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LINEAR INDEPENDENCE OF CERTAIN LAMBERT SERIES

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ABSTRACT. We prove that if $q \neq 0, \pm 1$ and $\ell \geq 1$ are fixed integers, then the numbers

$$1, \quad \sum_{n=1} \frac{1}{q^n - 1}, \quad \sum_{n=1}^{\infty} \frac{1}{q^{n^2} - 1}, \quad \ldots, \quad \sum_{n=1}^{\infty} \frac{1}{q^{n^\ell} - 1}$$

are linearly independent over $\mathbb{Q}$. This generalizes a result of Erdős, who treated the case $\ell = 1$. The method is based on the original approaches of Chowla and Erdős, together with some results about primes in arithmetic progressions with large moduli of Ahlford, Granville and Pomerance.

1. INTRODUCTION

Let q be an integer with $|q| \geq 2$. In 1948, Erdős [7] proved that the q-adic expansion of the number

$$\sum_{n=1}^{\infty} \frac{1}{q^n - 1} \tag{1}$$

contains any arbitrarily long string of zeros (without being identically zero from some point on), and so he deduced the irrationality of the values of Lambert series

$$f(z) := \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n} = \sum_{n=1}^{\infty} d(n) z^n \quad (|z| < 1)$$

at the rational numbers $z = q^{-1}$, where $d(n)$ denotes the number of positive divisors of n. Later, Erdős and Graham [8] pointed out that the number $\sum_{n=1}^{\infty} 1/(2^n - 3)$ was not known to be irrational. Borwein [2], [3] answered this question in the affirmative by proving the irrationality of the numbers $\sum_{n=1}^{\infty} 1/(q^n - r)$, where r is a rational number with $r \neq 0, q^n$ $(n \geq 1)$. The second author (see [10]) modified Borwein's contour-integral constructions from [3] and showed the linear independence of the three numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{(q^n - 1)}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{(q^n - 1)}$$

over $\mathbb{Q}$. Later, Bundschuh and Väänänen [4] and Zudilin [12], independently, gave quantitative results concerning the linear independence measure for the three above-mentioned numbers over $\mathbb{Q}$. The best known upper bound of the irrationality measure of the number shown in (1) is $2.46497\ldots$ and is due to Zudilin [11].

In this paper, we generalize the original result of Erdős by proving the linear independence over $\mathbb{Q}$ of a set of certain Lambert series containing the number shown

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in (1). The method is based on the original approaches of Chowla [5] and Erdős [7], together with a deep result about primes in arithmetic progressions with large moduli of Ahlford, Granville, and Pomerance [1], which played an important role in their celebrated proof of the existence of infinitely many Carmichael numbers. Our result is the following:

Theorem 1. *Let q be an integer with $|q| \geq 2$. For any integer $\ell \geq 1$, the $\ell + 1$ numbers*

$$1, \quad \sum_{n=1}^{\infty} \frac{1}{q^n - 1}, \quad \sum_{n=1}^{\infty} \frac{1}{q^{n^2} - 1}, \quad \ldots, \quad \sum_{n=1}^{\infty} \frac{1}{q^{n^\ell} - 1}$$

are linearly independent over $\mathbb{Q}$. In particular, the number

$$\sum_{n=1}^{\infty} \frac{1}{q^{n^\ell} - 1}$$

is irrational.

2. Lemmas

For any integer $\ell \geq 1$, we have

$$\sum_{n=1}^{\infty} \frac{1}{q^{n^\ell} - 1} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{q^{kn^\ell}} = \sum_{n=1}^{\infty} \frac{a_\ell(n)}{q^n}, \tag{2}$$

where $a_\ell(n) := \sum_{d^\ell | n} 1$. Then the arithmetical function $a_\ell(n)$ is multiplicative and written by

$$a_\ell(n) = \prod_i \left(1 + \left\lfloor \frac{e_i}{\ell} \right\rfloor\right) \tag{3}$$

for a positive integer n having the prime factorization $\prod_i p_i^{e_i}$ (cf. [6]).

