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BLOCH'S CONJECTURE FOR INOUE SURFACES WITH $p_g = 0$, $K^2 = 7$

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Abstract. The aim of this paper is to prove Bloch's conjecture for Inoue surfaces with $p_g = 0$ and $K^2 = 7$. These surfaces can also be described as bidouble covers of the four nodal cubic, which allows one to use the method of "enough automorphisms" due to Inose and Mizukami.

Introduction

Let S be a smooth projective complex surface and let

$$A_0(S) = \bigoplus_{i \in \mathbb{Z}}^{\infty} A_0^i(S)$$

be the group of rational equivalence classes of zero cycles on S. Then *Bloch's conjecture* asserts the following:

Conjecture 0.1 (S. Bloch, [5]). *Let S be a smooth surface with $p_g(S) = 0$. Then the kernel $T(S)$ of the natural morphism*

$$A_0^0(S) \to \mathrm{Alb}(S)$$

is trivial.

The conjecture has been proven for surfaces S with $\kappa(S) < 2$ by Bloch, Kas and Liebermann (cf. [6]), and has been verified for several examples (cf. e.g. [1], [9], [13], [15], [20]). It has been observed recently that by a beautiful result due to S. Kimura (cf. [16]) all product quotient surfaces (i.e., minimal models of $C_1 \times C_2/G$, where G is a finite group acting on the product of two curves of respective genera at least 2) with $p_g = 0$ satisfy Bloch's conjecture (cf. [2]).

Even if nowadays a substantial number of examples are known to fulfill Bloch's conjecture, there is still no idea how to prove the result in general. Also worth mentioning is that to our knowledge Bloch's conjecture has not been verified for any fake projective plane, i.e., a surface of general type with $p_g = 0$ and $K_S^2 = 9$.

The main result of this note is verification of Bloch's conjecture for Inoue surfaces with $p_g = 0$ and $K^2 = 7$. Inoue surfaces are up to now[1] the only known family with $p_g = 0$ and $K_S^2 = 7$. They form a four dimensional irreducible connected

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[1]In the meantime a new family of surfaces of general type with $K_S^2 = 7$, $p_g = 0$ has been constructed by Yifan Chen; cf. [10].

component $\mathfrak{N}_I$ in the Gieseker moduli space $\mathfrak{M}^{can}_{1,7}$ of canonical models of surfaces of general type with $p_g = 0$, $K^2 = 7$, as was shown among other things in [3].

These surfaces were first constructed by M. Inoue in [14] as quotients of complete intersections of two divisors (explicitly given by equations) of respective multi-degrees $(2,2,2,0)$ and $(0,0,2,2)$ by a free $(\mathbb{Z}/2\mathbb{Z})^5$-action.

They can also be described as bidouble covers of the four nodal cubic surface (cf. [17]). This description will be crucial for the proof of our main result.

Theorem 0.2. *Let S be an Inoue surface with $K_S^2 = 7$ and $p_g = 0$. Then*

$$T(S) = A_0^0(S) = 0;$$

i.e., S satisfies Bloch's conjecture.

The proof will in fact use the method of "enough automorphisms" introduced by Inose and Mizukami (cf. [13]) and refined by Barlow (cf. [1]), but in a much simplified form.

Remark 0.3. As shown in [3], if $[S] \in \mathfrak{N}_I$, then S is an Inoue surface. Therefore our result shows Bloch's conjecture for *each surface* in the irreducible connected component $\mathfrak{N}_I$.

1. Bloch's conjecture for surfaces with a $(\mathbb{Z}/2\mathbb{Z})^2$-action

The aim of this note is to prove Bloch's conjecture for Inoue surfaces using the method of "enough automorphisms" introduced by Inose and Mizukami (cf. [13]) and refined by Barlow (cf. [1]).

We need the following notation.

Definition 1.1. Let G be a finite group and $H \leq G$ be a subgroup. Then we set

$$z(H) := \sum_{h \in H} h \in \mathbb{C}G.$$

We recall Barlow's reformulation of the criterion of Inose and Mizukami in [13].

