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THE HARMONIC MAP PROBLEM WITH MIXED BOUNDARY CONDITIONS

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ABSTRACT. Given two polygons $S \subset \mathbb{R}^2$ and $\Sigma \subset \mathbb{R}^m$ with the same number of sides, we prove the existence and uniqueness of a smooth harmonic map $u : S \to \mathbb{R}^m$ satisfying the mixed boundary conditions for S and Σ. This solution is constructed and characterized as a minimizer of the Dirichlet's energy in the class of maps which satisfy the first mixed boundary condition. Several properties of the solution are established. We also discuss the mixed boundary conditions for harmonic maps defined in smooth domains of the plane.

1. INTRODUCTION

Given two smooth Jordan domains Ω_1 and Ω_2 in the plane, the Riemann mapping theorem ensures the existence of bijective conformal maps f between Ω_1 and Ω_2 which extend to homeomorphisms of $\overline{\Omega_1}$ onto $\overline{\Omega_2}$, and moreover satisfy the boundary condition:

$$(1)\quad x \in \partial\Omega_1 \Rightarrow \frac{\partial f}{\partial n}(x) \perp \partial\Omega_2 \text{ , where } n \text{ is the outer unit normal vector to } \partial\Omega_1.$$

A computation shows that these conformal maps are the only minimizers of the Dirichlet's energy

$$J_{\Omega_1}(h) = \int_{\Omega_1} \frac{1}{2}|\nabla h|^2 \mathrm{d}x, \tag{2}$$

in the class H of harmonic maps in Ω_1, $C^1(\overline{\Omega_1}, \mathbb{R}^2)$ smooth, and whose restrictions to $\partial\Omega_1$ are sense-preserving homeomorphisms onto $\partial\Omega_2$. In Corollary 6.4, we also give another characterization of conformal diffeomorphisms between Ω_1 and Ω_2. We show that a map $h \in H$ satisfies the boundary condition (1) if and only if it is conformal. Another interesting approach consists of proving directly the existence of a minimizer of the Dirichlet's energy in the class H, and deducing that it is a conformal diffeomorphism between Ω_1 and Ω_2. This is done when solving the Plateau problem in the particular case of a plane curve (cf. [4]).

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In the present paper, we follow this approach when the domain $S \subset \mathbb{R}^2$ and the target $\Sigma \subset \mathbb{R}^m$ (with $m \geq 2$) are two polygons with the same number of sides, $n \geq 3$. By minimizing the Dirichlet's energy in a suitable class of maps we prove the existence and uniqueness of a smooth solution $u : S \to \mathbb{R}^m$ to the problem:

$$\begin{cases} \Delta u = 0 \text{ in } S, \quad \text{with mixed boundary conditions:} \\ x \in S_i \Rightarrow u(x) \in \Pi_i \,, \\ x \in \operatorname{int}(S_i) \Rightarrow \dfrac{\partial u}{\partial r_i}(x) \perp \Pi_i \,, \end{cases} \tag{3}$$

where S_i denotes the ith side of S, r_i its outer unit normal vector, and Π_i the supporting line of the ith side of Σ. Note that the first boundary condition is different from the corresponding one for smooth domains. It imposes that u maps the vertices of S onto the corresponding vertices of Σ, but does not require for the restriction of u to ∂S, to be a homeomorphism onto $\partial\Sigma$. The second boundary condition resembles the conformal boundary condition (1). However, the solution u is conformal only for triangles ($n = 3$), and in some particular cases when $n \geq 4$. Indeed, in the case of triangles, we can compose the conformal diffeomorphisms of S and Σ onto the unit disk with a Möbius transformation to ensure that the vertices of S go to the vertices of Σ. When $n \geq 4$ and Σ is flat, the polygonal constraints impose extra conditions which prevent the solution from being conformal. For instance, if S and Σ are two rectangles, the solution u is an affine map. With the help of the Schwarz-Christoffel formula, it is possible to obtain a necessary and sufficient condition (depending on the shape of S and Σ) for the solution to be conformal. This condition however, is not of any practical use (cf. [1]).

The mixed boundary conditions have been studied by Hamilton in [11], where an existence result is established for harmonic maps between Riemannian manifolds with boundary. In [7], harmonic maps between Riemannian polyhedra are also treated thoroughly. It seems however that variational problems with the polygonal constraints are not frequently encountered in the literature. Problem (3) is similar to the Plateau problem, since the solution of (3) is constructed and characterized as the only minimizer of the Dirichlet's energy in the class of maps which satisfy the first mixed boundary condition. The polygonal constraints are interesting because they provide a unique solution which in general is not conformal.

The plan of the paper is as follows. In section 2, we introduce the notation and the relevant class of maps for problem (3). In section 3, we construct and characterize the solution of problem (3) as a minimizer of the Dirichlet's energy. In section 4, we establish properties of the solution when Σ is flat, with the help of the tools available in two dimensions (e.g. the Radó–Kneser–Choquet theorem among others, cf. [6]). We do not utilize the Riemann mapping theorem in this section and give alternative proofs of the results which can be deduced from it (cf. Proposition 4.5). Sections 5 and 6 are independent and examine the case of smooth domains. In section 5, we emphasize the similarity between the solution of problem (3) and conformal maps between smooth domains of the plane. We give a simple proof of the fact that conformal maps are also characterized as minimizers of the Dirichlet's energy. Finally, in section 6, we prove that for a harmonic map $u : \Omega \to \mathbb{R}^m$ (with $m \geq 1$) defined in a smooth domain of the plane, the conformal boundary conditions propagate to the interior of the set.

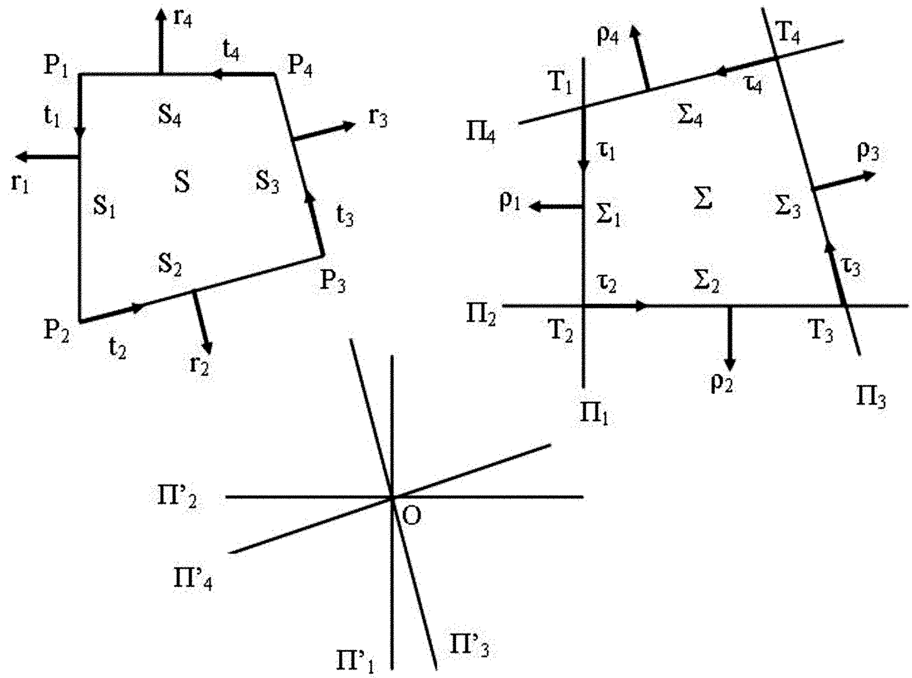

FIGURE 1. The polygons S and Σ.

