DOUBLE EXPONENTIAL GROWTH OF THE VORTICITY GRADIENT FOR THE TWO-DIMENSIONAL EULER EQUATION

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ABSTRACT. For the two-dimensional Euler equation on the torus, we prove that the L^∞–norm of the vorticity gradient can grow as double exponential over arbitrary long but finite time provided that at time zero it is already sufficiently large. The method is based on the perturbative analysis around the singular stationary solution studied by Bahouri and Chemin in 1994. Our result on the growth of the vorticity gradient is equivalent to the statement that the operator of Euler evolution is not bounded in the linear sense in Lipschitz norm for any time $t > 0$.

1. INTRODUCTION AND SOME UPPER BOUNDS

Consider the two-dimensional Euler equation for the vorticity

(1) $$\dot{\theta} = \nabla\theta \cdot \nabla^\perp \psi, \quad \psi = \Delta^{-1}\theta, \quad \theta(x, y, 0) = \theta_0(x, y)$$

and θ is 2π–periodic in both x and y (that is, the equation is considered on the torus $\mathbb{T}^2$). We assume that θ_0 has zero average over $\mathbb{T}^2$ and then Δ^{-1} is well defined since the Euler flow is area-preserving and the average of $\theta(\cdot, t)$ is zero as well. Denote the operator of Euler evolution by $\mathcal{E}_t$, i.e.

$$\theta(t) = \mathcal{E}_t \theta_0.$$

The global existence of the smooth solution for smooth initial data is well known and is due to Wolibner [13] (see also [10]). The estimates on the possible growth of the Lipschitz [10] or Sobolev ([2, Section 3]) norms, however, are double exponential. We sketch the proof of this bound for H^2–norm first. The estimates for $H^s, s > 2$ can be obtained similarly. More general results on regularity can be found in [4]. Let

$$j_k(t) = \|\theta(t)\|_{H^k}.$$

Lemma 1.1. *If θ is the smooth solution of* (1) *and $\|\theta_0\|_\infty \sim 1$, then*

(2) $$j_2(t) \leq \exp\Big(\frac{(1 + 2\log^+ j_2(0))\exp(Ct) - 1}{2}\Big).$$

Proof. Acting on (1) with Laplacian we get

$$\Delta\dot{\theta} = \Delta\theta_x \psi_y + 2\nabla\theta_x \cdot \nabla\psi_y - \Delta\theta_y \psi_x - 2\nabla\theta_y \cdot \nabla\psi_x.$$

Multiply by $\Delta\theta$ and integrate over $\mathbb{T}^2$ to get

$$(3) \qquad \partial_t \|\theta\|_{H^2}^2 \lesssim \|H(\psi)\|_\infty \|\theta\|_{H^2}^2$$

where $H(\psi)$ denotes the Hessian of ψ. The next inequality is a simple exercise in Harmonic analysis (see [4], proposition 1.4 for a more general result)

$$(4) \qquad \|H(\psi)\|_\infty < C(\sigma)(1 + \log^+ \|\theta\|_{H^\sigma})$$

for any $\sigma > 1$ assuming that $\|\theta\|_\infty \sim 1$. Notice that $\|\theta\|_\infty$ is invariant under the dynamics so combine (3) and (4) to get (2). $\qquad\square$

Remark 1. In the same way one can prove bounds for higher Sobolev norms, e.g.,

$$(5) \qquad \log j_4(t) \lesssim (1 + \log^+ j_4(0)) \exp(Ct) - 1$$

as long as $\|\theta_0\|_\infty \sim 1$.

The natural questions one can ask then are the following: first, how fast can the Sobolev norms grow in time and what is the mechanism that leads to their growth? Secondly, for fixed t, how does $\|\mathcal{E}(t)\theta_0\|_{H^s}$ depend on $\|\theta_0\|_{H^s}$ when the last expression grows to infinity? For example, given $\|\theta_0\|_\infty \sim 1$, the right hand side in (2) grows as a power function in $j_2(0)$, the degree grows exponentially in t and is more than one for any $t > 0$.

Instead of working with Sobolev norms, we will be studying the uniform norm of the vorticity gradient (or Lipschitz norm) as this norm is more natural for the method used in the proof. It allows a similar upper bound. We again give a sketch of the proof.