For a positive real number x, we write $\pi(x)$ and $\pi(x; d, a)$ for the number of primes $p \leq x$ and the number of primes $p \leq x$ with $p \equiv a \pmod{d}$, respectively. The following lemma is quite technical and played an important role in the proof of the existence of infinitely many Carmichael numbers from [1]. It basically says that arithmetic progressions of numbers up to y with the first term a coprime to the modulus d contain a number of primes at least as large as the expected one when the modulus d is in suitable large ranges with respect to y (say $d \leq y^{2/3}$), except for a set of potentially bad values of the modulus d, which must be divisible by some number from a set of bounded cardinality all of whose elements are large. The next formulation is taken straight from [1]. The notation $\phi(m)$ stands for the Euler function of the positive integer m.

Lemma 1. *There exist numbers x_0 and D such that the inequality*

$$\pi(y; d, a) \geq \frac{\pi(y)}{2\phi(d)}$$

holds whenever a and d are coprime integers, $x > x_0$, $d \leq \min\{x^{2/3}, y/x^{1/3}\}$, and d is not divisible by any number from $\mathcal{D}(x)$, a set of at most D elements each of which exceeds $\log x$.

Applying Lemma 1, we obtain the following result.

Lemma 2. *There exists an integer n_0 with the following property. Let $n \geq n_0$ be an integer. Let A and a be positive integers such that A is odd, $2A$ and a are coprime, and $2A \leq \sqrt{n}$. If $2A$ is not divisible by any number from $\mathcal{D}(n^{3/2})$, then we have*

$$\pi(An/2^{\ell+5}; 2A, a) > \frac{n}{2^{\ell+7}\log n}.$$

Proof. By Lemma 1 with $d = 2A$, there exist numbers x_0 and D such that the inequality

$$\pi(y; 2A, a) \geq \frac{\pi(y)}{2\phi(2A)}$$

holds if $x > x_0$, $2A \leq \min\{x^{2/3}, y/x^{1/3}\}$, and $2A$ is not divisible by any number from $\mathcal{D}(x)$. Put $n_0 := \max\{x_0, 2^{3\ell+18}\}$. Letting $n \geq n_0$, $x := n^{3/2}$, and $y := An/2^{\ell+5}$, we obtain the inequality

$$2A \leq \min\{x^{2/3}, y/x^{1/3}\}$$

because of our assumption that $2A \leq \sqrt{n}$. Hence, Lemma 1 tells us that

$$\begin{aligned} \pi(An/2^{\ell+5}; 2A, a) &\geq \frac{\pi(An/2^{\ell+5})}{2\phi(2A)} \geq \frac{An}{2^{\ell+6}\phi(A)\log(An/2^{\ell+5})} \\ &> \frac{n}{2^{\ell+7}\log n}, \end{aligned}$$

where we used the fact that $\pi(x) > x/\log x$ for all $x \geq 17$ (see Corollary 1 in [9]), which for us holds because $An/2^{\ell+5} \geq n_0/2^{\ell+5} \geq 2^{2\ell+13} > 17$, as well as the fact that $An/2^{\ell+5} < An < n^2$; therefore $\log(An/2^{\ell+5}) < 2\log n$. □

In what follows, let A and B be positive coprime integers such that A is odd and $1 \leq B \leq A$. Define

$$S_{A,B}(n) := \{Ai + B \mid i = 1, 2, \ldots, n\}.$$

Lemma 3. *Let n_0 be the integer appearing in Lemma 2 and let $n \geq n_0$ be an integer with $2A \leq \sqrt{n}$. Assume that $2A$ is not divisible by any number from $\mathcal{D}(n^{3/2})$. Then for each integer $j = 0, 1, 2, \ldots, \ell - 1$, the set $S_{A,B}(n)$ contains at least $\lfloor n/(2^{\ell+7}\log n)\rfloor$ numbers m (depending on j) such that if $j = 0$, then*

$$a_1(m) = 2, \quad a_s(m) = 1 \quad \textit{for} \quad s = 2, 3, \ldots, \ell,$$

and if $j \geq 1$, then

$$a_1(m) = 2(j+1), \quad a_s(m) = \begin{cases} 1 + \lfloor j/s \rfloor & \textit{for} \quad s = 2, 3, \ldots, j, \\ 1 & \textit{for} \quad s = j+1, \ldots, \ell. \end{cases}$$

