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REGULARITY OF MULTIVARIATE BIRKHOFF INTERPOLATION SCHEMES

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Abstract. In this short note we address a conjecture of Ron-Qing Jia and A. Sharma regarding regularity of certain multivariate Birkhoff interpolation schemes. While originally conjectured for polynomials over reals, the conjecture turns out to be true over the complex field and false over the real field.

1. Introduction

In this short note we will settle a conjecture of Ron-Qing Jia and A. Sharma regarding regularity of certain multivariate Birkhoff interpolation schemes. While originally conjectured for polynomials over the field of real numbers, we will prove the complex version of the conjecture and provide a counterexample to the conjecture over the reals.

Some notation and terminology: The symbol $\Bbbk$ will stand for either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. We let $\Bbbk[\mathbf{x}] := \Bbbk\,[x_1, \dots, x_d]$ be the algebra of polynomials in d variables with coefficients in $\Bbbk$. Thus every $p \in \Bbbk[\mathbf{x}]$ could be written as a finite sum

$$p(\mathbf{x}) = \sum \hat{p}(\alpha)\mathbf{x}^\alpha,$$

where $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$, $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and the coefficients $\hat{p}(\alpha) \in \Bbbk$. For every $p \in \Bbbk[\mathbf{x}]$ we use $\bar{p}$ to denote the complex conjugate of the polynomial p. Thus if $p(\mathbf{x}) = \sum \hat{p}(\alpha)\mathbf{x}^\alpha$, then $\bar{p}(\mathbf{x}) = \sum \overline{\hat{p}}(\alpha)\mathbf{x}^\alpha$, where $\bar{c}$ is the complex conjugate of c. We use D_j to denote the partial derivative with respect to x_j, and for every $p \in \Bbbk[\mathbf{x}]$ the symbol $p(D)$ denotes the differential operator

$$p\,(D_1, \dots, D_d) = \sum \hat{p}(\alpha) D_1^{\alpha_1} \dots D_d^{\alpha_d}$$

acting on $\Bbbk[\mathbf{x}]$. Finally for $\mathbf{z} \in \Bbbk^d$ we use $\delta_\mathbf{z}$ to denote the point-evaluation functional: $\delta_\mathbf{z}(p) = p(\mathbf{z})$. Following [2], a subspace $P \subset \Bbbk[\mathbf{x}]$ will be called scale-invariant if for any $p \in P, t \in \Bbbk, p_t \in P$ where $p_t(\mathbf{x}) := p(t\mathbf{x})$.

Let Q be a subspace of $\Bbbk[\mathbf{x}]$ and let $P_1, \dots, P_k$ be subspaces of Q. As in [2], for a given sequence of points $\mathcal{Z} = (\mathbf{z}_1, \dots, \mathbf{z}_k) \subset \Bbbk^d$ we consider the problem of interpolation from the space Q with "interpolation conditions"

$$A(\mathcal{Z}, P_j, j = 1, \dots, k) := \sum_{j=1}^{k} \{\delta_{\mathbf{z}_j} \circ \bar{p}(D) : p \in P_j\}; \tag{1.1}$$

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i.e., for any $f \in \Bbbk[\mathbf{x}]$ we wish to find $q \in Q$ such that $\lambda(f) = \lambda(q)$ for all $\lambda \in A(\mathcal{Z}, P_j, j = 1, \dots, k)$. We follow G. Lorentz [3] (cf. also [2]) and say that the interpolation scheme $(Q, P_1, \dots, P_k)$ is *regular* if the interpolation problem from Q with interpolation conditions $A(\mathcal{Z}, P_j, j = 1, \dots, k)$ is uniquely solvable for any sequence of *pairwise distinct* points $\mathcal{Z}$.