Lemma 1.2. *If θ_0 is smooth and $\|\theta_0\|_\infty \sim 1$, then*

$$(6) \qquad \|\nabla\mathcal{E}_t\theta_0\|_\infty \lesssim \exp\left(C(1 + \log^+ \|\nabla\theta_0\|_\infty)e^{Ct}\right).$$

Proof. If $\Psi(z,t)$ is area-preserving Euler diffeomorphism, then

$$(\mathcal{E}_t\theta_0)(z) = \theta_0(\Psi^{-1}(z,t)).$$

On the other hand, $\Psi(z,t)$ solves

$$\dot{\Psi} = -u(\Psi, t), \quad \Psi(z,0) = z$$

where $u(z,t) = \nabla^\perp \Delta^{-1}\theta(\cdot, t)$. For the Riesz transform we have a trivial estimate

$$(7) \qquad \|H(\Delta^{-1}\theta)\|_\infty \lesssim 1 + \|\theta\|_\infty(1 + \log^+ \|\nabla\theta\|_\infty).$$

(Indeed, without loss of generality we can evaluate the integral at zero and assume $\theta(0) = 0$. Then, e.g.,

$$\left| \int_{B_1(0)} \frac{\xi_1 \xi_2}{|\xi|^4} \theta(\xi) d\xi \right| \leq \int_{B_\delta(0)} \frac{1}{|\xi|^2} |\theta(\xi)| d\xi + \int_{\delta < |z| < 1} \frac{1}{|\xi|^2} |\theta(\xi)| d\xi$$

where $\delta^{-1} = \max\{\|\nabla\theta\|_\infty, 2\}$. Apply now the mean value theorem to the first term to get (7).) So

$$|u(w_1, t) - u(w_2, t)| \lesssim |w_1 - w_2| b, \quad b = 1 + \log^+ \|\nabla\theta(t)\|_\infty.$$

Therefore, we have

$$|\dot{f}| \lesssim fb, \quad f(t) = |\Psi(z_2, t) - \Psi(z_1, t)|^2.$$

After integration

$$|z_2 - z_1| \exp\left(-C \int_0^t b(\tau)d\tau\right) \le |\Psi(z_2,t) - \Psi(z_1,t)| \le |z_2 - z_1| \exp\left(C \int_0^t b(\tau)d\tau\right).$$

Since

$$\|\nabla\theta(z,t)\|_\infty = \sup_{z_1,z_2} \frac{|\theta_0(\Psi^{-1}(z_2,t)) - \theta_0(\Psi^{-1}(z_1,t))|}{|z_2 - z_1|}$$

we get the inequality

$$\|\nabla\theta(z,t)\|_\infty \lesssim \|\nabla\theta_0\|_\infty \exp\left(C \int_0^t b(\tau)d\tau\right).$$

Taking log of both parts and applying the Gronwall-Bellman, we get (6). $\square$

In this paper, we will work only with large $\|\nabla\theta_0\|_\infty$. For that case, we will show that, given arbitrarily large λ, the estimate $\max_{t\in[0,T]} \|\nabla\theta(\cdot,t)\|_\infty > \lambda^{e^T-1}\|\nabla\theta_0\|_\infty$ can hold for some infinitely smooth initial data. This is far from showing that (2) or (6) are sharp. However, it already is equivalent to the statement that $\mathcal{E}_t$ is not bounded in the standard linear sense. The question of whether $\|\nabla\theta_0\|_\infty$ can be taken ~ 1 is left wide open; see the discussion in the last section.

Remark 2. The simple embedding inequality

$$\|\nabla\theta\|_\infty \lesssim \|\theta\|_{H^3(\mathbb{T}^2)}$$

along with careful estimation of the $\|\theta_0\|_{H^3(\mathbb{T}^2)}$ and a minor modification in the proof of Theorem 6.1 below show that the double-exponential growth for higher Sobolev norms is also possible.

Our results rigorously confirm the following observation: if the 2D incompressible inviscid fluid dynamics gets into a certain "instability mode", then the Sobolev norms can grow very fast in local time (i.e. counting from the time the "instability regime" was reached). Can Sobolev or Lipschitz norms grow infinitely as $t \to \infty$? The answer to this question is positive; see [5] and [8,9,11,12,14]. The important questions of linear and nonlinear instabilities were addressed before (see, e.g., [6] and the references there). In the recent paper [7], it was proved that Euler evolution of the velocity field is not uniformly continuous on the unit ball in Sobolev spaces.

