SMOOTH GLUING OF GROUP ACTIONS
AND APPLICATIONS

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ABSTRACT. Let M_1 and M_2 be two n-dimensional smooth manifolds with boundary. Suppose we glue M_1 and M_2 along some boundary components (which are, therefore, diffeomorphic). Call the result N. If we have a group G acting continuously on M_1, and also acting continuously on M_2, such that the actions are compatible on glued boundary components, then we get a continuous action of G on N that stitches the two actions together. However, even if the actions on M_1 and M_2 are smooth, the action on N probably will not be smooth.

We give a systematic way of smoothing out the glued G-action. This allows us to construct interesting new examples of smooth group actions on surfaces and to extend a result of Franks and Handel (2006) on distortion elements in diffeomorphism groups of closed surfaces to the case of surfaces with boundary.

1. INTRODUCTION

For any manifold with boundary M, we denote the boundary by ∂M. We assume manifolds are smooth. We write $\mathrm{Homeo}(M)$ and $\mathrm{Diff}^r(M)$ for homeomorphisms or C^r diffeomorphisms of M, respectively. If G is a discrete group, a *continuous (smooth) action* of G on M is a homomorphism $G \to \mathrm{Homeo}(M)(\mathrm{Diff}^\infty(M))$.

Let M_1 and M_2 be manifolds with boundary, and let f_1 and f_2 be homeomorphisms of M_1 and M_2, respectively. Suppose we glue in pairs some diffeomorphic boundary components of M_1; call the result N. If f_1 is compatible on the boundary components that we glue, we get an induced homeomorphism $\mathfrak{g}(f_1) \in \mathrm{Homeo}(N)$. Similarly, if we glue boundary components of M_1 to boundary components of M_2, calling the result N, and if f_1 and f_2 are compatible on corresponding boundary components, we get a homeomorphism $\mathfrak{g}(f_1, f_2) \in \mathrm{Homeo}(N)$. However, even if $f_1, f_2 \in \mathrm{Diff}^\infty(M_i)$, it does not follow that $\mathfrak{g}(f_1)$ or $\mathfrak{g}(f_1, f_2) \in \mathrm{Diff}^\infty(N)$: they will probably fail to be smooth across glued boundary components.

The goal of this article is to give a systematic procedure to apply to f_1 and f_2, so that when the resulting maps are glued together we get a diffeomorphism of N. The idea is simple: to make points near shared boundaries move approximately like points on the boundaries, we apply a topological conjugacy which "crushes" nearby points very strongly towards the shared boundaries. This has the effect of removing (up to all orders of derivatives) any motion under f_1 or f_2 that is transverse to the shared boundaries.

Our conjugacy is an example of what Kloeckner [9] calls *stretching*. He considers the question of when smooth or analytic group actions which are topologically conjugated by a stretch are in fact smoothly conjugate.

Some authors have considered the gluing of group actions. Katok-Lewis [8] considered the linear action of $SL(n, \mathbb{Z})$ on T^n. They blow up at the global fixed point 0, introducing a copy of S^{n-1} at that point. Since the original action was linear, two of these actions can be glued along S^{n-1}, and the result is real-analytic. They even make it volume-preserving. Farb-Shalen [4] did a similar construction in dimension 3; they consider a finite $SL(3, \mathbb{Z})$-invariant set. To our knowledge, our construction smoothing out general glued group actions (not assumed to originate from linear actions) is new. The proof is based on an article of Tsuboi [12].

Our result has applications to the study of smooth group actions. If a group G acts smoothly on M_1 and also acts smoothly on M_2, and the actions agree on common boundary components, by applying our result we get a smooth action of G on N; since our procedure involves conjugation, it respects the group structure.

An important example of a group-theoretic property to which our result applies is the notion of *distortion*.

Definition 1. If G is a finitely generated group, and we choose the generating set $\{g_1, \ldots, g_s\}$, then $f \in G$ is said to be a *distortion element* of G if f has infinite order and
$$\liminf_{n \to \infty} \frac{|f^n|}{n} = 0,$$
where $|f^n|$ is the word length of f^n in the generators $\{g_1, \ldots, g_s\}$. It is easy to see that this property is independent of the generating set chosen. If G is not finitely generated, we say $f \in G$ is a distortion element if it is a distortion element in some finitely generated subgroup.