Lemma 1.2 (Precise version of Inose's "enough automorphisms", [1])**.** *Let S be a nonsingular surface and G a finite subgroup of* $\mathrm{Aut}(S)$*. Let H, $H_1, \dots, H_r$ be subgroups of G. We denote by $\mathcal{I}$ the two-sided ideal of $\mathbb{C}G$ generated by $z(H_1), \dots,$ $z(H_r)$. Assume that*

(1) $z(H) \in I$,
(2) $T(S/H_i) = 0,\ \forall i \in \{1, \dots\}$.

Then $T(S/H) = 0$.

Using the above we can show the following:

Proposition 1.3. *Let S be a surface of general type with $p_g(S) = 0$. Assume that $G = (\mathbb{Z}/2\mathbb{Z})^2 \leq$* $\mathrm{Aut}(S)$*. Then S satisfies Bloch's conjecture if and only if for each $\sigma \in G \setminus \{0\}$ the quotient S/σ satisfies Bloch's conjecture.*

Remark 1.4. Note that S/σ is a surface with at most nodes, and denoting by $X_\sigma \to S/\sigma$ its resolution of singularities, X_σ is minimal and has $p_g = 0$. Moreover, since nodes are rational singularities, $T(S/\sigma) = T(X_\sigma)$.

Proof. If S satisfies Bloch's conjecture, then obviously each quotient by an involution also.

For the other direction we apply Lemma 1.2 for $G = (\mathbb{Z}/2\mathbb{Z})^2$, $H = 0$, $H_1 = \langle \gamma_1 \rangle$, $H_2 = \langle \gamma_2 \rangle$, $H_3 = \langle \gamma_3 \rangle$, where $\gamma_1, \gamma_2, \gamma_3 = \gamma_1\gamma_2$ are the three nontrivial elements of G. Then by assumption, S/H_i satisfies Bloch's conjecture, i.e. $T(S/H_i) = T(X) = 0$.

Therefore it remains to verify that $1 = z(H) \in \mathcal{I}$, where $\mathcal{I}$ is the ideal in $\mathbb{C}G$ generated by $z(H_1), z(H_2), z(H_3)$. Observe that $z(H_i) = 1 + (\gamma_i)_*$, i.e. $\gamma_i \equiv -1 \mod \mathcal{I}$. On the other hand, $(\gamma_3)_* = (\gamma_1)_*(\gamma_2)_* \equiv 1 \mod \mathcal{I}$, whence $1 = z(H) \in \mathcal{I}$. □

Corollary 1.5. *Let S be a surface of general type with $p_g(S) = 0$ and assume that $G = (\mathbb{Z}/2\mathbb{Z})^2 \leq \mathrm{Aut}(S)$. Assume that for each $\sigma \in G \setminus \{0\}$ the quotient S/σ has $\kappa(S/\sigma) \leq 1$; then S satisfies Bloch's conjecture, i.e., $T(S) = A_0^0(S) = 0$.*

Proof. Follows immediately by combining Proposition 1.3 and [6]. □

Remark 1.6. The above criterion can obviously be used to prove Bloch's conjecture for several other surfaces of general type with $p_g = 0$ having $(\mathbb{Z}/2\mathbb{Z})^2$ as a subgroup of their automorphism group, e.g., *Burniat surfaces* (cf. [8]) and *extended Burniat surfaces* (cf. [4]). For Burniat surfaces Bloch's conjecture is known by [13], whereas for extended Burniat surfaces the proof is similar to the proof we give below for Inoue surfaces.

2. Inoue surfaces with $p_g = 0$ and $K_S^2 = 7$ as bidouble covers of the four nodal cubic

In [14] the author constructs a family of minimal surfaces of general type S with $p_g = 0$, $K_S^2 = 7$ as quotients of a complete intersection of two divisors (explicitly given by equations) of respective multi-degrees $(2, 2, 2, 0)$ and $(0, 0, 2, 2)$ by a free $(\mathbb{Z}/2\mathbb{Z})^5$-action.

In order to prove our main result we use a different description of the Inoue surfaces, given by Mendes Lopes and Pardini in [17], as $(\mathbb{Z}/2\mathbb{Z})^2$-Galois covers of the four nodal cubic.