Proof. We fix $j \geq 0$. Since A is odd, there exists an integer $b = b(j)$ such that

$$2^j(2b+1) \equiv B \pmod{A} \quad 1 \leq b \leq A. \tag{4}$$

Note that the integers $2A$ and $2b+1$ are coprime because so are A and B. Consider the numbers

$$m_i = m_i(j) := 2^j(2Ai + 2b + 1), \quad i = 0, 1, \ldots, \lfloor n/2^{\ell+5}\rfloor,$$

such that $2Ai + 2b + 1$ is prime. Since

$$2A\lfloor n/2^{\ell+5}\rfloor + 2b + 1 > An/2^{\ell+5},$$

it follows, from Lemma 2, that the number of such primes is at least

$$\pi(An/2^{\ell+5}, 2A, 2b+1) > \frac{n}{2^{\ell+7}\log n}.$$

For such numbers m_i, we have, by (3), that if $j = 0$, then

$$a_1(m) = 2, \quad a_s(m) = 1 \quad \text{for} \quad s = 2, 3, \ldots, \ell,$$

and if $j \geq 1$, then

$$a_1(m) = 2(j+1), \quad a_s(m) = \begin{cases} 1 + \lfloor j/s \rfloor & \text{for} \quad s = 2, 3, \ldots, j, \\ 1 & \text{for} \quad s = j+1, \ldots, \ell. \end{cases}$$

On the other hand, since $n > 4^{\ell+2}$, we have

$$m_i = A(2^{j+1}i) + 2^j(2b+1) \leq An + B.$$

Therefore, by the congruence (4), we see that $m_i \in S_{A,B}(n)$ for every $i \geq 1$ and for every $j = 0, 1, 2, \ldots, \ell - 1$. The proof of Lemma 3 is completed. □

We next modify [6, Lemma 7] as follows.

Lemma 4. *Suppose that $n \geq 2A$. Then for any $s = 1, 2, 3, \ldots, \ell$, we have*

$$\sum_{i=1}^{n} a_s(Ai + B) \leq 8n \log n.$$

Proof. We have

$$\sum_{i=1}^{n} a_s(Ai+B) = \sum_{i=1}^{n} \sum_{d^s \mid Ai+B} 1 = \sum_{d=1}^{\lfloor (An+B)^{1/s} \rfloor} \left(\sum_{\substack{1 \leq i \leq n \\ d^s \mid Ai+B}} 1 \right). \tag{5}$$

Suppose that d^s divides $Ai+B$. Since A and B are coprime, so are A and d. Hence, we have $i \equiv -BA^{-1} \pmod{d^s}$, and so

$$\sum_{\substack{1 \leq i \leq n \\ d^s \mid Ai+B}} 1 \leq 1 + \frac{n}{d^s}. \tag{6}$$

By the assumptions $B \leq A \leq n$ and $2A \leq n$, we see that $\sqrt{An+B} \leq n$. Hence, we have, by (5) for $s \geq 2$,

$$\begin{aligned} \sum_{i=1}^{n} a_s(Ai+B) &\leq \sum_{d=1}^{\lfloor (An+B)^{1/s} \rfloor} \left(1 + \frac{n}{d^s}\right) \leq (An+B)^{1/s} + n \sum_{d=1}^{\lfloor (An+B)^{1/s} \rfloor} d^{-s} \\ &\leq \sqrt{An+B} + n \sum_{d=1}^{\sqrt{An+B}} d^{-2} \leq n + \frac{n\pi^2}{6} \\ &\leq 3n, \end{aligned}$$