See Gromov [7] for a good discussion with many examples. Many authors, including Ávila [1], Militon [11], and Calegari and Freedman [3], have exhibited examples of distortion elements in transformation groups. Franks and Handel [5] have shown that the possibilities for distortion elements among area-preserving surface diffeomorphisms are severely limited. Namely, they prove:

Theorem 2. *Let S be a closed surface. Let $\mathrm{Diff}^1(S)_0$ denote C^1 diffeomorphisms isotopic to the identity. Let $f \in \mathrm{Diff}^1(S)_0$ be distorted. If $S = S^2$, assume f has at least 3 fixed points; if $S = T^2$, assume f has at least 1 fixed point. Then (*) for any f-invariant Borel probability measure $\mu, \mathrm{supp}(\mu) \subset \mathrm{Fix}(f)$.*

In particular, (*) says that f cannot be area-preserving, since the support of area is the whole surface, so f would be the identity, which is not a distortion element.

Remark 3. The assumption on fixed points is necessary. By [3], an irrational rotation of S^2 is distorted in $\mathrm{Diff}^\infty(S^2)$, and by [1] or [11] an irrational rotation of T^2 is distorted in $\mathrm{Diff}^\infty(T^2)$.

In this paper, we apply our main result in two ways: First, we exhibit new examples of distortion elements in diffeomorphism groups of surfaces, by constructing actions of the Heisenberg group on surfaces. Second, we generalize Franks and Handel's theorem to the case of surfaces with boundary.

Corollary 4. *Let S be a compact surface with nonempty boundary. Let $f \in \mathrm{Diff}^1(S)_0$ be distorted. If $S = D$ is the closed disk, assume that f has at least two fixed points in the interior of the disk, or at least one fixed point on the boundary. If $S = A$, assume f has at least one fixed point. Then (*) holds.*

2. Main result

Let M_1, M_2 be smooth n-manifolds with boundary. We will glue boundary components of M_1 and M_2. Our proof also implies the corresponding result if we are identifying boundary components of a single manifold.

Let $\{(C_i)_1\}$, $\{(C_i)_2\}$ be components of ∂M_1 and ∂M_2, respectively. There are at most countably many; they are smooth $(n-1)$-manifolds without boundary. Suppose there exist diffeomorphisms $\alpha_i \colon (C_i)_1 \to (C_i)_2$. Write $C_1 = \bigcup_i (C_i)_1$ and $C_2 = \bigcup_i (C_i)_2$. Define $\alpha \colon C_1 \to C_2$ by $\alpha(x) = \alpha_i(x)$ for $x \in (C_i)_1$. We may form a new manifold N by gluing the boundaries according to these diffeomorphisms: $N = M_1 \sqcup M_2 / \sim$, where $x \sim \alpha(x)$ for $x \in C_1$. Let $\pi \colon M_1 \sqcup M_2 \to N$ be the projection.

We may endow N with smooth structure as follows. Away from the glued boundaries, we use the smooth structure of M_1 or M_2. On the boundary, we rely on product neighborhoods. Specifically, choose a neighborhood U_1 of C_1 diffeomorphic to $C_1 \times [0,1)$; let $\eta_1 \colon U_1 \to C_1 \times [0,1)$ be a diffeomorphism with the property that for $x \in C_1, \eta_1(x) = (x,0)$. Note that $(\alpha \times id) \circ \eta_1$ sends U_1 to $C_2 \times [0,1)$. Similarly, take U_2 and η_2 such that $\eta_2 \colon U_2 \to C_2 \times (-1,0]$ is a diffeomorphism and for $x \in C_2, \eta_2(x) = (x,0)$. By putting together $(\alpha \times id) \circ \eta_1$ and η_2, we get a homeomorphism $\eta \colon \pi(U_1 \cup U_2) \to C_2 \times (-1,1)$. We declare it to be a diffeomorphism.

