1 + i \alpha_t dx^4, \quad \varphi_t^2 := dx^2 + i dx^5, \quad \varphi_t^3 := dx^3 + i dx^6;$$

therefore, the structure equations are

$$d\varphi_t^1 = i d\alpha_t \wedge dx^4, \quad d\varphi_t^2 = 0, \quad d\varphi_t^3 = 0.$$

Straightforward computations give that the J -anti-invariant d-closed 2-forms are of the type

$$\begin{aligned} \psi = & \frac{C}{\alpha_t} (dx^{13} - \alpha_t dx^{46}) + D (dx^{16} - \alpha_t dx^{34}) + E (dx^{23} - dx^{56}) \\ & + F (dx^{26} - dx^{35}), \end{aligned}$$

where $C, D, E, F \in \mathbb{R}$ (we shorten $dx^j \wedge dx^k$ by dx^{jk}). Moreover, the forms $dx^{23} - dx^{56}$ and $dx^{26} - dx^{35}$ are clearly harmonic with respect to the standard flat metric $\sum_{j=1}^6 dx^j \otimes dx^j$, while the classes of $dx^{16} - \alpha_t dx^{34}$ and $dx^{13} - \alpha_t dx^{46}$ are non-zero, their harmonic parts being non-zero. Hence, we get that $h_{J_0}^- = 6$ and

$$h_{J_t}^- = 4 \quad \text{for small } t \neq 0.$$

In the general case, we ask the following natural question.

Question 5.2. *Are there examples of non-integrable almost-complex structures J on a compact $2n$ -dimensional manifold with $\dim_\mathbb{R} H_J^-(M) > n(n-1)$?*

Now consider a solvmanifold $M = \Gamma \backslash G$, namely, a compact quotient of a connected simply-connected solvable Lie group G by a co-compact discrete subgroup Γ . Denote the Lie algebra naturally associated to G by \mathfrak{g} , and consider $(\wedge^\bullet \mathfrak{g}^*, d)$ to be the subcomplex of the de Rham complex $(\wedge^\bullet M, d)$ given by the left-invariant differential forms. The following result by Nomizu [25] and Hattori [19] holds.

Theorem 5.3 ([25, Theorem 1], [19, Theorem 4.2]). *Let M be a nilmanifold or, more in general, a completely-solvable solvmanifold. Then $H^\bullet(\wedge^\bullet \mathfrak{g}^*, d) \simeq H_{dR}^\bullet(M; \mathbb{R})$.*

Let J be a left-invariant almost-complex structure on M , namely, an almost-complex structure on M induced by an almost-complex structure on G that is invariant under the action of G on itself given by left-translations. Given $p, q \in \mathbb{N}$, denote by

$$H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} := \left\{ \mathfrak{a} = [\alpha] \in H^\bullet(\wedge^\bullet \mathfrak{g}^*, d) : \alpha \in \wedge_J^{(p,q),(q,p)} \mathfrak{g}^* \right\} \subseteq H_{dR}^\bullet(M; \mathbb{R})$$

the subgroup (see, e.g., [8, Lemma 9]) of $H_{dR}^\bullet(M; \mathbb{R})$ that consists of classes admitting a left-invariant representative of type $(p, q) + (q, p)$, where $\wedge_J^{(p,q),(q,p)} \mathfrak{g}^* := (\wedge^{p,q}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^* \oplus \wedge^{q,p}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^*) \cap \wedge^\bullet \mathfrak{g}^*$.

Using Belgun’s symmetrization trick [4, Theorem 7], one can prove the following Nomizu type result, which relates the subgroups $H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}$ with their left-invariant part $H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}}$.

Theorem 5.4. *Let $M = \Gamma \backslash G$ be a solvmanifold endowed with a left-invariant almost-complex structure J , and denote the Lie algebra naturally associated to G by \mathfrak{g} . For any $p, q \in \mathbb{N}$, the map*

$$j : H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \rightarrow H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}$$

induced by left-translations is injective, and if $H_{dR}^\bullet(\wedge^\bullet \mathfrak{g}^, d) \simeq H_{dR}^\bullet(M; \mathbb{R})$ (for instance, if M is a completely-solvable solvmanifold), then $j : H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \rightarrow H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}$ is in fact an isomorphism.*

Proof. Since J is left-invariant, left-translations induce the map $j : H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \rightarrow H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}$.

Since by J. Milnor’s Lemma [24, Lemma 6.2] G is unimodular, one can take in particular a bi-invariant volume form η on M such that $\int_M \eta = 1$. Consider the Belgun symmetrization map in [4, Theorem 7], namely,

$$\mu : \wedge^\bullet M \rightarrow \wedge^\bullet \mathfrak{g}^*, \quad \mu(\alpha) := \int_M \alpha \lrcorner_m \eta(m).$$

Since μ commutes with d by [4, Theorem 7], it induces the map $\mu : H_{dR}^\bullet(M; \mathbb{R}) \rightarrow H^\bullet(\wedge^\bullet \mathfrak{g}^*, d)$, and since μ commutes with J , it preserves the bi-graduation; therefore it induces the map $\mu : H_J^{(p,q),(q,p)}(M) \rightarrow H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}}$. Moreover, since μ is the identity on the space of left-invariant forms by [4, Theorem 7], we get the commutative diagram

$$\begin{array}{ccccc} H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} & \xrightarrow{j} & H_J^{(p,q),(q,p)}(M)_{\mathbb{R}} & \xrightarrow{\mu} & H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \\ & & \searrow & \swarrow & \\ & & \text{id} & & \end{array}$$

hence $j : H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \rightarrow H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}$ is injective and $\mu : H_J^{(p,q),(q,p)}(M)_{\mathbb{R}} \rightarrow H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}}$ is surjective.

Furthermore, when $H^\bullet(\wedge^\bullet \mathfrak{g}^*, d) \simeq H_{dR}^\bullet(M; \mathbb{R})$ (for instance, when M is a completely-solvable solvmanifold, by Hattori’s theorem [19, Theorem 4.2]; see Theorem 5.3), since $\mu \lrcorner_{\wedge^\bullet \mathfrak{g}^*} = \text{id} \lrcorner_{\wedge^\bullet \mathfrak{g}^*}$ by [4, Theorem 7], we get that $\mu : H_{dR}^\bullet(M; \mathbb{R}) \rightarrow H^\bullet(\wedge^\bullet \mathfrak{g}^*, d)$ is the identity map, and hence $\mu : H_J^{(p,q),(q,p)}(M)_{\mathbb{R}} \rightarrow H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}}$ is also injective and hence an isomorphism. \square

In particular, if $M = \Gamma \backslash G$ is a $2n$ -dimensional completely-solvable solvmanifold endowed with a left-invariant almost-complex structure J , then

$$\dim_{\mathbb{R}} H_J^-(M) \leq n(n-1) \quad \text{and} \quad \dim_{\mathbb{R}} H_J^+(M) \leq n^2;$$

this provides a partial negative answer to Question 5.2.

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, LARGO BRUNO PONTECORVO 5, 56127, PISA, ITALY

E-mail address: `angella@mail.dm.unipi.it`

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI PARMA, PARCO AREA DELLE SCIENZE 53/A, 43124 PARMA, ITALY

E-mail address: `adriano.tomassini@unipr.it`

DEPARTMENT OF MATHEMATICS, 1825 EAST HALL, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109

E-mail address: `wy Zhang@umich.edu`