Remark 3. It must be mentioned here that 2D Euler allows rescaling which provides the tradeoff between the size of θ_0 and the speed of the process, i.e. if $\theta(x,y,t)$ is a solution, then $\mu\theta(x,y,\mu t)$ is also a solution for any $\mu > 0$. However, in our construction we will always have $\|\theta\|_p \sim 1, \quad \forall p \in [1,\infty]$.

Remark 4. If one replaces Δ^{-1} in (1) by $\Delta^{-\alpha}$ with $\alpha > 1$, then the growth of the vorticity gradient is at most exponential, e.g.

$$\|\nabla\mathcal{E}_t^{(\alpha)}\theta_0\|_\infty \lesssim \|\nabla\theta_0\|_\infty \exp(C\|\theta_0\|_\infty t).$$

Moreover, the lower exponential bound can hold for all times as long as θ_0 is properly chosen (see [5]).

The idea of constructing the smooth initial data for a double exponential scenario is quite simple and roughly can be summarized as follows. We will first identify a special stationary singular solution which generates the hyperbolic dynamics with double-exponential contraction around some stationary points. Then, given any $T > 0$, we will smooth out this singular steady state such that the dynamics are still

double-exponential over $[0, T]$ in a certain domain away from the separatrices. Then we will place a small but steep bump in the area of double exponential behavior and will let it evolve hoping that the vector field generated by this bump itself is not going to ruin the double-exponential contraction. The rest of the paper verifies that this indeed is the case.

Below, we will use the following standard notation:

$$\langle f, g \rangle = \int_{\mathbb{T}^2} f(x, y) g(x, y) dx dy.$$

2. The singular stationary solution and dynamics on the torus

The following singular stationary solution was studied before (see, e.g., [1, 3] in the context of $\mathbb{R}^2$). We consider the following function:

$$\theta_0^s(x, y) = sgn(x) \cdot sgn(y), \quad |x| \le \pi, |y| \le \pi.$$

This is a steady state. Indeed, the function $\psi_0 = \Delta^{-1} \theta_0^s$ is odd with respect to each variable as can be verified on the Fourier side. That, in particular, implies that ψ_0 is zero on the coordinate axes so its gradient is orthogonal to them. This steady state, of course, is a weak solution, a vortex-patch steady state. Another consequence of ψ_0 being odd is that the origin is a stationary point of the dynamics.

By the Poisson summation formula, we have

$$\sum_{n \in \mathbb{Z}^2, n \ne (0,0)} |n|^{-2} e^{in \cdot z} = C \ln |z| + \phi(z), \quad z \sim 0$$

where $\phi(z)$ is smooth and even.

Therefore, around the origin we have

$$\nabla \psi_0(x, y) \sim \int \int_{B_{0.5}(0)} \frac{(x - \xi_1, y - \xi_2)}{(x - \xi_1)^2 + (y - \xi_2)^2} sgn(\xi_1) sgn(\xi_2) d\xi_1 d\xi_2 + (O(y), O(x)).$$

Due to symmetry, it is sufficient to consider the domain $D = \{0 < x < y < 0.001\}$. Then, taking the integrals, we see that

$$(8) \qquad\qquad \mu(x, y) = (\mu_1, \mu_2) = \left(\nabla^\perp \psi_0 \right)(x, y)$$

$$= c_1 \left(-\int_0^x \ln(y^2 + \xi^2) d\xi + x r_1(x, y), \int_0^y \ln(x^2 + \xi^2) d\xi + y r_2(x, y) \right)$$
$$= c_2(-x \log y + x O(1), y \log y + y O(1)) \quad \text{if } (x, y) \in D.$$

The correction terms $r_{1(2)}$ are smooth. Without loss of generality we will later assume that $c_2 = 1$ in the last formula (so $c_1 = 0.5$). That can always be achieved by time-rescaling. Notice also that the flow given by the vector-field μ is area-preserving.