Let $\mathcal{A} = \{(f_1, f_2) \colon f_1 \in \mathrm{Homeo}(M_1), f_2 \in \mathrm{Homeo}(M_2)$, and $\alpha \circ f_1|_{C_1} = f_2|_{C_2} \circ \alpha\}$. Note that we do not require $(C_i)_j$ to be f_j-invariant. If $(f_1, f_2) \in \mathcal{A}$, they agree on their glued boundaries, so we get a glued map $\mathfrak{g}(f_1, f_2) \in \mathrm{Homeo}(N)$.

Theorem 5. *There exist homeomorphisms Ψ_1 and Ψ_2 of M_1 and M_2 with the following property. For any $f_i \in \mathrm{Diff}^\infty(M_i)$, $\Psi_i^{-1} f_i \Psi_i \in \mathrm{Diff}^\infty(M_i)$. Furthermore, if $(f_1, f_2) \in \mathcal{A}$, then $\mathfrak{g}(\Psi_1^{-1} f_1 \Psi_1, \Psi_2^{-1} f_2 \Psi_2) \in \mathrm{Diff}^\infty(N)$.*

Proof. We will define $X_1 \colon C_1 \times [0,1) \to C_1 \times [0,1)$ below. Then $X_2 \colon C_2 \times (-1,0]$ will be given by $X_2 = (\alpha \times -id) \circ X_1 \circ (\alpha \times -id)^{-1}$, and we will set

$$\Psi_i(x) = \begin{cases} \eta_i^{-1}(X_i(\eta_i(x))), & x \in U_i \\ x, & x \in M_i \setminus U_i \end{cases}$$

for i = 1, 2.

We will construct X_1 so that it satisfies the following properties:

(1) It is of the form $X_1(x,y) = (x, \chi(y))$.
(2) χ is a C^∞ diffeomorphism of $(0,1)$ with $\chi(y) = \phi^2(y)$ (i.e. $\phi(\phi(y))$) in a neighborhood of 0, where $\phi(y) = e^{-1/y}$, and $\chi(y) = y$ in a neighborhood of 1.

In all that follows, we will not worry about domains and codomains. This is because we are only concerned with local behavior; we could if desired specify (co)domains, but which choices we made would not affect our calculations. Technically, we are dealing with *germs* of maps, but we leave this implicit to avoid cumbersome notation.

Let $p \in C_1$. Let Φ be defined near $(p, 0)$ in $C_1 \times [0, 1)$ to be $\Phi(x, y) = (x, \phi(y))$. We claim that (1) if g is C^∞ at $(p, 0)$, then so is $\Phi^{-1}g\Phi$. Since locally $X_1^{-1}gX_1 = \Phi^{-2}g\Phi^2$, applying (1) twice implies that $X_1^{-1}gX_1$ is also C^∞ at $(p, 0)$. Let g_x and g_y denote the first and second components of g (which is defined on some open set in $C_i \times [0, 1)$). Then we further claim that (2) $X_1^{-1}gX_1$ looks to all orders like $(x, y) \mapsto (g_x(x, 0), y)$ at $(p, 0)$. These claims follow Tsuboi [12].

Let us first show why claims (1) and (2) are enough to establish the theorem. Let $(f_1, f_2) \in \mathcal{A}$. The map $\eta_1 f_1 \eta_1^{-1}$ is C^∞ at $(p, 0)$, and $X_1^{-1}\eta_1 f_1 \eta_1^{-1}X_1$ looks like $(x, y) \mapsto ((\eta_1 f_1 \eta_1^{-1})_x(x, 0), y)$ to all orders at $(p, 0)$. Similarly, $\eta_2 f_2 \eta_2^{-1}$ is C^∞ at $(\alpha(p), 0)$, and $X_2^{-1}\eta_2 f_2 \eta_2^{-1}X_2$ looks like $(x, y) \mapsto ((\eta_2 f_2 \eta_2^{-1})_x(x, 0), y)$ to all orders at $(\alpha(p), 0)$. Therefore, when we glue $(\alpha \times id)X_1^{-1}\eta_1 f_1 \eta_1^{-1}X_1(\alpha \times id)^{-1}$ and $X_2^{-1}\eta_2 f_2 \eta_2^{-1}X_2$, the result is C^∞. The fact that $\mathfrak{g}(f_1, f_2) \in \text{Diff}^\infty(N)$ follows, since we declared η to be a diffeomorphism.