2. Notation and preliminaries

Let S be an open domain of the plane, bounded by a polygonal line, whose vertices, in order from the line being described with S to the left, are P_1, P_2,..., P_n. For convenience, we set $P_{n+1} = P_1$. For $i = 1, \dots, n$, we denote that by S_i the ith side of S, which is the closed segment P_iP_{i+1}. r_i, will be the outer unit normal vector to the side S_i while $t_i = \frac{1}{|\overrightarrow{P_iP_{i+1}}|}\overrightarrow{P_iP_{i+1}}$ will be the tangential one.

Let us now consider for every $i = 1, \dots, n$, a line $\Pi_i \subset \mathbb{R}^m$ $(m \geq 2)$, and suppose that two consecutive lines of this family intersect at a point, that is, $\Pi_n \cap \Pi_1 =: T_1$, and $\Pi_i \cap \Pi_{i+1} =: T_{i+1}$, for $i = 1, \dots, n-1$. Let Π'_i be the line containing the origin O and parallel to Π_i. When the points $T_1, \dots, T_n$ are distinct, they define a polygon $\Sigma \subset \mathbb{R}^m$. Setting $T_{n+1} = T_1$, we can define for every $i = 1, \dots, n$ the vector $\tau_i = \frac{1}{|\overrightarrow{T_iT_{i+1}}|}\overrightarrow{T_iT_{i+1}}$, and denote by Σ_i the closed segment T_iT_{i+1}. If in addition $m = 2$, we can also define the vector ρ_i such that (ρ_i, τ_i) is a positively oriented orthonormal basis.

In what follows, we shall utilize the Euclidean inner product denoted by $\langle\cdot,\cdot\rangle$, and the Euclidean norm $|\cdot|$. Given a map $F : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^m$, we shall denote by F_{x_i} the partial derivative $\frac{\partial F}{\partial x_i}$, and by $F_r := \nabla F \cdot r$, the derivative of F in the direction of the vector $r \in \mathbb{R}^2$.

To construct a solution to problem (3) we will minimize the Dirichlet's energy in the class

$$K := \{u \in W^{1,2}(S, \mathbb{R}^m) \mid (Tr\, u)(S_i) \subset \Pi_i \;, \forall i \;\; 1 \leq i \leq n\} \tag{4}$$

of $W^{1,2}(S,\mathbb{R}^m)$ maps such that the restriction of the trace (denoted by Tr) to each of the sides S_i of S, takes its values almost everywhere in the corresponding line Π_i. We shall also consider the linear subspace of $W^{1,2}(S,\mathbb{R}^m)$,

$$E := \{v \in W^{1,2}(S,\mathbb{R}^m) \mid (Tr\, v)(S_i) \subset \Pi_i' \;, \forall i \;\; 1 \leq i \leq n\}. \tag{5}$$

The class K is non empty according to

Proposition 2.1. *There exists a map $u_0 \in C^\infty(\overline{S},\mathbb{R}^m) \cap K$.*

Proof. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a C^∞ increasing function such that

$$\rho(\alpha) = \begin{cases} 0 & \text{for } \alpha \leq 1/4, \\ 1 & \text{for } \alpha \geq 3/4. \end{cases}$$

For each side S_i, we consider $R_i := \{P_i + \alpha\overrightarrow{P_iP_{i+1}} + \beta r_i \mid -\epsilon < \alpha < 1+\epsilon \;, |\beta| < \epsilon\}$ an open rectangle of width $2\epsilon > 0$ containing S_i, and define $\phi_i : R_i \to \mathbb{R}^m$ by $\phi_i(P_i + \alpha\overrightarrow{P_iP_{i+1}} + \beta r_i) := T_i + \rho(\alpha)\overrightarrow{T_iT_{i+1}}$. ϕ_i is C^∞, equal to T_i (respectively T_{i+1}) in a neighborhood of P_i (respectively P_{i+1}), and satisfies $\phi_i(S_i) = \Sigma_i$. Furthermore, if ϵ is small enough, $\phi_i \equiv \phi_{i+1} \equiv T_{i+1}$ in $R_i \cap R_{i+1}$. Thus in $\Omega := R_1 \cup \ldots \cup R_n$, which is a neighborhood of ∂S, we can define a smooth map ϕ by setting $\phi \equiv \phi_i$ in R_i, and take $u_0 := \sigma\phi$, where $\sigma : \mathbb{R}^2 \to [0,1]$ is C^∞, with support in Ω such that $\sigma \equiv 1$ on ∂S. Clearly, the map u_0 has all the desired properties. $\square$

The next propositions establish some properties of the class K.

Proposition 2.2. *Let $u_0 \in K$ be fixed. Then $K = u_0 + E$, and K is an affine subspace which is strongly as well as weakly closed in $W^{1,2}(S,\mathbb{R}^m)$.*

Proof. The only point to check is that E is strongly closed. Let $(v_n) \subset E$ be a sequence such that v_n converges to v in $W^{1,2}(S,\mathbb{R}^m)$. Then $(Tr\, v_n)$ converges to $(Tr\, v)$ in $L^2(\partial S,\mathbb{R}^m)$, and for a subsequence $(Tr\, v_n)$ converges to $(Tr\, v)$ a.e. in ∂S. This implies that $(Tr\, v)(S_i) \subset \Pi_i'$ a.e. and $v \in E$. Since K is an affine subspace of $W^{1,2}(S,\mathbb{R}^m)$, it is convex and therefore it is also weakly closed in $W^{1,2}(S,\mathbb{R}^m)$. $\square$

Proposition 2.3. *There exists a positive constant C such that for every $v \in E$,*

$$\|v\|^2_{L^2(S,\mathbb{R}^m)} \leq C\, \|\nabla v\|^2_{L^2(S,\mathbb{R}^{2m})} \,. \tag{6}$$

Proof. By homogeneity it is sufficient to prove the statement for $v \in E$ such that $\|v\|_{W^{1,2}(S,\mathbb{R}^m)} = 1$. Suppose by contradiction that there exists a sequence $(v_n) \subset E$ such that

$$\|v_n\|_{W^{1,2}(S,\mathbb{R}^m)} = 1 \text{ and } \|v_n\|^2_{L^2(S,\mathbb{R}^m)} \geq n\, \|\nabla v_n\|^2_{L^2(S,\mathbb{R}^{2m})} \,. \tag{7}$$

Since $\|v_n\|_{W^{1,2}(S,\mathbb{R}^m)}$ is bounded, there exists a subsequence still called v_n, such that $v_n \rightharpoonup v$ weakly in $W^{1,2}(S,\mathbb{R}^m)$, and $v_n \to v$ strongly in $L^2(S,\mathbb{R}^m)$. Utilizing (7), we see that $\|\nabla v_n\|^2_{L^2(S,\mathbb{R}^{2m})} \to 0$ since $\|v_n\|^2_{L^2(S,\mathbb{R}^m)} \to \|v\|^2_{L^2(S,\mathbb{R}^m)}$. But we also have $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^2(S,\mathbb{R}^{2m})$, thus $\|\nabla v\|_{L^2(S,\mathbb{R}^{2m})} \leq \liminf \|\nabla v_n\|_{L^2(S,\mathbb{R}^{2m})} = 0$ and we conclude that v is constant in S. This implies that $v \equiv 0$ by definition of E, and contradicts the fact that $\|v\|_{L^2(S,\mathbb{R}^m)} = 1$ (since $\|v_n\|^2_{L^2} + \|\nabla v_n\|^2_{L^2} = 1$). $\square$

3. The construction of the solution

Utilizing a direct method, we begin by proving the existence and uniqueness of a minimizer of the Dirichlet's energy in K.