which is even better than what is required. For $s = 1$, we modify the definition of the function $a_1(m)$ by looking only at the divisors d of m which are at most $\sqrt{m}$. That is,

$$a_1(m) = \sum_{d \mid m} 1 \leq 2 \sum_{d \leq \sqrt{m}} 1.$$

Inserting this into formula (5) and using inequality (6), we get

$$\begin{aligned}\sum_{i=1}^{n} a_1(Ai+B) &\le 2\sum_{d=1}^{\sqrt{An+B}} \sum_{\substack{1\le i\le n\\ d|Ai+B}} 1 \le 2\sum_{d=1}^{\sqrt{An+B}} \left(1+\frac{n}{d}\right)\\ &\le 2\sum_{d=1}^{\sqrt{An+B}} \frac{2n}{d},\end{aligned}$$

where the last inequality follows because $d \le \sqrt{An+B} \le n$. Thus, for $n \ge 2A > 2B \ge 2$, we have

$$\sum_{i=1}^{n} a_1(Ai+B) \le 4n \sum_{d=1}^{\sqrt{An+B}} \frac{1}{d} \le 4n(1+\log n) < 8n\log n.$$

□

3. Proof of Theorem 1

To prove Theorem 1, we use the following theorem of Duverney [6].

Theorem 2. *Let q be an integer with $|q| \ge 2$ and $\{\theta(n)\}_{n\ge1}$ be a sequence of integers. Assume that there exists a sequence of nonnegative integers $\{n_k\}_{k\ge1}$ with $n_k \ge 2k$ such that, for every k sufficiently large,*

$$\begin{cases} q|\theta(n_k-k+1),\ q^2|\theta(n_k-k+2),\dots,q^{k-1}|\theta(n_k-1),\\ q^{k+1}|\theta(n_k+1),\ q^{k+2}|\theta(n_k+2),\dots,q^{2k}|\theta(n_k+k),\end{cases}$$

and satisfying

$$\lim_{k\to+\infty} \frac{1}{|q|^k} \sum_{n=0}^{\infty} \frac{|\theta(n+n_k+k+1)|}{|q|^n} = 0.$$

Assume that $\sum_{n=0}^{\infty} \frac{\theta(n)}{q^n}$ is convergent and is a rational number. Then $q^k|\theta(n_k)$ for every large k.

Let q and ℓ be as in Theorem 1. Suppose on the contrary that the numbers

$$1,\quad \sum_{n=1}^{\infty}\frac{1}{q^{n^1}-1},\quad \sum_{n=1}^{\infty}\frac{1}{q^{n^2}-1},\quad \dots,\quad \sum_{n=1}^{\infty}\frac{1}{q^{n^\ell}-1}$$

are linearly dependent over $\mathbb{Q}$. Then, by (2), there exist rational integers b_0, $b_1,\dots,b_\ell$, not all zero, such that

$$-b_0 = \sum_{j=1}^{\ell} b_j \left(\sum_{n=1}^{\infty}\frac{1}{q^{n^j}-1}\right) = \sum_{n=1}^{\infty}\frac{\theta(n)}{q^n} \in \mathbb{Z}, \tag{7}$$

with $\theta(n) := b_1a_1(n)+\cdots+b_\ell a_\ell(n)$. In what follows, we denote by $c_1,c_2,\dots$ positive constants dependent on q, ℓ, and b_j, but not on k. Let $p_1,p_2,\dots$ be an increasing sequence of prime numbers. We recall the following upper bound of the nth prime p_n:

$$p_n \le 2n\log n, \tag{8}$$

valid for all $n \geq 6$ (see inequality (3.13) in [9]). Now we choose a sufficiently large k and put