Let us consider claim (1). We must show that $\Phi^{-1}g\Phi$ is C^∞. The only difficulty is that Φ^{-1} is not differentiable when $y = 0$ (it stretches too strongly there). The first component of $\Phi^{-1}g\Phi$ – the one in C_i, which we may denote $(\Phi^{-1}g\Phi)_x$ – is not affected by Φ^{-1}, so it is automatically C^∞.

Consider the second component, $(\Phi^{-1}g\Phi)_y$. Since $g_y(x, 0)$ vanishes for all $x \in C_i$, by Taylor's theorem $\frac{g_y}{y}$ extends continuously to $y = 0$; it is C^∞ and nonvanishing at $y = 0$. Let us denote this by h, so $g_y = y \cdot h$. Then

$$
\begin{aligned}
(\Phi^{-1}g\Phi)_y &= -1/\log(g_y(x, \phi(y))) \\
&= -1/\log(\phi(y)h(x, \phi(y))) \\
&= y/(1 - y\log(h(x, \phi(y)))).
\end{aligned}
$$

Therefore, $(\Phi^{-1}g\Phi)_y$ is C^∞. So we have established claim (1).

Now we consider claim (2). If $\bar{g}$ denotes the restriction of g to $C_1 \times \{0\}$, then we must show that $\Phi^{-2}g\Phi^2$ and $\bar{g} \times Id$ are C^∞-close at $(p, 0)$. Introducing local coordinates about p and about $\bar{g}(p)$ in C_1 which send p and $\bar{g}(p)$ respectively to the origin in $\mathbb{R}^{n-1}$, we can assume we are looking at maps (locally defined around the origin) of $\mathbb{R}^{n-1} \times [0, 1)$ (points of which can be denoted $(\vec{x}, y)$, where $\vec{x} = (x_1, \ldots, x_{n-1})$). In this context, we want to show that $\Phi^{-2}g\Phi^2 - \bar{g} \times Id =: G$ has all partial derivatives of all orders equal to 0 at the origin. (Note that it does not matter which local coordinates we chose.)

We use the following notation: subscript y means the y-component as before; subscript i for $1 \leq i \leq n - 1$ means the x_i-component. Assume that G_y does not have all partials equal to 0 at the origin. Then there is some term $x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} y^{i_n}$ in the Taylor series for G_y with nonzero coefficient. Let $N = i_1 + \cdots + i_n$. By Taylor's theorem, $G_y(\vec{x}, y) = P_N(\vec{x}, y) + R_N(\vec{x}, y)$, where P_N is the Taylor polynomial up to degree N, and R_N can be written in the form

$$
\sum_{j_1 + \cdots + j_n = N} c_{j_1, \ldots, j_n}(\vec{x}, y)x_1^{j_1} \cdots x_{n-1}^{j_{n-1}} y^{j_n},
$$

where each $c_{j_1, \ldots, j_n} \to 0$ as $(\vec{x}, y) \to 0$. Therefore, $\max_{r(\vec{x}, y) = r_0} G_n \sim r_0^N$ as $r_0 \to 0$.

On the other hand, when $(\vec{x}, y)$ is close to 0, $g_y(\vec{x}, y)$ is close to ay, where $a = \frac{\partial g_y}{\partial y}(0)$ (which is greater than 0); in particular, in a neighborhood of 0,

$$
\frac{a}{2}y \leq g_y(\vec{x}, y) \leq 2ay.
$$

This implies that

$$\frac{1}{\log(e^{1/y} - \log(\frac{a}{2}))} - y \le G_y(\vec{x}, y) \le \frac{1}{\log(e^{1/y} - \log(2a))} - y.$$

Since $\frac{1}{\log(e^{1/y} - C)} - y \sim Cy^2/e^{\frac{1}{y}}$ as $y \to 0$, it follows that for $(\vec{x})$ in a small enough interval about 0, $G_y(\vec{x}, y) \to 0$ faster than any polynomial as $y \to 0$.