Theorem 3.1. *There exists a unique $u \in K$ such that $J_S(u) = \min_K J_S$, where*

$$J_S(v) := \frac{1}{2}\int_S |\nabla v|^2 \mathrm{d}x_1 \mathrm{d}x_2 = \frac{1}{2}\int_S (|v_{x_1}|^2 + |v_{x_2}|^2)\mathrm{d}x_1 \mathrm{d}x_2. \tag{8}$$

Furthermore,

$$\int_S \nabla u \nabla \xi \mathrm{d}x = 0 \quad \text{for every } \xi \in E. \tag{9}$$

Proof. Let $M := \inf_K J_S$, and let $(v_n) \subset K$ be a sequence such that $J_S(v_n) \to M$. First, we check that $\|v_n\|_{L^2(S,\mathbb{R}^m)}$ is bounded. Indeed, we can write

$$\begin{aligned} \|v_n\|_{L^2}^2 &\leq 2\|v_n - u_0\|_{L^2}^2 + 2\|u_0\|_{L^2}^2 \ \text{(where } u_0 \text{ is given by Proposition 2.1)} \\ &\leq 2C\|\nabla(v_n - u_0)\|_{L^2}^2 + 2\|u_0\|_{L^2}^2 \ \text{(where } C \text{ is given by Proposition 2.3)} \\ &\leq 4C\|\nabla v_n\|_{L^2}^2 + 4C\|\nabla u_0\|_{L^2}^2 + 2\|u_0\|_{L^2}^2. \end{aligned}$$

As a consequence $\|v_n\|_{W^{1,2}(S,\mathbb{R}^m)}$ is also bounded, and for a subsequence still called v_n, $v_n \rightharpoonup u$ weakly in $W^{1,2}(S,\mathbb{R}^m)$, with $u \in K$ (cf. Proposition 2.2). Furthermore, $\nabla v_n \rightharpoonup \nabla u$ weakly in $L^2(S,\mathbb{R}^{2m})$, and thus by lower semicontinuity we deduce that $\|\nabla u\|_{L^2(S,\mathbb{R}^m)}^2 \leq \liminf \|\nabla v_n\|_{L^2(S,\mathbb{R}^m)}^2$. This proves that $J_S(u) = M$. Since u is a minimizer, and for every $\xi \in E$ and every $\lambda \in \mathbb{R}$, $u + \lambda\xi \in K$, we obtain that $\frac{\mathrm{d}}{\mathrm{d}\lambda}\big|_{\lambda=0} J_S(u+\lambda\xi) = \int_S \nabla u \nabla \xi \mathrm{d}x = 0$. The uniqueness of the minimizer follows from this relation. Indeed, if u and v are two minimizers, taking $\xi = v - u$ in (9) for the minimizers u and v, we have $\int_S \nabla u \nabla(v-u)\mathrm{d}x = \int_S \nabla v \nabla(v-u)\mathrm{d}x = 0$, which implies that $\|\nabla(v-u)\|_{L^2}^2 = 0$, and by Proposition 2.3 that $u \equiv v$. ☐

Next, we shall establish with the help of the following lemmas that the minimizer u is smooth.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth domain symmetric with respect to the x_1 coordinate axis and let $\Omega_0 = \{(x_1,x_2) \in \Omega \mid x_2 = 0\}$, $\Omega_+ = \{(x_1,x_2) \in \Omega \mid x_2 > 0\}$, $\Omega_- = \{(x_1,x_2) \in \Omega \mid x_2 < 0\}$. Suppose that $u \in W^{1,2}(\Omega_+,\mathbb{R}^m)$ is such that $(Tr\, u)\big|_{\Omega_0}(\Omega_0) \subset \mathbb{R} \times \{0\}^{m-1}$, where $\big|_{\Omega_0}$ denotes the restriction of the trace to Ω_0. Then, the symmetric extension of u to Ω defined by*

$$U(x) := \begin{cases} u(x_1,x_2) := (f(x_1,x_2), g(x_1,x_2)) \in \mathbb{R} \times \mathbb{R}^{m-1} & \text{for } (x_1,x_2) \in \Omega_+ \\ v(x_1,x_2) := (f(x_1,-x_2), -g(x_1,-x_2)) \in \mathbb{R} \times \mathbb{R}^{m-1} & \text{for } (x_1,x_2) \in \Omega_- \end{cases}$$

belongs to $W^{1,2}(\Omega,\mathbb{R}^m)$.

Proof. It is obvious that $v \in W^{1,2}(\Omega_-,\mathbb{R}^m)$. Furthermore, if the sequence

$$\phi_m(x_1,x_2) := (\alpha_m(x_1,x_2), \beta_m(x_1,x_2)) \in \mathbb{R} \times \mathbb{R}^{m-1}$$

of $C^\infty(\overline{\Omega_+},\mathbb{R}^m)$ smooth maps converges to u in $W^{1,2}(\Omega_+,\mathbb{R}^m)$, and we define for $(x_1,x_2) \in \Omega_-$ the sequence

$$\psi_m(x_1,x_2) := (\alpha_m(x_1,-x_2), -\beta_m(x_1,-x_2)) \in \mathbb{R} \times \mathbb{R}^{m-1},$$

it is clear that ψ_m also converges to v in $W^{1,2}(\Omega_-,\mathbb{R}^m)$. Since by assumption $(Tr\, u)\big|_{\Omega_0}(\Omega_0)\subset\mathbb{R}\times\{0\}^{m-1}$, $\lim_{m\to\infty}\beta_m\big|_{\Omega_0}=0$ holds in $L^2(\Omega_0,\mathbb{R}^{m-1})$ and thus

$$(Tr\, v)\big|_{\Omega_0}=\lim_{m\to\infty}(\alpha_m\big|_{\Omega_0},-\beta_m\big|_{\Omega_0}) \text{ in } L^2(\Omega_0,\mathbb{R}^m)$$

coincides with $(Tr\, u)\big|_{\Omega_0}$. This implies that U defines a $W^{1,2}(\Omega,\mathbb{R}^m)$ map. □

Lemma 3.3. *Let Ω, Ω_+, Ω_-, Ω_0, u and U be as in Lemma 3.2. If u, in addition, satisfies*

$$\int_{\Omega_+}\nabla u\,\nabla\xi\,\mathrm{d}x=0 \quad \textit{for every } \xi\in C_c^\infty(\Omega,\mathbb{R}^m) \textit{ such that } \xi(\Omega_0)\subset\mathbb{R}\times\{0\}^{m-1},$$

$$\textit{then} \int_{\Omega}\nabla U\,\nabla\phi\,\mathrm{d}x=0 \quad \textit{for every } \phi\in C_c^\infty(\Omega,\mathbb{R}^m).$$

Proof. Every map $\phi(x_1,x_2):=(\alpha(x_1,x_2),\beta(x_1,x_2))\in\mathbb{R}\times\mathbb{R}^{m-1}$, belonging to $C_c^\infty(\Omega,\mathbb{R}^m)$, can be written $\phi=o+e$ with

$$o=(\alpha,\gamma)\in\mathbb{R}\times\mathbb{R}^{m-1}, \text{ where } \gamma(x_1,x_2):=1/2(\beta(x_1,x_2)-\beta(x_1,-x_2)),$$

$$e=(0,\delta)\in\mathbb{R}\times\mathbb{R}^{m-1}, \text{ where } \delta(x_1,x_2):=1/2(\beta(x_1,x_2)+\beta(x_1,-x_2)).$$