$$t_k = \frac{k(k+1)}{2}, \qquad r_k = t_k + 1.$$

Let $L := \mathrm{lcm}[1, 2, 3, \ldots, \ell]$, $N_k = 2^{k^3}$, $x = N_k^{3/2}$. Let $\mathcal{D} = \mathcal{D}(x)$. For each $d \in \mathcal{D}$, let p_d be some prime factor of d, and let $\mathcal{P} = \{p_d : d \in \mathcal{D}\}$. Clearly, $\mathcal{P}$ has at most D elements. Let $q_1, q_2, \ldots, q_{t_{2k}}$ be the first t_{2k} odd prime numbers greater than $4k^3$ and which do not belong to $\mathcal{P}$. By the Chinese Remainder Theorem, we get a natural number β_k satisfying

$$\begin{cases} \beta_k - k + 1 \equiv q_1^{|q|L-1} & (\mathrm{mod} \quad q_1^{|q|L}), \\ \beta_k - k + 2 \equiv (q_2 q_3)^{|q|L-1} & (\mathrm{mod} \quad (q_2 q_3)^{|q|L}) \\ \vdots & \vdots \\ \beta_k - 1 \equiv (q_{r_{k-2}} \cdots q_{t_{k-1}})^{|q|L-1} & (\mathrm{mod} \quad (q_{r_{k-2}} \cdots q_{t_{k-1}})^{|q|L}), \\ \beta_k + 1 \equiv (q_{r_k} \cdots q_{t_{k+1}})^{|q|L-1} & (\mathrm{mod} \quad (q_{r_k} \cdots q_{t_{k+1}})^{|q|L}) \\ \vdots & \vdots \\ \beta_k + k \equiv (q_{r_{2k-1}} \cdots q_{t_{2k}})^{|q|L-1} & (\mathrm{mod} \quad (q_{r_{2k-1}} \cdots q_{t_{2k}})^{|q|L}). \end{cases} \tag{9}$$

In particular, putting

$$A_k := \prod_{\substack{i=1 \\ i \neq r_{k-1}, \ldots, t_k}}^{t_{2k}} q_i^{|q|L},$$

we obtain an integer β_k uniquely subject to

$$1 \leq \beta_k \leq A_k.$$

Since $q_i > 4k^3$, we have, by using (8),

$$\beta_k \leq A_k \leq \prod_{i=1}^{t_{2k}} p_{i+D+4k^3}^{|q|L} \leq e^{c_1 k^2 \log k}, \tag{10}$$

where we can take $c_1 = 7|q|L$, and then the last inequality holds for all sufficiently large k. Let

$$S(k) := S_{A_k, \beta_k}(N_k) = \{u_{k,i} := A_k i + \beta_k \mid i = 1, 2, \ldots, N_k\}.$$

Define the subsets $T_j(k)$ for $j = 0, 1, \ldots, \ell - 1$ of $S(k)$ by

$$T_0(k) = \left\{ u \in S(k) \;\middle|\; a_1(u) = 2, \quad a_s(u) = 1 \quad \text{for} \quad s = 2, 3, \ldots, \ell \right\}, \tag{11}$$

and for $j = 1, 2, \ldots, \ell - 1$ by

$$T_j(k) = \left\{ u \in S(k) \;\middle|\; \begin{array}{c} a_1(u) = 2(j+1), \quad a_s(u) = 1 + \lfloor j/s \rfloor, \quad 2 \leq s \leq j, \\ a_s(u) = 1, \quad j+1 \leq s \leq \ell \end{array} \right\}. \tag{12}$$

We note that A_k and β_k are coprime. Otherwise, there exists some prime q_i which divides β_k. Hence, by (9), q_i divides a nonzero integer t with $-k+1 \leq t \leq k$. This is impossible since $0 < |t| \leq k < q_i$. Note that A_k is odd. Further, for large k, the hypotheses of Lemma 3 are satisfied. Indeed, the inequality $2A_k \leq \sqrt{N_k}$ clearly holds for large k by inequality (10), while the fact that $2A_k$ is not divisible by

any number from the set $\mathcal{D}(N_k^{3/2})$ follows from the way we have chosen our primes $q_1, \dots, q_{2k}$. By Lemma 3, we see that there exist at least $\left\lfloor \frac{N_k}{2^{\ell+7}\log N_k} \right\rfloor \geq \left\lfloor \frac{N_k}{2^{\ell+7}k^3} \right\rfloor$ numbers in $T_j(k)$ for each $j = 0, 1, 2, \dots, \ell - 1$.