We can use the same type of reasoning to show that G_i ($1 \le i \le n-1$) vanishes to all orders at 0. If it did not, then we could find $(\vec{x}, y)$ nearby such that $G_i(\vec{x}, y)$ is only polynomially small in terms of $r(\vec{x}, y)$. We will show that this is not the case.

For any $\vec{x}$, for y sufficiently close to 0, the difference between $g_i(\vec{x}, y) - g_i(\vec{x}, 0)$ and $\frac{\partial g_i}{\partial y}(\vec{x}, 0) \cdot y$ is small relative to y. Indeed, if N is a sufficiently small neighborhood of the origin, we will have

$$\left| g_i(\vec{x}, y) - g_i(\vec{x}, 0) - \frac{\partial g_i}{\partial y}(\vec{x}, 0) \cdot y \right| < y$$

for all $(\vec{x}, y) \in N$.

Now $G_i(\vec{x}, y) = g_i(\vec{x}, \phi^2(y)) - g_i(\vec{x}, 0)$; for $(\vec{x}, y) \in N$,

$$\left| g_i(\vec{x}, \phi^2(y)) - g_i(\vec{x}, 0) - \frac{\partial g_i}{\partial y}(\vec{x}, 0) \cdot \phi^2(y) \right| < \phi^2(y),$$

so

$$\left| g_i(\vec{x}, \phi^2(y)) - g_i(\vec{x}, 0) \right| < \phi^2(y) + \left| \frac{\partial g_i}{\partial y}(\vec{x}, 0) \right| \cdot \phi^2(y).$$

But $\left| \frac{\partial g_i}{\partial y}(\vec{x}, 0) \right|$ is bounded above on N, so we have what we sought: $G_i(\vec{x}, y) \to 0$ faster than any polynomial as $(\vec{x}, y) \to 0$. This finishes the proof of claim (2). $\square$

Remark 6. We may replace Diff^∞ with Diff^r in the statement of the theorem. Most of the proof is unchanged. We must do slightly more for claim (1), that is, showing that conjugation of a C^r map by Φ is still C^r. This is because the map "h" may be only C^{r-1}. But it can be seen by an inductive argument that $(x, y) \mapsto h(x, \phi(y))$ is C^r.

3. APPLICATION: NEW DISTORTION ELEMENTS VIA HEISENBERG GROUP ACTIONS

In this section, we use our main result to exhibit new examples of distortion elements in C^∞ diffeomorphism groups of surfaces. To do this, we construct actions of the Heisenberg group on surfaces.

Definition 7. The *(discrete) Heisenberg group* is the group of matrices

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

H is generated by the elements $X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. The commutator is $Z = [X, Y] = XYX^{-1}Y^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Since X and Y commute with the commutator Z, H is a 2-step nilpotent group. Intuitively, H is "close to abelian". This gives another motivation for studying Heisenberg group actions: it is a natural first step towards extending what we know about the dynamics of abelian groups of diffeomorphisms (e.g. the question of whether there exist global fixed points) to the non-abelian setting.

There are no relations besides two generators commuting with their commutator, so whenever we have a group with three elements f, g, and h of infinite order such that $[f, g] = h$, $fh = hf$, and $gh = hg$, this group is isomorphic to the Heisenberg group.

Proposition 8. *The central element $Z \in H$ is a distortion element.*

Proof. Notice that $[X^m, Y^n] = Z^{mn}$ for any $m, n \in \mathbb{Z}$, and in particular $[X^n, Y^n] = Z^{n^2}$. The length of the word $[X^n, Y^n]$ is $4n$, so if we take $\{X, Y\}$ as the generating set of the Heisenberg group, the word length of Z^{n^2} is at most $4n$. The quantity in the definition of distortion is $\frac{4n}{n^2}$, which goes to 0 as $n \to \infty$. $\qquad\square$

Any time we give a faithful action $\phi \colon H \to \text{Diff}^\infty(M)$ for a manifold M, the diffeomorphism $\phi(Z)$ is a distortion element in $\text{Diff}^\infty(M)$. In this section, we show how to construct new actions of the Heisenberg group on surfaces.