Clearly γ is odd and δ is even with respect to x_2. We also set $\epsilon(x_1,x_2):=\alpha(x_1,-x_2)$ and $\psi:=(\epsilon,\gamma)\in C_c^\infty(\Omega,\mathbb{R}^m)$. Because of the symmetry and the fact that $o(\Omega_0)\subset\mathbb{R}\times\{0\}^{m-1}$, $\psi(\Omega_0)\subset\mathbb{R}\times\{0\}^{m-1}$, one can check that

$$\int_{\Omega_-}\nabla v\,\nabla e\,\mathrm{d}x=-\int_{\Omega_+}\nabla u\,\nabla e\,\mathrm{d}x,$$

$$\text{and} \int_{\Omega_-}\nabla v\,\nabla o\,\mathrm{d}x=\int_{\Omega_+}\nabla u\,\nabla\psi\,\mathrm{d}x=0=-\int_{\Omega_+}\nabla u\,\nabla o\,\mathrm{d}x.$$

Thus $\displaystyle\int_{\Omega_-}\nabla U\,\nabla\phi\,\mathrm{d}x=-\int_{\Omega_+}\nabla U\,\nabla\phi\,\mathrm{d}x$, which establishes the Lemma. □

Lemma 3.4. *Let S_1 and S_2 be two open sectors of the plane with center at the origin O, with radii R_1 and R_2, and angles α_1 and α_2, respectively. We suppose that $\overline{S_1}\subset S_2\cup\{O\}$ and we consider $u:S_2\to\mathbb{R}^m$, a harmonic map in S_2 such that $|\nabla u|\in L^2(S_2,\mathbb{R})$. Then, $|x||\nabla u(x)|\to 0$ as $S_1\ni x\to O$.*

Proof. There exists $\epsilon>0$ such that for every $x\in S_1$, $\overline{B(x,\epsilon|x|)}\subset S_2$, where $B(x,r)$ denotes the open ball of center x and radius r. Utilizing the fact that $\dfrac{\partial u}{\partial x_i}\in L^2(S_2,\mathbb{R}^m)$ we obtain, thanks to the mean value theorem applied to the derivatives of u, that for $x\in S_1$ and $i=1,2$:

$$\left|\frac{\partial u}{\partial x_i}(x)\right|\leq\frac{1}{\pi\epsilon^2|x|^2}\int_{B(x,\epsilon|x|)}\left|\frac{\partial u}{\partial x_i}(x)\right|\mathrm{d}x\leq\frac{1}{\sqrt{\pi}\epsilon|x|}\left(\int_{B(x,\epsilon|x|)}\left|\frac{\partial u}{\partial x_i}(x)\right|^2\mathrm{d}x\right)^{1/2}$$

$$\leq\frac{1}{\sqrt{\pi}\epsilon|x|}\left(\int_{B(O,(1+\epsilon)|x|)\cap S_2}\left|\frac{\partial u}{\partial x_i}(x)\right|^2\mathrm{d}x\right)^{1/2},$$

and noting that the expression in parenthesis goes to 0 as $|x|\to 0$, we deduce the Lemma. □

Now we can state

Theorem 3.5. *There exists in* $C(\overline{S}, \mathbb{R}^m) \cap C^\infty(\overline{S} \setminus \{P_1, \cdots, P_n\}, \mathbb{R}^m)$ *a unique solution to problem* (3) *which is the minimizer* u*. Furthermore, in a neighborhood of each vertex* P_i*,* $i = 1, \ldots, n$*, the following estimate holds:*

$$|x - P_i||\nabla u(x)| \to 0 \text{ as } \overline{S} \setminus \{P_i\} \ni x \to P_i. \tag{10}$$ [1]

Proof. To prove that the minimizer u has the required smoothness we will proceed step by step. Choosing in (9) $\xi \in C_c^\infty(S, \mathbb{R}^m) \subset E$, we immediately see that u is a weak solution to $\Delta u = 0$ in S and therefore a $C^\infty(S, \mathbb{R}^m)$ classical one. Next, let us choose a point $x \in \operatorname{int}(S_i)$. Thanks to Lemma 3.2, we can extend u symmetrically with respect to the Π_i line, to a Sobolev map defined in a neighborhood Ω of x. According to Lemma 3.3, and since (9) holds for every $\xi \in C_c^\infty(\Omega, \mathbb{R}^m)$ such that $\xi(S_i \cap \Omega) \subset \Pi_i'$, we deduce that the extension of u is harmonic in Ω, and thus u is smooth in the interior of the sides of S. Furthermore, since we have extended u symmetrically, u satifies the second boundary condition of problem (3), that is $x \in \operatorname{int}(S_i) \Rightarrow \dfrac{\partial u}{\partial r_i}(x) \perp \Pi_i$. It remains to show the continuity of u up to the vertices. The points $x \in S$ near the vertex P_i belong to a sector with center P_i and angle α_i, bounded by the sides S_{i-1} and S_i. Let us describe them with the help of the polar coordinates $r = |x - P_i|$ and $\theta = (t_i, x - P_i)$, and the positively oriented orthonormal basis $(e_r := (x - P_i)/r, e_\theta)$. Since u can be extended symmetrically on both sides S_{i-1} and S_i, to a harmonic and Sobolev map defined in a larger sector, Lemma 3.4 applies and gives (10). As a consequence,

$$|u(r, \phi) - u(r, 0)| \leq \left| \int_0^\phi \frac{\partial u}{\partial \theta}(r, \theta) \mathrm{d}\theta \right| \leq \int_0^\phi r \left| \frac{\partial u}{\partial e_\theta}(r, \theta) \right| \mathrm{d}\theta \to 0 \text{ as } r \to 0, \tag{11}$$

the convergence being uniform in $\phi \in [0, \alpha_i]$. Taking $\phi = \alpha_i$, and recalling that $u(r, \alpha_i) \in \Pi_{i-1}$ and $u(r, 0) \in \Pi_i$, we obtain that necessarily $u(r, 0) \to T_i$ as $r \to 0$. This, combined with (11) again, shows the continuity of u at P_i. To prove uniqueness, suppose that u and v are two solutions in $C(\overline{S}, \mathbb{R}^m) \cap C^\infty(\overline{S} \setminus \{P_1, \cdots, P_n\}, \mathbb{R}^m)$ and apply the maximum principle to $|v - u|^2$. We have $\Delta |v - u|^2 = 2|\nabla(v - u)|^2 \geq 0$, $|v - u|^2(P_i) = 0$, and $\dfrac{\partial |v - u|^2}{\partial r_i}(x) = 2 \left\langle (v - u)(x), \dfrac{\partial v}{\partial r_i}(x) - \dfrac{\partial u}{\partial r_i}(x) \right\rangle = 0$ for $x \in \operatorname{int}(S_i)$, because of the two boundary conditions. Thus, necessarily $u \equiv v$. □

4. Some properties of the solution when the target is a flat polygon

In this section we suppose that Σ is an open domain of the plane, bounded by a polygonal line, whose vertices, in order from the line being described with Σ to the left, are T_1, T_2,..., T_n. We will establish some properties of the solution $u : S \to \mathbb{R}^2$ to problem (3) when the target Σ is such a flat polygon.

The Proposition below states that the Dirichlet's energy of u is always greater or equal to the area of Σ. For instance when S and Σ are two rectangles, u is an affine map and it is possible to compute $J_S(u) = \frac{|\Sigma_1|^2 |S_2|^2 + |\Sigma_2|^2 |S_1|^2}{2|S_1||S_2|}$, where we have

[1] (10) is a general estimate which holds for all polygons and will be sufficient for our present purpose. A more precise estimate depending on the ratio of the angles at the vertices T_i and P_i can be obtained for similar elliptic problems in corner domains (cf. [5]). In our context, this sharper estimate can be established with the help of complex analysis, observing that a harmonic map $h : S \to \mathbb{R}^2$ can be written $h = \phi + \overline{\psi}$, with ϕ and ψ holomorphic.

denoted by $|.|$ the length of the sides of the two rectangles. One can also see that $J_S(u)$ equals the area of Σ if and only if S is similar to Σ, in which case u is a similarity transformation.