We briefly recall their construction here; for details we refer to the article [17], example 4.1.

Let Λ in $\mathbb{P}^2$ be a complete quadrilateral and denote the vertices by $P_1, \ldots, P_6$.

We have labeled the vertices in a way that

- the intersection point of the line $\overline{P_1P_2}$ and the line $\overline{P_3P_4}$ is P_5,
- the intersection point of $\overline{P_1P_4}$ and $\overline{P_2P_3}$ is P_6.

Let $Y \to \mathbb{P}^2$ be the blowup in $P_1, \ldots, P_6$, denote by L the total transform of a line in $\mathbb{P}^2$, and let E_i, $1 \leq i \leq 6$, be the exceptional curve lying over P_i. Moreover, we denote by S_i, $1 \leq i \leq 4$, the strict transforms on Y of the sides $S_i := \overline{P_iP_{i+1}}$ for $1 \leq i \leq 3$, $S_4 := \overline{P_4P_1}$, of the quadrilateral Λ.

We denote by Δ_i, $1 \leq i \leq 3$, the strict transforms of the three diagonals of the complete quadrilateral on Y, i.e.,

- $\Delta_1 \equiv L - E_1 - E_3$,
- $\Delta_2 \equiv L - E_2 - E_4$,
- $\Delta_3 \equiv L - E_5 - E_6$.

Observe that the four (-2) curves S_i come from the resolution of the 4 nodes of the cubic surface Σ which is the anticanonical image of Y, and the curves E_i are the strict transforms of the 6 lines in Σ connecting pairs of nodal points.

The surface Σ also contains a triangle of lines. These are the 3 strict transforms Δ_1, Δ_2, Δ_3 of the three diagonals of the complete quadrilateral Λ.

For each line Δ_i in the cubic surface Σ we consider the pencil of planes containing them and the base point free pencil of residual conics, which we denote by $|f_i|$. Hence we have

$$|f_i| = |(-K_Y) - \Delta_i|, \quad \Delta_i + f_i \equiv (-K_Y).$$

In the plane realization we have:

- f_1 is the strict transform on Y of a general element of the pencil of conics Γ_1 through P_2, P_4, P_5, P_6,
- f_2 is the strict transform on Y of a general element of the pencil of conics Γ_2 through P_1, P_3, P_5, P_6,
- f_3 is the strict transform on Y of a general element of the pencil of conics Γ_3 through P_1, P_2, P_3, P_4.

It is then easy to see that each curve S_h is disjoint from the other curves S_j $(j \neq h)$, Δ_i, and f_i, if f_i is smooth. Moreover,

$$\Delta_i \cdot f_i = 2, \quad \Delta_i \cdot f_j = 0, i \neq j, \quad f_i^2 = 0, \quad f_j f_i = 2, i \neq j.$$

Definition 2.1. We define the *Inoue divisors* on Y as follows:

- $D_1 := \Delta_1 + f_2 + S_1 + S_2$, where $f_2 \in |f_2|$ smooth;
- $D_2 := \Delta_2 + f_3$, where $f_3 \in |f_3|$ smooth;
- $D_3 := \Delta_3 + f_1 + f_1' + S_3 + S_4$, where $f_1, f_1' \in |f_1|$ smooth.

Let $\pi\colon \tilde{S} \to Y$ be the bidouble covering with branch divisors D_1, D_2, D_3 (associated to the 3 nontrivial elements $\gamma_1, \gamma_2, \gamma_3$ of the Galois group $G \cong (\mathbb{Z}/2\mathbb{Z})^2$).

Then $\tilde{S}$ is smooth, and by the previous remarks we see that over each S_i there are two disjoint (-1)-curves. Contracting these eight exceptional curves we obtain a minimal surface with $p_g = 0$ and $K_S^2 = 7$.

Moreover, S is a smooth $(\mathbb{Z}/2\mathbb{Z})^2$-covering of the four nodal cubic Σ, obtained from Y by contracting the four (-2)-curves S_j, and by [17] these are exactly the Inoue surfaces.