H obviously acts linearly on $\mathbb{R}^3$. We may "projectivize" this to get an action on S^2 which is faithful and real-analytic. Namely, let S^2 be the unit sphere in $\mathbb{R}^3$. Define $\phi \colon H \to \text{Diff}(S^2)$ as follows: for $x \in S^2$ and $A \in H$, define $\phi(A)(x) = \frac{Ax}{|Ax|}$. This action has two *global fixed points*, points fixed by every element of the group: $(\pm 1, 0, 0)$.

We may puncture at one of these fixed points, so we get an action of H on the open disk. If we also puncture at the other fixed point, we get an action of H on the open annulus. In fact, there is a canonical way of compactifying to get a closed disk D or closed annulus A, called "blowing up", which we will describe. By doing a blow up at one or both fixed points, we get an action of H on the closed disk and on the closed annulus.

Blowing up. For reference, see Melrose [10]. Let us first consider blowing up the origin in $\mathbb{R}^n$. Intuitively, we will remove the origin, and insert an $(n-1)$-sphere there. Define the blow up to be $\beta(\mathbb{R}^n, 0) = S^{n-1} \times [0, \infty)$, together with the blow-down map $\beta \colon \beta(\mathbb{R}^n, 0) \to \mathbb{R}^n$ given by $\beta(\theta, r) = r\theta$ (S^{n-1} is identified with the unit sphere in $\mathbb{R}^n$). This is a diffeomorphism of $S^{n-1} \times (0, \infty)$ to $\mathbb{R}^n \setminus \{0\}$. Given a smooth map $f \colon \mathbb{R}^n \to \mathbb{R}^n$ fixing the origin, we get an induced map $\tilde{f} \colon \beta(\mathbb{R}^n, 0) \to \beta(\mathbb{R}^n, 0)$, defined as follows:

$$\tilde{f}(x) = \begin{cases} \beta^{-1} f \beta(x), & x \in S^{n-1} \times (0, \infty) \\ (\frac{D_0 f(\theta)}{|D_0 f(\theta)|}, 0), & \text{else.} \end{cases}$$

Here S^{n-1} is being identified with the unit tangent space of $\mathbb{R}^n$ at 0. It is a standard result that if f was C^∞, then so is $\tilde{f}$; see [10]. Blow ups can also be done on manifolds. To blow up at $x \in M$, take a diffeomorphism $\phi \colon U \to \mathbb{R}^n$ for a neighborhood $U \ni x$, such that $\phi(x) = 0$. Define

$$\beta(M, x) = ((M \setminus x) \cup S^{n-1} \times [0, \infty)) / \sim,$$

where for any $y \in U \setminus x, y \sim \beta^{-1}(\phi(y))$, β being the blow-down defined above. $\beta(M, x)$ is a smooth manifold with boundary. For a map $f \colon M \to M$ fixing x, we
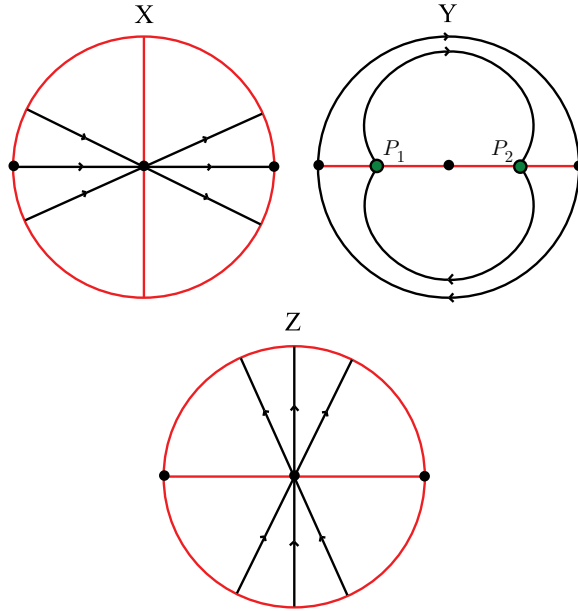
FIGURE 1. Action of H on the disk. All drawn curves and points are invariant under the given element; those without arrows are pointwise fixed.