Proposition 4.1.

$$\int_S \det(\nabla u) = |\Sigma| \leq J_S(u), \ \textit{where } |\Sigma| \textit{ denotes the area of } \Sigma, \tag{12}$$

and equality holds if and only if u is holomorphic in S.

Proof. We first note that since $2|\det(\nabla u)| \leq |\nabla u|^2$, $\det(\nabla u)$ is integrable in S. Furthermore, if (f, g) are the two components of u, we have

$$|\nabla u|^2 - 2\det(\nabla u) = (f_{x_1} - g_{x_2})^2 + (f_{x_2} + g_{x_1})^2,$$

and from this expression it follows that equality holds in (12) if and only if the Cauchy-Riemann equations are satisfied by u in S. To prove that the integral over S of the Jacobian of u is equal to the area of Σ, we need to introduce the vector field $X := (\det(u, u_{x_2}), \det(u_{x_1}, u))$ defined in $\overline{S} \setminus \{P_1, \cdots, P_n\}$. For every positively oriented orthonormal basis (r, t), we have: $X \cdot r = \det(u, u_t)$, and in addition $\operatorname{div} X = 2\det(\nabla u)$. Now, we consider for every vertex P_i, the arc $\Gamma_i(\epsilon) := \{x \in \overline{S} \mid |x - P_i| = \epsilon\}$ where $\epsilon > 0$ is small. The arcs $\Gamma_i(\epsilon)$ and the sides S_i bound an open domain called $S(\epsilon)$. We also define the segments $S_i(\epsilon) := S_i \cap \overline{S(\epsilon)}$. To each point of $\partial S(\epsilon)$, we associate the positively oriented orthonormal basis (r, t), where r is the outer unit normal vector and t the tangential one. The divergence theorem applied to X in $S(\epsilon)$ gives:

$$\int_{S(\epsilon)} 2\det(\nabla u) = \sum_{i=1}^n \int_{\Gamma_i(\epsilon)} \det(u, u_t) + \sum_{i=1}^n \int_{S_i(\epsilon)} \det(u, u_{t_i}), \tag{13}$$

and we note that by (10), the first term in the right-hand side of (13)

$$A(\epsilon) = \sum_{i=1}^n \int_{\Gamma_i(\epsilon)} \det(u, u_t) \to 0 \ \text{ as } \epsilon \to 0.$$

Let us compute the other term in (13):

$$\begin{aligned} B(\epsilon) &= \sum_{i=1}^n \int_{S_i(\epsilon)} \det(u, u_{t_i}) = \sum_{i=1}^n \int_{S_i(\epsilon)} |u||u_{t_i}| \sin(u, u_{t_i}) = \sum_{i=1}^n \int_{S_i(\epsilon)} \langle u, \rho_i \rangle \langle u_{t_i}, \tau_i \rangle \\ &= \sum_{i=1}^n h_i \int_{S_i(\epsilon)} \langle u_{t_i}, \tau_i \rangle, \ \text{ since } \langle u, \rho_i \rangle \equiv h_i \text{ in } S_i(\epsilon). \end{aligned}$$

From this expression we have as $\epsilon \to 0$, denoting by $|\Sigma_i|$ the length of the side Σ_i:

$$B(\epsilon) \to \sum_{i=1}^n h_i |\Sigma_i| = 2|\Sigma|, \text{ by the divergence theorem applied to the identity.}$$

$$\text{Thus, } \lim_{\epsilon \to 0} \int_{S(\epsilon)} 2\det(\nabla u) = \int_S 2\det(\nabla u) = 2|\Sigma|. \qquad \square$$

We will need the next Lemma to establish properties of the conjugate map of u.

Lemma 4.2. *For every $i = 1, \ldots, n$, the limit L_i of*

$$\int_{AB} \frac{\partial u}{\partial r_i} \quad \textit{as } \mathrm{int}(S_i) \ni A \to P_i \textit{ and } \mathrm{int}(S_i) \ni B \to P_{i+1} \textit{ exists.} \tag{14}$$

$$\textit{Furthermore, } \sum_{i=1}^{n} L_i = 0. \tag{15}$$

Proof. For every $i = 1, \ldots, n$, let $C_i(\epsilon, R) := \{x \in \overline{S} \mid \epsilon \leq |x - P_i| \leq R\}$ be the set bounded by the segments $S_{i-1}(\epsilon, R) \subset S_{i-1}$ and $S_i(\epsilon, R) \subset S_i$, and the arcs $\Gamma_i(\epsilon) := \{x \in \overline{S} \mid |x - P_i| = \epsilon\}$ and $\Gamma_i(R) := \{x \in \overline{S} \mid |x - P_i| = R\}$. Green's formula applied to u in $C_i(\epsilon, R)$ gives

$$0 = \int_{\Gamma_i(\epsilon)} \frac{\partial u}{\partial n} + \int_{\Gamma_i(R)} \frac{\partial u}{\partial n} + \int_{S_{i-1}(\epsilon,R)} \frac{\partial u}{\partial r_{i-1}} + \int_{S_i(\epsilon,R)} \frac{\partial u}{\partial r_i}, \tag{16}$$

where we have denoted by n the outer unit normal vector to the arcs $\Gamma_i(\epsilon)$ and $\Gamma_i(R)$. By (10), we note that the first integral in (16) goes to 0 as $\epsilon \to 0$ while the third and the fourth are scalar multiples of the vectors ρ_{i-1} and ρ_i, respectively. This implies that the limits in (14) exist. In the same way, by applying again Green's formula to u in the domain $S(\epsilon)$ defined in the proof of Proposition 4.1, one can prove (15). $\square$

Now, we can introduce the conjugate harmonic map of u, denoted by u^* and which is defined by the equations $u_{x_1} = u^*_{x_2}$ and $u_{x_2} = -u^*_{x_1}$. One can check that $|\nabla u^*| = |\nabla u|$, $\det(\nabla u^*) = \det(\nabla u)$, and that for every positively oriented orthonormal basis (r, t): $u_r = u^*_t$. Furthermore, it follows from Lemma 4.2 that $u^* \in C(\overline{S}, \mathbb{R}^2) \cap C^\infty(\overline{S} \setminus \{P_1, \cdots, P_n\}, \mathbb{R}^2)$, and has the following properties for every $i = 1, \ldots, n$:

(i) $u^*(P_i) := T^*_i$ with $T^*_{i+1} = T^*_i + L_i$,
(ii) $x \in S_i \Rightarrow (u^*(x) - T^*_i) \parallel \rho_i$,
(iii) $x \in \mathrm{int}(S_i) \Rightarrow \dfrac{\partial u^*}{\partial r_i}(x) \perp \rho_i$.

As a consequence of (ii) and (iii), u^* satisfies the mixed boundary conditions (cf. (3)). However, the points T^*_i may not always define a polygon. A sufficient condition to ensure this when Σ is convex, is that $\langle L_i, \rho_i \rangle > 0$ for every $i = 1, \ldots, n$. With the help of the conjugate harmonic map we prove

Theorem 4.3. *When Σ is convex, the following statements are equivalent:*

(i) $\langle u_{t_i}(x), \tau_i \rangle > 0$ *for every* $i = 1, \ldots, n$ *and every* $x \in \mathrm{int}(S_i)$,
(ii) u *is a diffeomorphism of* S *onto* Σ,
(iii) $u(\overline{S}) \subset \overline{\Sigma}$ *(this property is called 'positivity' of* u*),*
(iv) $\langle u_{r_i}(x), \rho_i \rangle > 0$ *for every* $i = 1, \ldots, n$ *and every* $x \in \mathrm{int}(S_i)$.