Starting with the observation that the condition

$$\dim Q = \sum_{j=1}^{k} \dim P_j \tag{1.2}$$

is necessary but not sufficient for regularity of the scheme $(Q, P_1, \dots, P_k)$, Jia and Sharma [2] proved that, for scale-invariant spaces Q and $P_1, \dots, P_k$ such that $P_i \subset Q$, the condition

$$Q = \bigoplus_{j=1}^{k} P_j \tag{1.3}$$

implies the regularity of the interpolation scheme $(Q, P_1, \dots, P_k)$ and conjectured that the converse also holds:

Conjecture 1.1. *Let Q and $P_1, \dots, P_k$ be scale-invariant spaces of polynomials on $\Bbbk^d$ such that $P_i \subset Q$. Then for $d \geq 2$, the interpolation scheme is regular if and only if* (1.3) *holds.*

In the next section we will affirm the conjecture for $\Bbbk = \mathbb{C}$ and present a counterexample for $\Bbbk = \mathbb{R}$.

The proof that (1.3) implies regularity of the scheme $(Q, P_1, \dots, P_k)$ in the complex case is identical to the proof given in [2] in the real case since $\bar{h}(D)h$ is a positive constant for any nontrivial homogeneous polynomial $h \in \mathbb{C}[\mathbf{x}]$. It remains to prove the converse. In fact we will do a little bit more: we will prove that for $\Bbbk = \mathbb{C}$ and $d \geq 2$, regularity of the scheme $(Q, P_1, \dots, P_k)$ implies (1.3) even without the assumption of scale-invariance.

Observe that if we forgo the assumption of scale invariance, condition (1.3) is not a sufficient condition for regularity even in one variable. Indeed consider the scheme (Q, P_1, P_2) with $Q = \operatorname{span}\{x, x^2\}$, $P_1 = \operatorname{span}\{x\}$ and $P_2 = \operatorname{span}\{x + x^2\}$. This scheme clearly satisfies (1.3), yet the interpolation problem with $\mathbf{z}_1 = 1$, $\mathbf{z}_2 = 0$ does not have a unique solution.

2. Main results

We will need the following rudimentary facts from algebraic geometry: A subset $\mathcal{V} \subset \Bbbk^N$ is called an affine variety if there exists a finite set of polynomials $f_1, \dots, f_s$ in $\Bbbk[x_1, \dots, x_N]$ such that

$$\mathcal{V} = \{\mathbf{x} \in \Bbbk^N : f_1(\mathbf{x}) = \cdots = f_s(\mathbf{x}) = 0\};$$

the union of affine varieties is an affine variety (cf. [1, p. 188, Theorem 15]), and with every affine variety $\mathcal{V}$ we can associate the integer $\dim \mathcal{V}$ (cf. [1, Chapter 9]).

Lemma 2.1. *For $j = 1, \dots, k$ let $\mathbf{z}_j = (z_{j,1}, \dots, z_{j,d}) \in \mathbb{C}^d$. Let*

$$\mathcal{U} := \left\{(z_{1,1}, \dots, z_{1,d}, \dots, z_{k,1}, \dots, z_{k,d}) \in \mathbb{C}^{kd} : \mathbf{z}_j \neq \mathbf{z}_m \text{ for } j \neq m\right\}. \tag{2.1}$$

Then $\mathcal{W} := \mathcal{U}^c = \mathbb{C}^{kd} \backslash \mathcal{U}$ *is an affine variety in* $\mathbb{C}^{kd}$ *and*

$$\dim \mathcal{W} := kd - d.$$

Proof. For every $j \neq m$, $j, m = 1, \ldots, k$, consider the linear subspace (hence an affine variety) $\mathcal{W}_{j,m} \subset \mathbb{C}^{kd}$ defined by the d linear equations: $\mathbf{z}_j - \mathbf{z}_m = 0$; i.e.,

$$\mathcal{W}_{j,m} = \left\{ (z_{1,1}, \ldots, z_{1,d}, \ldots, z_{k,1}, \ldots, z_{k,d}) \in \mathbb{C}^{kd} : z_{j,i} - z_{m,i} = 0,\ i = 1, \ldots, d \right\}.$$

Then $\dim \mathcal{W}_{j,m} = nd - d$ for $j \neq m$. Observe that $\mathcal{W}$ is a finite union of these varieties:

$$\mathcal{W} = \bigcup_{j,m=1}^{k} \mathcal{W}_{j,m};$$

hence $\mathcal{W}$ is an affine variety and (cf. [1, p. 462, Corollary 9])

$$\dim \mathcal{W} = \max \left\{ \dim \mathcal{W}_{j,m},\ j, m = 1, \ldots, k \right\} = kd - d.$$