Thus, the dynamics of the point $(\alpha, \beta) \in D, \alpha \ll \beta$ is

$$(9) \quad (C_1 \beta)^{e^t} \lesssim y(t) \lesssim (C_2 \beta)^{e^t}, \quad \alpha(C_1 \beta)^{-e^t+1} \lesssim x(t) \lesssim \alpha(C_2 \beta)^{-e^t+1}, \ t \in [0, t_0],$$

where t_0 is the time the trajectory leaves the domain D. These estimates therefore give a bound on t_0. The attraction to the origin, the stationary point, is double exponential along the vertical axis and the repulsion along the horizontal axis is also double-exponential.

3. The model equation

Consider the following system of ODE's:

$$(10) \qquad \begin{cases} \dot{x} = \mu_1(x,y) + \nu_1(x,y,t), & x(\alpha,\beta,0) = \alpha, \\ \dot{y} = \mu_2(x,y) + \nu_2(x,y,t), & y(\alpha,\beta,0) = \beta, \end{cases}$$

where $\mu_{1(2)}$ are defined in (8). Here we assume the following:

$$(11) \qquad |\nu_{1(2)}| < 0.0001\upsilon r, \quad r = \sqrt{x^2 + y^2}$$

and

$$(12) \qquad |\nabla \nu_{1(2)}| < 0.0001\upsilon$$

with small υ (to be specified later) and these estimates are valid in the area of interest

$$\aleph = \{y > \sqrt{x}\} \cap \{y < \epsilon_2\} \cap \{x > \epsilon_1\}$$

where

$$\upsilon \ll \epsilon_1 \ll \epsilon_2.$$

The functions $\nu_{1(2)}$ are infinitely smooth in all variables in $\aleph$ but we have no control over higher derivatives. We also assume that the flow given by (10) is area-preserving. Our goal is to study the behavior of trajectories within a time interval $[0, T]$. In this section, the parameters will eventually be chosen in the following order:

$$T \longrightarrow \epsilon_2 \longrightarrow \epsilon_1 \longrightarrow \upsilon.$$

Here are some obvious observations:

1. We have estimates

$$(13) \qquad \begin{aligned} x(-\log y - C) - \upsilon y &< \dot{x} < x(-\log y + C) + \upsilon y \\ -y(|\log y| + C) &< \dot{y} < -y(|\log y| - C). \end{aligned}$$

2. Assuming $\epsilon_{1(2)}$ to be fixed, let

$$(14) \qquad \upsilon < 0.1 \frac{\epsilon_1 |\log \epsilon_2|}{\epsilon_2}$$

and the initial data $(\alpha, \beta) \in \aleph$. Then, for $\epsilon_{1(2)}$ sufficiently small, equations (13) show that $x(t)$ increases and $y(t)$ decreases. This monotonicity persists as long as the trajectory stays within $\aleph$.

The second estimate in (13) yields

$$(15) \qquad e^{e^t(\log \beta + C)} > y(t) > e^{e^t(\log \beta - C)}.$$

Let us introduce

$$\kappa(T, \beta) = e^{e^T(\log \beta - C)}.$$

For $x(t)$, we have

$$x(t) \leq \alpha \exp\left(Ct - \int_0^t \log y(\tau) d\tau \right) + \upsilon \int_0^t y(\tau) \exp\left(C(t-\tau) - \int_\tau^t \log y(s) ds \right) d\tau$$

$$x(T) < (\alpha + \upsilon \beta T) \exp\left(T(C + |\log \kappa(T, \beta)|) \right).$$

Thus, the trajectory will stay inside $\aleph$ for any $t \in [0, T]$ as long as

$$\alpha < e^{2e^T (\log \beta - C) - T(C - \log \kappa)} - \upsilon \beta T$$

or

$$(16) \qquad\qquad \alpha < \kappa^{2+T} e^{-TC} - \upsilon \beta T.$$

Since

$$e^{e^T (\log \beta - C)} = \kappa < e^{-TC}$$

for large T and $\beta < \epsilon_2$, we see that (16) is satisfied if we require

$$\alpha < \kappa^{3+T} - \upsilon \epsilon_2 T$$

and if we have

$$(17) \qquad\qquad \upsilon < \kappa^{4+T}(T, \beta).$$

Then the condition

$$(18) \qquad\qquad \alpha < \beta^{8e^{2T}}$$

is sufficient for the trajectory to stay inside $\aleph$ for $t \in [0, T]$. Thus, we are taking

$$\epsilon_1 < \epsilon_2^{8e^{2T}}$$

and focusing on the nonempty domain

$$\Omega_0 = \{(\alpha, \beta) : \epsilon_1 < \alpha < \beta^{8e^{2T}}, \beta < \epsilon_2\}.$$