Remark 2.2. We immediately see that there is an open dense subset in the product

$$|f_1| \times |f_1'| \times |f_2| \times |f_3| \cong (\mathbb{P}^1)^4$$

parametrizing the family of Inoue surfaces.

Remark 2.3. Denoting by $\chi_i \in G^*$ the nontrivial character orthogonal to γ_i, the nontrivial character sheaves of this bidouble cover are

- $\mathcal{L}_1 = \mathcal{O}_Y(-K_Y + f_1 - E_4)$;
- $\mathcal{L}_2 = \mathcal{O}_Y(-2K_Y - E_5 - E_6)$;
- $\mathcal{L}_3 = \mathcal{O}_Y(-K_Y + L - E_1 - E_2 - E_3)$.

That is, G acts on $\mathcal{L}_i^{-1}$ via the character χ_i.

Then we have the following:

Proposition 2.4 (Mendes Lopes, Pardini [17]). *The following is the decomposition of $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}}))$ as a sum of isotypical components:*

$$H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}})) \cong H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}}))^G \oplus \bigoplus_{i=1}^{3} H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}}))^{\chi_i}, \text{ where}$$

(0) $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}}))^G = H^0(Y, \mathcal{O}_Y(-K_Y + f_1 + \sum S_j)) \cong \mathbb{C}^7$,
(1) $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}}))^{\chi_1} = H^0(Y, \mathcal{O}_Y(-K_Y + f_1 + \sum S_j - \mathcal{L}_1)) \cong \mathbb{C}$,
(2) $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}}))^{\chi_2} = H^0(Y, \mathcal{O}_Y(-K_Y + f_1 + \sum S_j - \mathcal{L}_2)) = 0$,
(3) $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}}))^{\chi_3} = H^0(Y, \mathcal{O}_Y(-K_Y + f_1 + \sum S_j - \mathcal{L}_3)) = 0$.

Proof. See [17], example 4.1, p. 271. □

Corollary 2.5. *The decomposition of $H^0(S, \mathcal{O}_S(2K_S))$ in an invariant and anti-invariant part with respect to γ_i is as follows:*

$$H^0(S, \mathcal{O}_S(2K_S))^{+,\gamma_i} \cong H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}}))^G \oplus H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}}))^{\chi_i},$$

$$H^0(S, \mathcal{O}_S(2K_S))^{-,\gamma_i} \cong H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}}))^{\chi_j} \oplus H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}}))^{\chi_k},$$

where $\{i, j, k\} = \{1, 2, 3\}$.

In particular, $h^0(S, \mathcal{O}_S(2K_S))^{-,\gamma_1} = 0$ and $h^0(S, \mathcal{O}_S(2K_S))^{-,\gamma_i} = 1$ for $i = 2, 3$.

3. Quotients of Inoue surfaces by an involution

In order to prove Theorem 0.2 we have to show that for each automorphism $\sigma \in G = (\mathbb{Z}/2\mathbb{Z})^2$ of an Inoue surface S we have

$$\kappa(S/\langle\sigma\rangle) \leq 1.$$

Before doing this, we have to fix some notation.

Let S be a minimal regular surface of general type and let σ be an involution on S. Then σ is biregular, and its fixed locus consists of k isolated points and a nonsingular (not necessarily connected) curve R. The quotient $T := S/\langle\sigma\rangle$ has k nodes, and resolving them we get a cartesian diagram of morphisms:

(3.1)
$$\begin{tikzcd} \hat{S} \arrow[r, "\epsilon"] \arrow[d, "\hat{\pi}"'] & S \arrow[d, "\pi"] \\ \hat{T} \arrow[r] & T \end{tikzcd}$$

with vertical maps finite of degree 2 and horizontal maps birational.

We denote by Δ the branch curve $\pi(R)$ and by $E_1, \ldots, E_k$ the exceptional curves of ϵ.

The action of σ on $\hat{S}$ yields a decomposition $\hat{\pi}_*\mathcal{O}_{\hat{S}} = \mathcal{O}_{\hat{T}} \oplus \mathcal{O}_{\hat{T}}(-\hat{\delta})$, with $2\hat{\delta} \equiv \Delta + \sum_1^k \hat{\pi}(E_i)$. Recall that $K_{\hat{S}} \equiv \hat{\pi}^*(K_{\hat{T}} + \hat{\delta})$.