Futhermore, if one of the conditions above holds, u is a homeomorphism of $\overline{S}$ onto $\overline{\Sigma}$, and $u^{-1} \in C^\infty(\overline{\Sigma} \setminus \{T_1, \cdots, T_n\}, \mathbb{R}^2)$.

Proof. *(i) ⇒ (ii)*. If (i) is satisfied, then the restriction of u to ∂S is a homeomorphism onto $\partial\Sigma$. According to the Radó–Kneser–Choquet theorem, this implies that u is a diffeomorphism of S onto Σ. *(ii) ⇒ (iii)* is clear since u is continuous in $\overline{S}$. To see that *(iii) ⇒ (iv)*, we shall apply Hopf's Lemma to the harmonic functions $g_i(x) = \langle u(x), \rho_i \rangle$ for $i = 1, \ldots, n$. Since u is supposed positive and Σ convex, the maximum of g_i is attained at any interior point x of S_i, and thus $\langle u_{r_i}(x), \rho_i \rangle > 0$.

$(iv) \Rightarrow (i)$ will be established with the help of the conjugate map u^*. When Σ is convex, (iv) is sufficient to ensure that the points T_i^* define a convex polygon Σ^*, such that $\tau_i^* := \frac{1}{|\overrightarrow{T_i^* T_{i+1}^*}|}\overrightarrow{T_i^* T_{i+1}^*} = \rho_i$. Since $\langle u^*_{t_i}(x), \tau_i^* \rangle = \langle u_{r_i}(x), \rho_i \rangle > 0$ for every $x \in \mathrm{int}(S_i)$, u^* satifies (i) and thus also (iv). Noting that the outer unit normal vector to the side $T_i^* T_{i+1}^*$ is $\rho_i^* = -\tau_i$, and writing (iv) for u^*, we obtain $\langle u^*_{r_i}(x), \rho_i^* \rangle = \langle u_{t_i}(x), \tau_i \rangle > 0$. Now that we have proved all the equivalences it remains to show that u^{-1} is defined and smooth. The fact that $u : \overline{S} \to \overline{\Sigma}$ is one-to-one and onto is a consequence of (i) and (ii). The smoothness of u^{-1} in Σ results from (ii). To see that u^{-1} is smooth in the interior of the sides of Σ, take $y \in \mathrm{int}(\Sigma_i)$ with preimage $x \in \mathrm{int}(S_i)$. Conditions (i) and (iv) ensure that the Jacobian of u at x does not vanish, and thus u and u^{-1} are local diffeomorphisms in neighborhoods of x and y, respectively. To prove the continuity of u^{-1} at the vertices T_i, we proceed by contradiction. Let us take a sequence $(y_n) \subset \overline{\Sigma}$ such that $y_n \to T_i$, and $u^{-1}(y_n) := x_n$ does not converge to P_i. Then, there exists a subsequence x_{n_k} which converges in $\overline{S}$ to a point $P \neq P_i$, and utilizing the continuity of u, we obtain $y_{n_k} \to u(P) \neq T_i$. □

Corollary 4.4. *If Σ is convex, and u is holomorphic in S, then u is a conformal diffeomorphism of S onto Σ.*

Proof. We shall prove by contradiction that $u(\overline{S}) \subset \overline{\Sigma}$ and then utilizing Theorem 4.3, the statement of the Corollary will follow. As in the proof of Theorem 4.3, we consider the harmonic functions $g_i(x) = \langle u(x), \rho_i \rangle$ for $i = 1, \ldots, n$. If $u(\overline{S}) \not\subseteq \overline{\Sigma}$, we can suppose that for instance the maximum of g_1 is positive. Let $x \in \overline{S}$ be a point where it is attained. We first note that x cannot be a vertex of S or belong to S_1 since $g_1 \equiv 0$ on S_1. If x is an interior point of S, then by the maximum principle g_1 is constant, and this contradicts the fact that $u(P_i) = T_i$ for every i. If $x \in \mathrm{int}(S_i)$ with $i \neq 1$, then $\dfrac{\partial u}{\partial t_i}(x) = 0$, and since u is holomorphic we have $\dfrac{\partial u}{\partial r_i}(x) = 0$, which contradicts Hopf's Lemma and completes the proof. We mention that Corollary 4.4 can also be deduced from the argument principle. □

In the case of triangles, for $n = 3$, the Riemann mapping theorem ensures the existence of a conformal diffeomorphism between S and Σ which is a solution to problem (3). We are now going to prove this result, utilizing the arguments developed before.

Proposition 4.5. *When S and Σ are triangles, the map u is a conformal diffeomorphism of S onto Σ.*

Proof. We first establish that for $i = 1, 2, 3$,

$$\int_S |\nabla u|^2 = h_i \langle L_i, \rho_i \rangle, \tag{17}$$

where $h_i := \langle \overrightarrow{T_{i-1} T_i}, \rho_i \rangle$ is the height of Σ corresponding to the basis Σ_i. We shall once again apply Green's formula to u in the domain $S(\epsilon)$ (cf. the proof of Proposition 4.1), supposing that the origin O coincides with the vertex T_{i-1}. Due to the boundary conditions, we obtain:

$$\int_{S(\epsilon)} |\nabla u|^2 = \sum_{j=1}^{3} \int_{\Gamma_j(\epsilon)} \langle u, u_n \rangle + \int_{S_i(\epsilon)} \langle u, u_{r_i} \rangle,$$

where we have denoted by n the outer unit normal vector to the arcs $\Gamma_j(\epsilon)$. Then, recalling (10), we deduce that

$$\int_S |\nabla u|^2 = \lim_{\epsilon \to 0} \int_{S_i(\epsilon)} \langle u, u_{r_i} \rangle = h_i \langle L_i, \rho_i \rangle.$$

As it was mentioned before, $\langle L_i, \rho_i \rangle > 0$, which results from (17), ensures that the conjugate map u^* is a solution to problem (3) with target a triangle Σ^*. Let us denote by h_i^*, its heights corresponding to the sides Σ_i^*, by $\rho_i^* = -\tau_i$ the outer unit normal vector to Σ_i^*, and by $\tau_i^* = \rho_i$ the tangential one. Now, if we consider the integrals L_i^* of u^* defined by (14), we see that $\langle L_i^*, \rho_i^* \rangle = |\Sigma_i|$, where$|.|$ denotes the length of a side. Similarly, we have $\langle L_i, \rho_i \rangle = |\Sigma_i^*|$, and since $J_S(u) = J_S(u^*)$, we obtain by (17):

$$2\, J_S(u) = h_i |\Sigma_i^*| = h_i^* |\Sigma_i| \quad \text{for } i = 1, 2, 3. \tag{18}$$

But according to Proposition 4.1, we have since the Jacobians of u and u^* are equal:

$$2 \int_S \det(\nabla u) = h_i |\Sigma_i| = h_i^* |\Sigma_i^*| \quad \text{for } i = 1, 2, 3. \tag{19}$$

Eliminating the heights in the equations (18) and (19), we deduce that $|\Sigma_i| = |\Sigma_i^*|$ for every i and that equality holds in (12). Therefore u is holomorphic and as a consequence of Corollary 4.4, u is a conformal diffeomorphism. □

We also give another application of Theorem 4.3 when $n = 4$.