In what follows, we fix j ($j = 0, 1, \dots, \ell - 1$) and estimate an upper bound for

$$\alpha_{k,j} := \sum_{\substack{1 \leq i \leq N_k \\ u_{k,i} \in T_j(k)}} \sum_{m=0}^{3k^3-1} |\theta(m + u_{k,i} + k + 1)|. \tag{13}$$

We see that A_k and $m + \beta_k + k + 1$ are coprime similarly to the case of A_k and β_k. Hence, by Lemma 4 with $A = A_k$ and $n = N_k$ and for all sufficiently large k and uniformly in $m \in \{0, 1, \dots, 3k^3 - 1\}$, we have

$$\begin{aligned} \alpha_{k,j} &\leq \sum_{m=0}^{3k^3-1} \sum_{i=1}^{N_k} \left(\sum_{s=1}^{\ell} |b_s| a_s(A_k i + m + \beta_k + k + 1) \right) \\ &\leq \sum_{m=0}^{3k^3-1} \sum_{s=1}^{\ell} |b_s| \left(\sum_{i=1}^{N_k} a_s(A_k i + m + \beta_k + k + 1) \right) \\ &\leq 24k^3(|b_1| + \cdots + |b_\ell|) N_k \log N_k \\ &\leq c_2 k^6 2^{k^3}. \end{aligned} \tag{14}$$

Here, for large k we can take $c_2 = 24(|b_1| + \cdots + |b_\ell|)$. Define

$$m_{k,j} := \min_{\substack{1 \leq i \leq N_k \\ u_{k,i} \in T_j(k)}} \left(\sum_{m=0}^{3k^3-1} |\theta(m + u_{k,i} + k + 1)| \right).$$

By Lemma 3, (13), and (14), we have

$$\left\lfloor \frac{N_k}{2^{\ell+7}k^3} \right\rfloor m_{k,j} \leq \alpha_{k,j} \leq c_2 k^6 2^{k^3},$$

so that $m_{k,j} \leq c_3 k^9$ for large k, where we can take $c_3 = 2^{\ell+8} c_2$. In particular, $m_{k,j} \leq k^{10}$ for all large k. Hence, for large k, there exists an $i_{k,j} \in \{1, 2, \dots, N_k\}$ such that $u_{k,i_{k,j}} \in T_j(k)$ and

$$\sum_{m=0}^{3k^3-1} |\theta(m + u_{k,i_{k,j}} + k + 1)| \leq k^{10}. \tag{15}$$

Define $n_{k,j} := u_{k,i_{k,j}} \in T_j(k)$. Then

$$n_{k,j} = u_{k,i_{k,j}} = A_k i_{k,j} + \beta_k \geq A_k \geq q_1 \geq 4k^3 > 2k. \tag{16}$$

On the other hand, we have, by (3),

$$a_s(q_i^{|q|L-1}) = 1 + \left\lfloor \frac{|q|L - 1}{s} \right\rfloor = 1 + \left(\frac{L}{s}|q| - 1 \right) = \frac{L}{s}|q| \equiv 0 \pmod{q}$$

for every prime number q_i with $i = 1, \dots, t_{2k}$ and for every $s = 1, 2, \dots, \ell$. Hence, $\theta(q_i^{|q|L-1})$ is a multiple of q, and so we obtain, by (9),

$$\begin{cases} q|\theta(n_{k,j} - k + 1),\ q^2|\theta(n_{k,j} - k + 2), \dots, q^{k-1}|\theta(n_{k,j} - 1), \\ q^{k+1}|\theta(n_{k,j} + 1),\ q^{k+2}|\theta(n_{k,j} + 2), \dots, q^{2k}|\theta(n_{k,j} + k). \end{cases} \tag{17}$$