□

Theorem 2.2. *Let* $d \geq 2$ *and* $Q, P_1, \ldots, P_k \subset \mathbb{C}[x_1, \ldots, x_d]$ *be spaces of polynomials (scale-invariant or not) such that* $P_j \subset Q$. *If the interpolation scheme* $(Q, P_1, \ldots, P_k)$ *is regular, then* (1.3) *holds.*

Proof. Let $N := \dim Q$ and $N_j = \dim P_j$. Let $(q_1, \ldots, q_N)$ be a basis in Q and let $(p_{j,m}, m = 1, \ldots, \dim P_j)$ be a basis in P_j. For any sequence of (not necessarily distinct) points $\mathcal{Z} = (\mathbf{z}_1, \ldots, \mathbf{z}_k) \subset \Bbbk^d$ consider the sequence of linear functionals

$$\left(\delta_{\mathbf{z}_1} \circ \bar{p}_{1,1}(D), \ldots, \delta_{\mathbf{z}_1} \circ \bar{p}_{1,N_1}(D), \ldots, \delta_{\mathbf{z}_k} \circ \bar{p}_{k,1}(D), \ldots, \delta_{\mathbf{z}_k} \circ \bar{p}_{k,N_k}(D) \right).$$

Since the scheme $(Q, P_1, \ldots, P_k)$ is regular, (1.2) holds and there are exactly N terms (functionals) in this sequence. We will denote them by $(\lambda_1[\mathcal{Z}], \ldots, \lambda_N[\mathcal{Z}])$. The interpolation problem with the interpolation conditions (1.1) is uniquely solvable if and only if the determinant

$$f(\mathbf{z}_1, \ldots, \mathbf{z}_k) := \det \left(\lambda_j[\mathcal{Z}](q_k) \right)_{j,k=1}^{N} \tag{2.2}$$

is not zero. This determinant is a polynomial in $d \times k$ variables:

$$(z_{1,1}, \ldots, z_{1,d}, \ldots, z_{k,1}, \ldots, z_{k,d}) \in \mathbb{C}^{kd},$$

where

$$\mathbf{z}_j = (z_{j,1}, \ldots, z_{j,d}),\ j = 1, \ldots, k. \tag{2.3}$$

Since, by the regularity assumption, the interpolation problem is uniquely solvable for some $\mathcal{Z}$, it follows that the polynomial (2.2) is not identically zero. We now prove that the regularity assumption implies that the polynomial (2.2) is a nonzero constant, i.e., independent of the choice of $\mathcal{Z}$.

Indeed, if f is not a constant, then the zero-locus of f,

$$\mathcal{V} := \left\{ \mathbf{x} \in \mathbb{C}^{kd} : f(\mathbf{x}) = 0 \right\},$$

is an affine variety, $\mathcal{V} \subset \mathbb{C}^{kd}$ with $\dim \mathcal{V} = kd - 1$ (cf. [1, p. 463, Exercise 2]). The regularity of the scheme $(Q, P_1, \ldots, P_k)$ means that f does not vanish on $\mathcal{U}$ as defined by (2.1); hence

$$\mathcal{V} \subset \mathcal{U}^c = \mathcal{W}. \tag{2.4}$$

Condition (2.4) implies (cf. [1, p. 457, Proposition 1]) that

$$kd - 1 = \dim \mathcal{V} \leq \dim \mathcal{W} = kd - d, \tag{2.5}$$

by Lemma 2.1; this is not possible for $d \geq 2$. Thus the determinant (2.2) is a nonzero constant independent of $\mathcal{Z}$. In particular, it is nonzero when all points in $\mathcal{Z}$ are zeros, i.e., $\mathcal{Z} = (\mathbf{0}, \ldots, \mathbf{0})$, and the interpolation problem is uniquely solvable in this case. This, in particular, implies that the dimension of the space of interpolation conditions $\delta_{\mathbf{0}} \circ \{\bar{p}(D) : p \in \left(+_{j=1}^{k} P_j\right)\}$ equals $\dim Q$, which implies that

$$\dim Q \leq \dim \left(+_{j=1}^{k} P_j\right) \leq \sum_{j=1}^{k} \dim P_j = \dim Q, \tag{2.6}$$

where the equality is by (1.2). Therefore $\dim \left(+_{j=1}^{k} P_j\right) = \sum_{j=1}^{k} \dim P_j$; thus $+_{j=1}^{k} P_j = \bigoplus_{j=1}^{k} P_j$, and, since $P_j \subset Q$, (1.2) implies (1.3). □