The condition on υ is (17), so taking the smallest possible $\kappa(T, \beta)$ within Ω_0 we get, e.g.,

$$(19) \qquad\qquad \upsilon < \epsilon_1^{10}.$$

Then, any point from Ω_0 stays inside $\aleph$ over $[0, T]$, $x(t)$ grows monotonically and $y(t)$ monotonically decays with the double-exponential rate given in (15).

Now, we will prove that the derivative in α of $x(\alpha, \beta, t)$ grows with the double-exponential rate and this will be the key calculation. For any $t \in [0, T]$, (8) yields

$$(20)$$
$$\begin{cases} \dot{x}_\alpha = 0.5 \left(-x_\alpha \log(x^2 + y^2) + x_\alpha r_1 \right. \\ \qquad\qquad \left. + xx_\alpha r_{1x} + xy_\alpha r_{1y} \right) + \nu_{1x} x_\alpha + \nu_{1y} y_\alpha - y_\alpha \arctan(xy^{-1}), \\ \dot{y}_\alpha = 0.5 \left(y_\alpha \log(x^2 + y^2) + y_\alpha r_2 + yx_\alpha r_{2x} \right. \\ \qquad\qquad \left. + yy_\alpha r_{2y} \right) + \nu_{2x} x_\alpha + \nu_{2y} y_\alpha + x_\alpha \arctan(yx^{-1}) \end{cases}$$

and $x_\alpha(\alpha, \beta, 0) = 1$, $y_\alpha(\alpha, \beta, 0) = 0$. Let

$$f_{11}(t) = \nu_{1x} - 0.5 \log(x^2 + y^2) + 0.5 r_1 + 0.5 x r_{1x}$$
$$f_{12}(t) = 0.5 x r_{1y} + \nu_{1y} - \arctan(xy^{-1})$$
$$f_{21}(t) = 0.5 y r_{2x} + \nu_{2x} + \arctan(yx^{-1})$$
$$f_{22}(t) = 0.5 \log(x^2 + y^2) + 0.5 r_2 + 0.5 y r_{2y} + \nu_{2y}$$
$$x_\alpha = \exp\left(\int_0^t f_{11}(\tau) d\tau \right) \hat{x}, \quad y_\alpha = \exp\left(\int_0^t f_{22}(\tau) d\tau \right) \hat{y}.$$

If

$$g = f_{11} - f_{22},$$

then

$$\hat{x}(t) = 1 + \int_0^t \hat{x}(s) f_{21}(s) \int_s^t f_{12}(\tau) \exp\left(-\int_s^\tau g(\xi) d\xi \right) d\tau ds.$$

Since the trajectory is inside $\aleph$, we have $y > \sqrt{x}$ and so

$$|f_{12}| \lesssim y + \upsilon, \quad |f_{21}| \lesssim 1, \quad f_{11} > e^t(-\log\beta + C), \quad g(t) > 1.$$

From (15), we get

$$|\hat{x}(t) - 1| \lesssim \upsilon \int_0^t |\hat{x}(\tau)| d\tau + \int_0^t |\hat{x}(s)| \left(\int_s^t e^{e^\tau(\log\beta + C)} e^{-(\tau - s)} d\tau \right) ds.$$

The following estimate is obvious:

$$\int_s^t e^{e^\tau(\log\beta + C)} e^{-(\tau - s)} d\tau \ll e^{-s}$$

as β is small. Assuming that

$$(21) \qquad \qquad \upsilon \ll (T + 1)^{-1}$$

and ϵ_2 is small, we have

$$\hat{x}(t) \sim 1$$

and

$$(22) \qquad \qquad x_\alpha(\alpha, \beta, T) > \left(\frac{1}{\beta} \right)^{(e^T - 1)/2}.$$

The estimate (22) is the key estimate that will guarantee the necessary growth.