Proposition 4.6. *If S is a quadrilateral and Σ a rectangle, the map u is a diffeomorphism of S onto Σ.*

Proof. We shall prove by contradiction that $u(\overline{S}) \subset \overline{\Sigma}$ and then utilizing Theorem 4.3, the statement of the Proposition will follow. As before, we consider the harmonic functions $g_i(x) = \langle u(x), \rho_i \rangle$ for $i = 1, \dots, 4$. If $u(\overline{S}) \not\subseteq \overline{\Sigma}$, we can suppose that for instance the maximum of g_1 is positive. Let $x \in \overline{S}$ be a point where it is attained. We first note that x cannot belong to $S_1 \cup S_3$ since $g_1 \le 0$ on $S_1 \cup S_3$. If x is an interior point of S, then by the maximum principle g_1 is constant, and this contradicts the fact that $u(P_i) = T_i$ for every i. If $x \in \operatorname{int}(S_2) \cup \operatorname{int}(S_4)$, then due to the boundary conditions we have $\dfrac{\partial g_1}{\partial r_2}(x) = 0$ which contradicts Hopf's Lemma. □

We are not aware if the solution u is always a diffeomorphism when Σ is convex. Arguments as those utilized in the proof of Proposition 4.6 apply to prove positivity only when all the angles of Σ are acute. In the general case however, they allow us to bound the lengths of the segments $u(S_i)$ and thus to bound $u(S)$.

5. A comparison with the case of smooth domains

When Ω_1 and Ω_2 are C^1 Jordan domains of the plane, the harmonic problem with mixed boundary conditions can be stated as below:

$$\begin{cases} h \in C^1(\overline{\Omega_1}, \mathbb{R}^2), \quad \text{satisfies} \\ \Delta h = 0 \ \text{ in } \Omega_1\,, \text{ with mixed boundary conditions:} \\ h\big|_{\partial\Omega_1} \text{ is a sense-preserving homeomorphism onto } \partial\Omega_2, \\ x \in \partial\Omega_1 \Rightarrow \dfrac{\partial h}{\partial n}(x) \perp \partial\Omega_2, \end{cases} \tag{20}$$

where n denotes the outer unit normal vector to $\partial\Omega_1$ and $\big|$ the restriction of a map. Supposing now that Ω_1 and Ω_2 are $C^{1,\alpha}$ Jordan domains (with $\alpha \in (0,1)$), the conformal diffeomorphisms between Ω_1 and Ω_2 provided by the Riemann mapping theorem are $C^1(\overline{\Omega_1}, \mathbb{R}^2)$ smooth (cf. [13]), and thus they solve problem (20) [2]. Their Dirichlet's energy is equal to the area of Ω_2. Furthermore, in the class H of harmonic maps in Ω_1, $C^1(\overline{\Omega_1}, \mathbb{R}^2)$ smooth, and whose restrictions to $\partial\Omega_1$ are sense-preserving homeomorphisms onto $\partial\Omega_2$, these conformal maps are the only minimizers of J_{Ω_1}. This property is established indirectly, by proving that for a map $h \in H$, $J_{\Omega_1}(h)$ is greater or equal to the area of Ω_2.

The Proposition below is the analog of Proposition 4.1 for smooth domains.

Proposition 5.1. *Let Ω_1 and Ω_2 be two C^1 Jordan domains, then for every map h in the class H,*

$$\int_{\Omega_1} \det(\nabla h) = |\Omega_2| \leq J_{\Omega_1}(h) \quad \text{where } |\Omega_2| \text{ denotes the area of } \Omega_2, \tag{21}$$

and equality holds if and only if h is holomorphic in Ω_1.

Proof. We will just prove that the integral over Ω_1 of the Jacobian of h is equal to the area of Ω_2. For the other statements we will refer to the proof of Proposition 4.1. Once again, we will utilize the vector field $X = (\det(h, h_{x_2}), \det(h_{x_1}, h))$, and associate to each point of $\partial\Omega_1$, the positively oriented orthonormal basis (n, t) where n is the outer unit normal vector to $\partial\Omega_1$ and t the tangential one. Similarly, for every point of $\partial\Omega_2$, we shall denote by (ν, τ) the corresponding basis. The divergence theorem applied to X in Ω_1 gives:

$$\begin{aligned}
\int_{\Omega_1} 2\det(\nabla h) &= \int_{\partial\Omega_1} \det(h, h_t) = \int_{\partial\Omega_1} |h||h_t| \sin(h, h_t) \\
&= \int_{\partial\Omega_1} \langle h, \nu\rangle |h_t|, \text{ and doing a change of variable we obtain:} \\
&= \int_{\partial\Omega_2} \langle y, \nu(y)\rangle \mathrm{d}\sigma(y) \quad (\text{where } \mathrm{d}\sigma(y) \text{ is the length element of } \partial\Omega_2) \\
&= 2|\Omega_2|, \text{ by the divergence theorem applied to the identity.}
\end{aligned}$$
□

Corollary 5.2. *Let Ω_1 and Ω_2 be two $C^{1,\alpha}$ Jordan domains. Then, the conformal diffeomorphisms of Ω_1 onto Ω_2 are the only minimizers of J_{Ω_1} in the class H.*

Proof. According to Proposition 5.1, the conformal diffeomorphisms of Ω_1 onto Ω_2 are minimizers of J_{Ω_1} in the class H, since their Dirichlet's energy is equal to the area of Ω_2. On the other hand, by Proposition 5.1, every minimizer of J_{Ω_1} in the class H is holomorphic in Ω_1, and thus, by the argument principle, conformal. □

We note that if Ω_2 is convex, Proposition 5.1, remains valid for every map $h \in C(\overline{\Omega_1}, \mathbb{R}^2)$, harmonic in Ω_1, and whose restriction to $\partial\Omega_1$ is a sense-preserving homeomorphism onto $\partial\Omega_2$. Indeed, in this case, we know by the Radó–Kneser–Choquet theorem that h is a diffeomorphism of Ω_1 onto Ω_2.

[2]It will be shown in the next section that they are also the only solutions of problem (20).

6. The propagation of conformal boundary conditions in smooth domains of the plane

In this section, we show how conformal boundary conditions for a harmonic map u defined in a smooth domain, propagate in the interior of the set. We prove a general theorem (Theorem 6.2) which applies to a pair (u, v) of harmonic maps and then, by considering the pair (u, u), particularize it to obtain the aforementioned results. In what follows Ω will always be a Jordan domain of the plane, at least C^1 smooth . To each point of $\partial\Omega$, we associate the positively oriented orthonormal basis (n, t) where n is the outer unit normal vector to $\partial\Omega$ and t the tangential one. We first establish

Proposition 6.1. *Let Ω be a C^1 Jordan domain of the plane, let $u, v \in C^1(\overline{\Omega}, \mathbb{R}^m)$ be two harmonic maps in Ω, and let $a := (a_1, a_2) \in \mathbb{R}^2$. Then,*

$$\int_{\partial\Omega} \big(\big[\langle u_n, v_n\rangle - \langle u_t, v_t\rangle\big]\langle x-a, n\rangle + \big[\langle u_n, v_t\rangle + \langle u_t, v_n\rangle\big]\langle x-a, t\rangle\big)\mathrm{d}\sigma(x) = 0.\tag{22}$$ [3]