Interestingly, for $d = 1$ the inequality (2.5) does not provide a contradiction. There is a good reason for it: As mentioned in [2], in one variable the regularity of a scheme does not imply (1.3). Indeed consider the scheme $(Q, P_1, \ldots, P_k)$ with $Q \subset \Bbbk[x]$ being the polynomials of degree at most $k - 1$ and $P_j = \operatorname{span}\{1\}$ for all $j = 1, \ldots, k$. The interpolation problem is reduced to the standard Lagrange interpolation which, for distinct $z_1, \ldots, z_k \in \Bbbk$, is uniquely solvable. Hence the scheme is regular, yet (1.3) obviously fails. For a change, this is a result that holds true in the multivariate case and fails in the univariate setting.

The proof of the theorem shows that the following corollary holds:

Corollary 2.3. *The following are equivalent:*

(i) *The interpolation scheme* $(Q, P_1, \ldots, P_k)$ *is regular.*
(ii) *The determinant* (2.2) *is a nonzero constant.*
(iii) *The problem of interpolation from* Q *with interpolation conditions* (1.1) *is uniquely solvable for any* $\mathcal{Z}$.

Next we will provide a counterexample to the conjecture in the real case:

Theorem 2.4. *Conjecture* 1.1 *is false for* $\Bbbk = \mathbb{R}$ *and* $d = 2$.

Proof by example. Consider the interpolation scheme (Q, P_1, P_2) in $\mathbb{R}[x, y]$ with $Q = \operatorname{span}\left\{1, x, y, y^2 - x^2\right\}$, $P_1 = \operatorname{span}\{1, x\}$ and $P_2 = \operatorname{span}\{1, y\}$. Clearly (1.3) does not hold, yet for any two distinct points $\mathbf{z}_1 = (z_{1,1}, z_{1,2})$ and $\mathbf{z}_2 = (z_{2,1}, z_{2,2})$ the determinant

$$\det\left(\lambda_j[\mathcal{Z}]\left(q_k\right)\right)$$

of the 4×4 matrix of interpolation with $q_1 = 1$, $q_2 = x$, $q_3 = y$, $q_4 = y^2 - x^2$ and $\lambda_1[\mathcal{Z}] = \delta_{z_1}, \lambda_2[\mathcal{Z}] = \delta_{z_1} \circ D_x, \lambda_3[\mathcal{Z}] = \delta_{z_2}, \lambda_4[\mathcal{Z}] = \delta_{z_2} \circ D_y$ is

$$\det \begin{pmatrix} 1 & z_{1,1} & z_{1,2} & z_{1,2}^2 - z_{1,1}^2 \\ 0 & 1 & 0 & -2z_{1,1} \\ 1 & z_{2,1} & z_{2,2} & z_{2,2}^2 - z_{2,1}^2 \\ 0 & 0 & 1 & 2z_{2,2} \end{pmatrix} = \left(z_{1,1} - z_{2,1}\right)^2 + \left(z_{1,2} - z_{2,2}\right)^2 \neq 0$$

if $\mathbf{z}_1 \neq \mathbf{z}_2$ and both $\mathbf{z}_1$ and $\mathbf{z}_2$ have real coordinates. □

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References

[1] David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, An introduction to computational algebraic geometry and commutative algebra. Second edition. Undergraduate Texts in Mathematics, Springer, New York, 1997. MR1417938 (97h:13024)

[2] Rong Qing Jia and A. Sharma, *Solvability of some multivariate interpolation problems*, J. Reine Angew. Math. **421** (1991), 73–81, DOI 10.1515/crll.1991.421.73. MR1129576 (92h:41007)

[3] G. G. Lorentz, *Solvability of multivariate interpolation*, J. Reine Angew. Math. **398** (1989), 101–104, DOI 10.1515/crll.1989.398.101. MR998474 (90j:41004)

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