Now, let us place a circle $S_\gamma(\tilde{x}, \tilde{y})$ with radius γ and center at $(\tilde{x}, \tilde{y})$ into the zone Ω_0. Consider also the line segment $l = [A_1, A_2], A_1 = (\tilde{x} - \gamma/2, \tilde{y}), A_2 = (\tilde{x} + \gamma/2, \tilde{y})$ in the center, parallel to OX. We will track the evolution of this disc and this line segment under the flow. We have by the mean value theorem

$$x(A_2, T) - x(A_1, T) > \beta^{-(e^T - 1)/2} \gamma.$$

From the positivity of $x_\alpha(\alpha, \beta, T)$ it follows that the image of l under the flow at time T is a curve given by the graph of a smooth function $\Gamma(x)$. Thus, the image of l (call it l') has length at least $\beta^{-(e^T - 1)/2} \gamma$. Denote the distance from l' to $S'_\gamma(\tilde{x}, \tilde{y})$, the image of the circle, by d. Then, the domain $\{\Gamma(x) - d < y(x) < \Gamma(x) + d, x \in (x(A_1, T), x(A_2, T))\}$ is inside $S'_\gamma(\tilde{x}, \tilde{y})$. The area of this domain is at least

$$d\beta^{-(e^T - 1)/2} \gamma.$$

Thus, assuming that the flow preserves the area, we have

$$d \lesssim \beta^{(e^T - 1)/2} \gamma.$$

Consequently, if we place a bump in Ω_0 such that the l and $S_\gamma(\tilde{x}, \tilde{y})$ correspond to level sets, say, h_2 and h_1 (**and, what is crucial, $h_{1(2)}$ are essentially arbitrary** $0 < h_1 < h_2 < 0.0001$), then the original slope of at least $\sim |h_2 - h_1|/\gamma$ will become not less than

$$(23) \qquad \qquad \beta^{-(e^T - 1)/2} \cdot (|h_2 - h_1|/\gamma),$$

thus leading to double-exponential growth of arbitrarily large gradients.

Remark 1. If β is a fixed small number, we have growth in T. If T is any positive fixed moment of time, we have the growth if $\beta \to 0$.

Remark 2. Let us reiterate the order in which the parameters are chosen: we first fix any T, then small ϵ_2, then $\epsilon_1 < \epsilon_2^{8e^{2T}}$. How small ϵ_2 must be taken will be determined by how large the parameter λ is chosen in Theorem 6.1 below. This defines the set Ω_0. For the whole argument to work we need to collect all conditions on υ: (14), (19), (21) which leads to

$$(24) \qquad\qquad\qquad \upsilon < \epsilon_1^{10}.$$

4. SMALL PERTURBATIONS OF A SINGULAR CROSS CAN ALSO GENERATE DOUBLE EXPONENTIAL CONTRACTION IN ℵ

Assume that the function θ_1 at any given time $t \in [0,T]$ is such that

$$\theta_1(x,y,t) = \theta_0^s(x,y)$$

outside the "cross"-domain

$$(25) \qquad\qquad A = \{|x - \pi k| < \tau\} \cup \{|y - \pi l| < \tau\}$$

where τ is small and $k, l \in \mathbb{Z}$. Inside the domain A we only assume that θ_1 is bounded by one in absolute value, is even, and has zero average. Notice here that the Euler flow preserves property of the function to be even. Given fixed $\epsilon_{1(2)}$ and the domain ℵ defined by these constants, we are going to show that the flow generated by θ_1 can be represented in ℵ in the form (10) with $\upsilon(\tau) \to 0$ as $\tau \to 0$. We assume of course that $\tau \ll \epsilon_1$.

For that, we only need to study

$$F_1 = \nabla\Delta^{-1}p, \quad p = \theta_1 - \theta_0^s.$$

Here are some obvious properties of F_1:

1. $F_1(0) = 0$ as θ_1 and θ_0^s are both even.
2. We have

$$F_1(z) \sim \int_A \left(\frac{\xi - z}{|\xi - z|^2} - \frac{\xi}{|\xi|^2} \right) p(\xi)d\xi.$$

Using the formula

$$\left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right| = \frac{|x - y|}{|x| \cdot |y|}$$

we get

$$|F_1(z)| \lesssim |z|\frac{\tau|\log\tau|}{\epsilon_1}$$

if $z \in ℵ$.