Then (cf. [18], [7]):

Lemma 3.1.

(3.2) $$0 \leq k = K_S^2 + 6\chi(\mathcal{O}_{\hat{T}}) - 2\chi(\mathcal{O}_S) - 2h^0(\mathcal{O}_{\hat{T}}(2K_{\hat{T}} + \hat{\delta})).$$

Moreover, if $p_g(\hat{T}) = 0$, then the biconical map of S factors through σ if and only if $h^0(\mathcal{O}_{\hat{T}}(2K_{\hat{T}} + \hat{\delta})) = 0$.

Combining the above lemma with Corollary 2.5 we obtain:

Proposition 3.2. *Let S be an Inoue surface with $p_g(S) = 0$ and $K_S^2 = 7$ and let γ_i be one of the nontrivial elements of $G \cong (\mathbb{Z}/2\mathbb{Z})^2$. Then we have for the number of isolated fixed points k_i of γ_i:*

- *$k_1 = 11$, in particular the biconical map factors through γ_1;*
- *$k_2 = k_3 = 9$.*

Proof. Note that $k_i = K_S^2 + 6\chi(\mathcal{O}_{\hat{T}}) - 2\chi(\mathcal{O}_S) - 2h^0(\mathcal{O}_{\hat{T}}(2K_{\hat{T}} + \hat{\delta})) = 7 + 4 - 2h^0(\mathcal{O}_{\hat{T}}(2K_{\hat{T}} + \hat{\delta}))$. Now use

$$h^0(\mathcal{O}_{\hat{T}}(2K_{\hat{T}} + \hat{\delta})) = h^0(\hat{S}, \mathcal{O}_{\hat{S}}(2K_{\hat{S}}))^{-,\gamma_i} = h^0(S, \mathcal{O}_S(2K_S))^{-,\gamma_i}.$$

The claim follows now from Corollary 2.5. □

We also need the following results of [11]:

Lemma 3.3 ([11], Lemma 4.2)**.** *Let S be a smooth surface with $p_g(X) = q(X) = 0$ and let σ be an automorphism of S of order 2. We denote again the divisorial part of the fixed locus of σ by R and by k the number of isolated fixed points. Moreover, let t be the trace of $\sigma|H^2(S, \mathbb{C})$. Then*

$$k = K_S \cdot R + 4, \quad t = 2 - R^2.$$

Furthermore, using the notation of diagram (3.1) we have the following relation for the Picard numbers:

$$\rho(S) + t = 2\rho(\hat{T}) - 2k.$$

Proposition 3.4 ([11], Prop. 4.1)**.** *Let Y be a surface with $p_g(Y) = q(Y) = 0$ and Kodaira dimension $\kappa(Y) \geq 0$. Moreover, let $C_1, \dots, C_k \subset Y$ be disjoint rational (-2)-curves. Then:*

(i) *$k \leq \rho(Y) - 2$;*
(ii) *if $k = \rho(Y) - 2$, then Y is minimal.*

In fact, we also need to consider the case $\rho(Y) = k - 3$. Here Y is not necessarily minimal, but using the same line of arguments as in Proposition 4.1 of [11] we can show the following:

Lemma 3.5. *If in Proposition 3.4 we have $k = \rho(Y) - 3$, then the minimal model $\bar{Y}$ of Y is either*

- *equal to Y or*
- *$Y \to \bar{Y}$ is the blow up in one point, in particular, $K_Y^2 = K_{\bar{Y}}^2 - 1$, or*
- *$Y \to \bar{Y}$ is the blow up in two infinitely near points, in particular, $K_Y^2 = K_{\bar{Y}}^2 - 2$.*

Proof. Assume that Y is not minimal. Let $E \subset Y$ be an irreducible (-1) curve and let Y' be the surface obtained by blowing down E.

If E does not intersect any of the nodal curves C_i, then Y' contains k disjoint nodal curves; hence $k = \rho(Y) - 3 = \rho(Y') - 2$. Therefore, by Proposition 3.4, Y' is minimal.