get a map $\tilde{f}$, as follows:

$$\tilde{f}(x) = \begin{cases} f(x), & x \in M \setminus x \\ \frac{D_0(\phi f \phi^{-1})(\theta)}{|D_0(\phi f \phi^{-1})(\theta)|}, & \text{else.} \end{cases}$$

Again, if $f \in \text{Diff}^\infty(M)$ and f fixes x, then $\tilde{f} \in \text{Diff}^\infty(\beta(M,x))$. Also note that if both f and g fix x, then $\widetilde{f \circ g} = \tilde{f} \circ \tilde{g}$. Therefore, the action of H we have described on S^2 yields smooth actions on the closed disk and on the closed annulus, by blowing up at global fixed points. For a picture on the closed disk, see Figure 1.

The points P_1 and P_2 in the action of Y have the following significance. Let e (for equator) be the horizontal line through the center of the disk, and ∂D the boundary; $D \setminus (e \cup \partial D)$ has two components, which we may denote D^u and D^l (upper and lower). For every $x \in D^u$, $\omega(x)$ is P_2, where $\omega(x)$ is the ω limit set of x. For every $x \in D^l$, $\omega(x)$ is P_1. The action of H on the closed annulus is similar; we take the action on the closed disk and blow up at the global fixed point at the center.

Since the given actions of H on D and A agree on the boundary components, we may glue them together, or glue the two boundary components of the annulus. The standard coordinates for the blow up may be defined by taking coordinates on the boundary sphere, and also recording r. In these coordinates, the above Heisenberg actions are C^1, but not C^2. But by our theorem, after a change of coordinates the gluing is C^∞. See Figure 2.

In fact, we may glue arbitrarily many annuli $A_1, A_2, \ldots, A_n$ concentrically and get a smooth action of H on the resulting annulus for which the A_i are H-invariant. We may then glue in a disk, so we get an action on D; two disks, so we get an
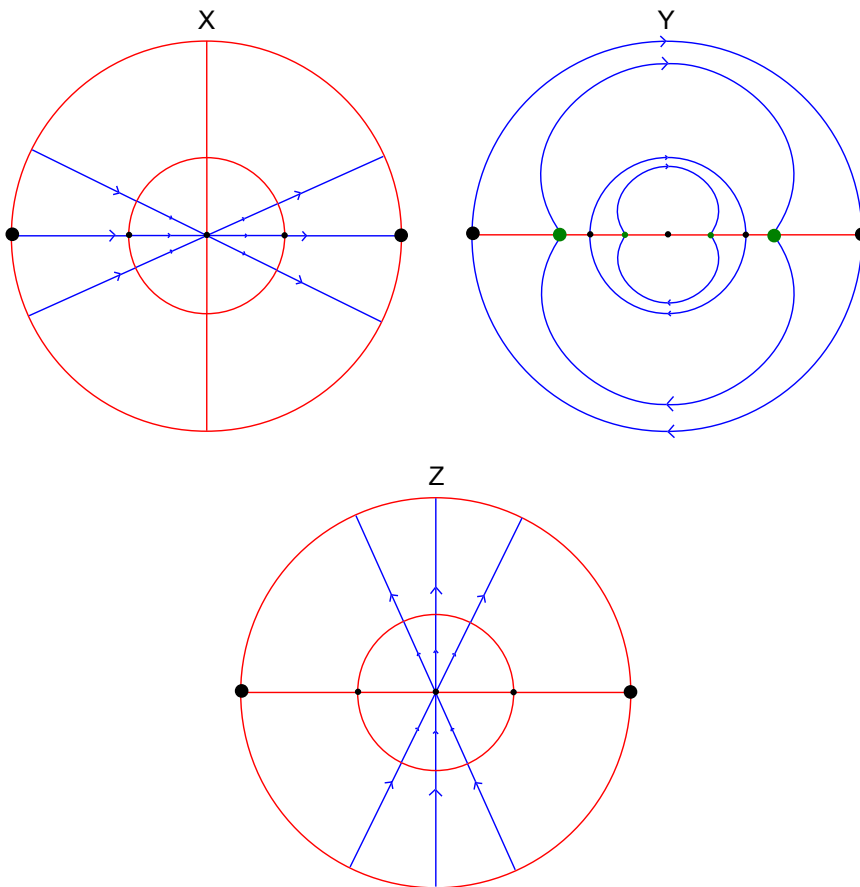
FIGURE 2. Action of H on the disk and annulus, glued together.

action on S^2; or a new annulus, so we get an action on T^2. Or, we may glue the two boundary components of $\bigcup_i A_i$, also yielding an action on T^2. Finally, in these glued examples there are lots of global fixed points; we can blow them up and glue to get even more examples.