Thus, by taking τ small, we can satisfy (11). How about (12)? For the Hessian, we have

$$|H\Delta^{-1}p| \lesssim \epsilon_1^{-2}\tau$$

and after combining we must have

$$(26) \qquad\qquad\qquad \epsilon_1^{-2}\tau|\log\tau| \lesssim \epsilon_1^{10}$$

by (24). Thus, this condition on the size of the cross guarantees that the arguments in the previous section work.

5. The flow generated by a small steep bump in $\aleph$

In this section, we assume that at a given moment $t \in [0, T]$, we have a smooth even function $b(x, y, t)$ with support in $\aleph \cup -\aleph$, with zero average, and

$$\|b\|_2 < \omega, \quad \|\nabla b\|_\infty < M$$

(here one should think about small ω and large M). We will study the flow generated by this function. Let

$$F_2 = \nabla \Delta^{-1} b.$$

Here are some properties of F_2:

1. $F_2(0) = 0$.

2. To estimate the Hessian of $\Delta^{-1} b$, consider the second order derivatives. For example,

$$(\Delta^{-1}b)_{xy}(x,y) \sim \int_{(x-\xi)^2+(y-\eta)^2<1} \frac{(x-\xi)(y-\eta)}{((x-\xi)^2+(y-\eta)^2)^2} b(\xi,\eta,t)\,d\xi d\eta$$

$$= \int_{1>(x-\xi)^2+(y-\eta)^2>\rho^2} \frac{(x-\xi)(y-\eta)}{((x-\xi)^2+(y-\eta)^2)^2} b(\xi,\eta,t)\,d\xi d\eta$$

$$+ \int_{(x-\xi)^2+(y-\eta)^2<\rho^2} \frac{(x-\xi)(y-\eta)}{((x-\xi)^2+(y-\eta)^2)^2} \left[b(x,y,t) \right.$$

$$\left. + \nabla b(\xi',\eta',t) \cdot (\xi-x,\eta-y) \right] d\xi d\eta.$$

The first term is controlled by $\omega\rho^{-1}$. By our assumption, the second term is dominated by $M\rho$. Optimizing in ρ we have

$$\|H\Delta^{-1}b\|_\infty \lesssim \sqrt{M\omega}.$$

To guarantee the conditions that lead to double-exponential growth with arbitrary a priori given M, we want to make ω so small that conditions (11) and (12) are satisfied with v as small as we need (i.e., (24)). The condition (12) is immediate and (11) follows from $F_2(0) = 0$, the mean value theorem and the estimate on the Hessian.

6. One stability result and the proof of the main theorem

It is well known that given $\theta_0 \in L^\infty(\mathbb{T}^2)$, the weak solution exists and the flow can be defined by the homeomorphic maps $\Psi_{\theta_0}(x, y, t)$ for all t so that $\theta(x, y, t) = \theta_0(\Psi_{\theta_0}^{-1}(x, y, t))$ where Ψ_{θ_0} itself depends on θ_0. The regularity of this map though is rather poor ([3, Theorem 2.3], p. 99). In this section, we will need to take smooth θ_0 such that

$$\max_{t\in[0,T]} \max_{z\in\mathbb{T}^2} |\Psi_{\theta_0}(z,t) - \Psi_{\theta_0^s}(z,t)|$$

is sufficiently small.

To this end, we will consider $\theta_0 = \theta_0^s$ outside the domain $\mathcal{D}$ of small area. Inside this domain we assume θ_0 to be bounded by some universal constant. The proof of Yudovich theorem (see, e.g., the argument on pp. 313–318, proof of Proposition 8.2, [2]) implies

$$(27) \qquad \max_{t\in[0,T]} \max_{z\in\mathbb{T}^2} |\Psi_{\theta_0^s}(z,t) - \Psi_{\theta_0}(z,t)| \to 0$$

as $|\mathcal{D}| \to 0$.

This is the only stability result with respect to initial data that we are going to need in the argument below.