Assume now that E intersects, say, C_1; i.e., $E \cdot C_1 = \alpha > 0$. Then, if C_1' denotes the image of C_1 in Y', C_1' is irreducible and

$$(C_1')^2 = -2 + \alpha^2, \quad C_1' \cdot K_{Y'} = -\alpha.$$

Suppose $\alpha \geq 2$; then $(C_1')^2 > 0$ and the image C_1'' in $\bar{Y}$ is a curve and satisfies $C_1'' \cdot K_{\bar{Y}} \leq C_1' \cdot K_{Y'} < 0$. This contradicts $K_{\bar{Y}}$ nef.

This implies that $\alpha = 1$; i.e., C_1' is a (-1)-curve. Moreover, $E \cdot C_i = 0$ for $i \neq 1$, since otherwise we would have on Y' two intersecting (-1)-curves, which is not possible on a surface with $\kappa(Y) \geq 0$. Denote by Y'' the surface obtained from Y' by blowing down C_1'. Then Y'' contains $k-1$ nodal curves and

$$k - 1 = \rho(Y) - 4 = \rho(Y'') - 2.$$

By Proposition 3.4 we get that Y'' is minimal. □

Now we are ready to prove our main result.

Proof of Theorem 0.2. We will in fact show the following more general result.

Proposition 3.6. *Let S be a minimal surface of general type with $p_g = 0$ and $K_S^2 = 7$. Let σ be an involution on S with k isolated fixed points. If $\kappa(S/\langle\sigma\rangle) = 2$, then $k = 5$ or $k = 7$.*

Proof of Proposition 3.6. Since S is a regular surface with $p_g = q = 0$, $\rho(S) = e(S) - 2$. Therefore, $K_S^2 = 7$ implies $\rho(S) = 3$.

Since the class of the canonical divisor in $H^2(S, \mathbb{C})$ is invariant under σ, we have for t the possibilities $t = 3, 1$ or -1.

Assume that $t = -1$; i.e. $\sigma | H^2(S, \mathbb{C})$ has eigenvalues $1, -1, -1$ and, in particular, $\dim H^2(S, \mathbb{C})^{inv} = 1$. This implies that K_S is numerically equivalent to rR, which contradicts $R^2 = 2 - t = 3$. Therefore this case does not occur.

<u>$t = 1$</u>: then $R^2 = 1$ and $K_S \cdot R = 2m + 1$, for some $m \geq 0$. This implies

$$k = K_S \cdot R + 4 = 5 + 2m \geq 5.$$

On the other hand, by Lemma 3.3,

$$4 = \rho(S) + 1 = 2\rho(\hat{T}) - 2k,$$

whence $\rho(\hat{T}) = k + 2$.

Since $\kappa(S/\langle\sigma\rangle) = 2$, it follows by Proposition 3.4 that $\hat{T}$ is minimal. In particular, $K_{\hat{T}}^2 > 0$. Therefore,

$$K_{\hat{T}}^2 = 12 - e(\hat{T}) = 8 - k > 0.$$

This implies that $k = 5$ or $k = 7$.

<u>$t = 3$</u>: in this case $R^2 = -1$, whence again $K_S \cdot R = 2m + 1$, for some $m \geq 0$. This implies

$$k = K_S \cdot R + 4 = 5 + 2m \geq 5.$$

On the other hand, by Lemma 3.3, $\rho(\hat{T}) = k + 3$.

Since $\kappa(S/\langle\sigma\rangle) = 2$, it follows by Lemma 3.5 that $K_{\hat{T}}^2 \geq -1$. Therefore,

$$K_{\hat{T}}^2 = 12 - e(\hat{T}) = 7 - k \geq -1.$$

This immediately implies that $k = 5$ or $k = 7$. □

Combining Proposition 3.6 with Proposition 3.2 we see that for an Inoue surface we have $k(S/\langle\gamma_i\rangle) \leq 1$. By Corollary 1.5 our main result is proven. □

Remark 3.7. We were kindly informed that Proposition 3.6 follows also from [12] (cf. e.g. the table on page 2 of [12]), as well as from [19].

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