4. APPLICATION: GENERALIZING A RESULT OF FRANKS-HANDEL

In this section, we prove the following corollary of Franks and Handel's theorem on distortion elements:

Corollary 9. *Let S be a compact surface with nonempty boundary. Let $f \in \mathrm{Diff}^1(S)_0$ be distorted. If $S = D$ is the closed disk, assume that f has at least two fixed points in the interior of the disk, or at least one fixed point on the boundary. If $S = A$, assume f has at least one fixed point. Then (*) holds.*

Proof. Assume, by way of contradiction, that (*) does not hold. Thus there is some f-invariant measure μ such that $\mathrm{supp}(\mu) \not\subset \mathrm{Fix}(f)$.

First, suppose that $S = D$ is the closed disk, and f has no fixed points in $\mathrm{Int}(D)$. Thus f does have at least one fixed point on ∂D, by the Brouwer fixed

point theorem. We must have $\mu(\partial D \setminus \text{Fix}(f)) = 0$, since if this set is nonempty it consists of intervals on which points wander towards fixed points. Therefore, by assumption, $\mu(\text{Int}(D)) > 0$. Therefore, we can apply the Poincaré recurrence theorem to get a recurrent point in the interior of the disk. But that implies, by the Brouwer plane translation theorem, that there is actually a fixed point in the interior of the disk, a contradiction.

Now suppose we are not in the above case (the disk with no fixed points in the interior). Let $M_i = S \times \{i\}$ for $i = 1, 2$. Glue each boundary component of M_1 to the corresponding one in M_2, in the obvious way: for $x \in \partial S, \alpha(x, 1) = (x, 2)$. Choose a smooth structure on the result, as described above. The result can be called the *double* of $S, D(S)$.

By Remark 6, we can choose conjugacies Ψ_1 and Ψ_2 to make the glued map $\tilde{f} = \mathfrak{g}(\Psi_1^{-1} f_1 \Psi_1, \Psi_2^{-1} f_2 \Psi_2)$ be C^1. In fact, since M_1 and M_2 are both copies of S, we can find a single conjugacy $\Psi \in \text{Homeo}(S)$ and let $\Psi_i = \Psi \times id, i = 1, 2$.

Any fixed point for f in the interior $\text{Int}(S)$ yields two fixed points for $\tilde{f}$, one on each side. A fixed point for f on ∂S yields one fixed point for $\tilde{f}$. The hypotheses guarantee that $\tilde{f}$ has as many fixed points as the theorem of Franks and Handel requires: one for T^2 and three for S^2.

There will be an invariant probability measure not supported on the fixed point set of $\tilde{f}$. Namely, $\Psi^{-1} f \Psi$ preserves the measure $\Psi_*^{-1}(\mu)$, whose support is $\text{supp}(\Psi_*^{-1}(\mu) = \Psi^{-1}(\text{supp}(\mu))$. On the other hand, $\text{Fix}(\Psi^{-1} f \Psi) = \Psi^{-1}(\text{Fix}(f))$. So $\text{supp}(\Psi_*^{-1}(\mu)) \not\subset \text{Fix}(\Psi^{-1} f \Psi)$. If we take $\Psi_*^{-1}(\mu)$ on both copies of S, we get an invariant measure for $\tilde{f}$ not supported in $\text{Fix}(\tilde{f})$.

Therefore, by Franks and Handel, $\tilde{f}$ is not distorted in $\text{Diff}^1(D(S))_0$. But the map $f \mapsto \tilde{f}$ is a 1-1 homomorphism, so it preserves the property of being a distortion element, a contradiction. □

This corollary leads to the following question:

Question 10. Is an irrational rotation of the disk distorted in $\text{Diff}^1(D)_0$?

Acknowledgments

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