Furthermore, since

$$\begin{aligned} 3k^3 + n_{k,j} + k &\leq 3k^3 + A_k i_{k,j} + \beta_k + k \\ &\leq 3k^3 + 2^{k^3} e^{c_1 k^2 \log k} + e^{c_1 k^2 \log k} + k \\ &\leq 2^{2k^3}, \end{aligned} \tag{18}$$

for all sufficiently large k, we obtain, by (15) and (18),

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{|\theta(m + n_{k,j} + k + 1)|}{|q|^m} &\leq k^{10} + \sum_{m=3k^3}^{\infty} \frac{|\theta(m + n_{k,j} + k + 1)|}{|q|^m} \\ &\leq k^{10} + c_4 \sum_{m=3k^3}^{\infty} \frac{m + n_{k,j} + k + 1}{|q|^m} \\ &\leq k^{10} + \frac{c_4}{|q|^{3k^3}} \sum_{m=0}^{\infty} \frac{m + 3k^3 + n_{k,j} + k + 1}{|q|^m} \\ &\leq k^{10} + \frac{c_4}{|q|^{3k^3}} \sum_{m=0}^{\infty} \frac{m + 2^{2k^3} + 1}{|q|^m} \\ &\leq k^{10} + c_4 \left(\frac{2^2}{|q|^3} \right)^{k^3} \sum_{m=0}^{\infty} \frac{m+1}{|q|^m} \leq k^{11}. \end{aligned} \tag{19}$$

Thus, by (16), (17), and (19), the assumptions in Theorem 2 are satisfied for the sequence $\{\theta(n)\}_{n \geq 1}$ and for each subsequence $n_k = n_{k,j}$ $(j = 0, 1, \ldots, \ell - 1)$.

Hence, by (12), the numbers

$$\begin{aligned} \theta(n_{k,j}) &= \sum_{s=1}^{\ell} b_s a_s(n_{k,j}) \\ &= \begin{cases} 2b_1 + b_2 + b_3 + \cdots + b_\ell, & (j = 0); \\ 4b_1 + b_2 + b_3 + \cdots + b_\ell, & (j = 1); \\ \sum_{s=1}^{j-1} b_s a_s(n_{k,j}) + 2b_j + b_{j+1} + \cdots + b_\ell & (2 \leq j \leq \ell - 1) \end{cases} \end{aligned} \tag{20}$$

are multiples of q^k for every large k, where $a_1(n_{k,j}) = 2(j+1)$ and

$$a_s(n_{k,j}) = 1 + \lfloor j/s \rfloor \quad (j = 2, 3, \ldots, \ell - 1,\ s = 2, \ldots, j - 1).$$

Since $a_s(n_{k,j})$ are bounded, the right-hand sides of (20) must be zero for every large k. Thus, we obtain $A\mathbf{b} = \mathbf{0}$ with $\mathbf{b} = {}^t(b_1 \ldots, b_\ell)$ and

$$A = \begin{pmatrix} 2 & 1 & 1 & \ldots & 1 & 1 \\ 4 & 1 & 1 & \ldots & 1 & 1 \\ 6 & 2 & 1 & \cdots & 1 & 1 \\ 8 & 2 & 2 & \cdots & 1 & 1 \\ \vdots & & & & \ddots & \vdots \\ 2\ell & 1 + \lfloor (\ell-1)/2 \rfloor & 1 + \lfloor (\ell-1)/3 \rfloor & \cdots & 2 & 1 \end{pmatrix}.$$

Noting that $\det A = 2(-1)^{\ell-1} \neq 0$, we have $b_1 = \cdots = b_\ell = 0$, and hence $b_0 = 0$ by (7). This is a contradiction, and the proof of Theorem 1 is completed.

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