Proof. We consider the following vector fields, defined on $\overline{\Omega}$:

$$X_1 := (x_1 - a_1)\big(\langle u_{x_1}, v_{x_1}\rangle - \langle u_{x_2}, v_{x_2}\rangle, \langle u_{x_1}, v_{x_2}\rangle + \langle u_{x_2}, v_{x_1}\rangle\big),$$
$$X_2 := (x_2 - a_2)\big(\langle u_{x_1}, v_{x_2}\rangle + \langle u_{x_2}, v_{x_1}\rangle, -\langle u_{x_1}, v_{x_1}\rangle + \langle u_{x_2}, v_{x_2}\rangle\big),$$
$$\text{and } X := X_1 + X_2.$$

We check that $\operatorname{div} X = 0$,

$$\langle X, n\rangle = \big[\langle u_n, v_n\rangle - \langle u_t, v_t\rangle\big]\langle x-a, n\rangle + \big[\langle u_n, v_t\rangle + \langle u_t, v_n\rangle\big]\langle x-a, t\rangle,$$

and apply the divergence theorem to X in Ω. □

With the help of Proposition 6.1, we prove

Theorem 6.2. *Let Ω be a $C^{1,\alpha}$ Jordan domain of the plane (with $0 < \alpha < 1$), let $u, v \in C^1(\overline{\Omega}, \mathbb{R}^m)$ be two harmonic maps in Ω, and suppose that one of the following is true:*

(i) $\forall x \in \partial\Omega \ : \ \langle u_n, v_t\rangle + \langle u_t, v_n\rangle = 0.$
(ii) $\forall x \in \partial\Omega \ : \ \langle u_n, v_n\rangle - \langle u_t, v_t\rangle = 0.$

Then, for every $x \in \overline{\Omega}$, we have

$$\langle u_{x_1}, v_{x_1}\rangle - \langle u_{x_2}, v_{x_2}\rangle = 0 \text{ and } \langle u_{x_1}, v_{x_2}\rangle + \langle u_{x_2}, v_{x_1}\rangle = 0.\tag{23}$$

Proof. We note that assumption (ii) (respectively (i)) holds for the pair of harmonic maps (u, v) if and only if assumption (i) (respectively (ii)) holds for the pair (u, v^*), where v^* is the conjugate harmonic map of v. Therefore, it suffices to prove the theorem only when assumption (i) is true.

We will first establish the theorem, supposing that (i) holds when Ω is the unit disk B, centered at the origin O. Applying Proposition 6.1 with $a = O$, we see that there exists a point $M \in \partial B$ such that

$$\langle u_n(M), v_n(M)\rangle - \langle u_t(M), v_t(M)\rangle = 0.$$

After a rotation of coordinates, we can suppose that $M = (1, 0)$. Next, we note that the relation $\big(\langle u_{x_1}, v_{x_1}\rangle - \langle u_{x_2}, v_{x_2}\rangle\big)_{x_2} = \big(\langle u_{x_1}, v_{x_2}\rangle + \langle u_{x_2}, v_{x_1}\rangle\big)_{x_1}$ holds in B

[3](22) is a Pohozaev-type identity (cf. [14]). Its proof is based on the technique utilized in [2] to establish the weak monotonicity formula for semilinear elliptic systems.

for every pair of harmonic maps (u, v), and thus ensures the existence of a function k defined by the compatibility conditions:

$$k_{x_1} = \langle u_{x_1}, v_{x_1}\rangle - \langle u_{x_2}, v_{x_2}\rangle \ \text{ and } \ k_{x_2} = \langle u_{x_1}, v_{x_2}\rangle + \langle u_{x_2}, v_{x_1}\rangle. \tag{24}$$

One can check that $\Delta k = 0$ in B and $k \in C^1(\overline{B}, \mathbb{R})$. If we set $n = (\cos\theta, \sin\theta)$, $t = (-\sin\theta, \cos\theta)$, and define the vectors $n' = (\cos 2\theta, \sin 2\theta)$, $t' = (-\sin 2\theta, \cos 2\theta)$, we also have on the boundary ∂B:

$$k_{n'} = \langle u_n, v_n\rangle - \langle u_t, v_t\rangle, \text{ and } \ k_{t'} = \langle u_n, v_t\rangle + \langle u_t, v_n\rangle = 0 \text{ by assumption.} \tag{25}$$

Since t is parallel to t' if and only if $x = M$ or $x = M' := (-1, 0)$, we see that for every $x \in \partial B \setminus \{M, M'\}$, there exists an outward vector N at x such that $k_N(x) = 0$. Recalling that $k_n(M) = 0$, we deduce thanks to the maximum principle, that whatever be the value of $k_n(M')$, k is constant in $\overline{B}$. This completes the proof when $\Omega = B$. We mention that the same arguments apply to prove the theorem when Ω is a C^1 smooth, convex domain.

Now, let us suppose that Ω is a $C^{1,\alpha}$ Jordan domain of the plane, and that the pair of harmonic maps (u, v) satisfies (i) on $\partial\Omega$. We consider a conformal map f of B onto Ω, which extends to a homeomorphism of $\overline{B}$ onto $\overline{\Omega}$. Furthermore, since Ω is $C^{1,\alpha}$ smooth, the extension of f is continuously differentiable on $\overline{B}$ and has a non-vanishing derivative at every point of ∂B (cf. [13]). That is, the conformality of f extends to ∂B. Setting $U := u \circ f$ and $V := v \circ f$, the pair of harmonic maps (U, V) satisfies assumption (i) on ∂B. Thus, (23) holds for the pair (U, V) in $\overline{B}$, and also for the pair (u, v) in $\overline{\Omega}$. ☐

Taking $v = u$ in the previous theorem, we immediately obtain

Corollary 6.3. *Let Ω be a $C^{1,\alpha}$ Jordan domain of the plane (with $0 < \alpha < 1$), let $u \in C^1(\overline{\Omega}, \mathbb{R}^m)$ be a harmonic map in Ω, and suppose that one of the following is true:*

(i) $\forall x \in \partial\Omega \ : \ \langle u_n, u_t\rangle = 0$.
(ii) $\forall x \in \partial\Omega \ : \ |u_n| = |u_t|$.

Then, u is isothermal in $\overline{\Omega}$, that is, for every $x \in \overline{\Omega}$ we have

$$\langle u_{x_1}, u_{x_2}\rangle = 0 \ \textit{and} \ |u_{x_1}| = |u_{x_2}|. \tag{26}$$

Finally, for harmonic maps taking their values in the plane we can give a more precise result.

Corollary 6.4. *Let Ω_1 be a $C^{1,\alpha}$ Jordan domain of the plane (with $0 < \alpha < 1$), let $u \in C^1(\overline{\Omega_1}, \mathbb{R}^2)$ be a harmonic map in Ω_1, and suppose that one of the following is true:*

(i) $\forall x \in \partial\Omega_1 \ : \ \langle u_n, u_t\rangle = 0$.
(ii) $\forall x \in \partial\Omega_1 \ : \ |u_n| = |u_t|$.

Then, u is either holomorphic in Ω_1, or antiholomorphic in Ω_1 (that is, $\overline{u}$ is holomorphic in Ω_1). If in addition, u carries $\partial\Omega_1$ in a sense-preserving manner onto a Jordan curve Γ bounding a domain Ω_2, then u is a conformal diffeomorphism of Ω_1 onto Ω_2. In particular, the only solutions of problem (20) *are the conformal diffeomorphisms between Ω_1 and Ω_2.*

Proof. Since u is isothermal in Ω_1, we have $2\det(\nabla u) = \pm|\nabla u|^2$ and at the points where $\det(\nabla u) > 0$ (respectively $\det(\nabla u) < 0$) u is holomorphic (respectively antiholomorphic). Noting that if the Cauchy-Riemann equations are satisfied in an open subset of Ω_1, they are also satisfied at every point of Ω_1, we deduce the first statement of the theorem. The second statement follows from the argument principle. □

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