Theorem 6.1. *For any large λ and any $T > 0$, we can find smooth initial data θ_0 so that $\|\theta_0\|_\infty < 2$ and*

$$\max_{t \in [0,T]} \|\nabla\theta(\cdot, t)\|_\infty > \lambda^{e^T - 1} \|\nabla\theta_0\|_\infty.$$

Proof. Fix any $T > 0$ and find $\epsilon_{1(2)}$. For larger λ, we have to take smaller ϵ_2 (see Remark 1 in the third section). Identify the domain Ω_0 and place a bump (call it $b(z)$) in $\Omega_0 \cup -\Omega_0$ so that the resulting function is even. Make sure that this bump has zero average, height h_2 and diameter of support h_1 so that the gradient initially is of the size $\sim h_2/h_1$. Here $h_1 \ll h_2 \ll 1$ will be adjusted later.

Take a smooth even function $\omega(x, y)$ supported on $B_1(0)$ such that

$$\int_{\mathbb{T}^2} \omega(x, y) dx dy = 1.$$

For positive small σ, consider

$$\tilde{\theta}_\sigma(x, y) = \theta_0^s * \omega_\sigma \in C^\infty, \quad \omega_\sigma = \sigma^{-2}\omega(x/\sigma, y/\sigma).$$

We take $\sigma \ll \epsilon_1$ so $\tilde{\theta}_\sigma(x, y)$ and $\theta_0^s(x, y)$ coincide in $\aleph$.

As the initial data for Euler dynamics we take a sum

$$\tilde{\theta}_\sigma(z) + b(z).$$

Then, since θ_0^s is stationary under the flow, the stability result (27) guarantees that given any τ and keeping the same value of h_2/h_1, we can find σ and h_1 so small that over the time interval $[0, T]$ we satisfy:

1. The "evolved bump" $b(z, t)$ stays in the domain $\aleph$ (e.g., $\Psi_{\theta_0}(t)\Big(\operatorname{supp} b(z)\Big) \subset \aleph$).

2. Outside the cross of size τ (the one considered in Section 5) and the support of the evolved bump b, the solution is identical to θ_0^s.

Fix σ and h_1' so small that for any $h_1 < h_1'$ we have τ, e.g. the size of A from (25), being as small as we wish. The value of τ must be small enough to ensure the double-exponential scenario, the conditions (11) and (12). For that, we need (26).

Next, we proceed by contradiction. Assume that for all $t \in [0, T]$ we have $\|\nabla\theta(z, t)\|_\infty < M = (h_2/h_1)\lambda^{e^T - 1}$. Then, because $\|b(z, t)\|_2$ is constant in time as the flow is area-preserving and $\|b(z, t)\|_2 \lesssim h_1 h_2$, we only need to take h_2 so small that $\sqrt{M h_1 h_2}$ is small enough to guarantee the double-exponential scenario and the estimate (23). This gives us a contradiction as the double-exponential scenario makes the gradient's norm more than M (provided that $\epsilon_2 \ll \lambda^{-2}$). For the initial value,

$$\|\nabla\theta_0\|_\infty \sim \sigma^{-1} + h_2/h_1 \sim h_2/h_1$$

by arranging $h_{1(2)}$ (and keeping $h_1 < h_1'$).

Here is an order in which parameters are chosen in this construction:

$$\{T, \lambda\} \longrightarrow \epsilon_2 \longrightarrow \epsilon_1 \longrightarrow \{\sigma, h_{1(2)}\}.$$

$\square$

7. The operator $\mathcal{E}_t$ does not allow a linear bound

Theorem 6.1 is equivalent to the following:

Proposition 1. *The operator $\mathcal{E}_t$ is not bounded in the linear sense in the Lipschitz norm for any $t > 0$, i.e.*

$$\sup_{\theta_0 \in C^\infty(\mathbb{T}^2), 0 < \|\theta_0\|_\infty \leq 1, \langle \theta_0, 1 \rangle = 0} \frac{\|\nabla \mathcal{E}_t \theta_0\|_\infty}{\|\nabla \theta_0\|_\infty} = +\infty.$$

Proof. The proof is immediate. Indeed, given any fixed t, we have

$$\sup_{\tau \in [0,t]} \left(\sup_{\theta_0 \in C^\infty(\mathbb{T}^2), \|\theta_0\|_\infty = 1, \langle \theta_0, 1 \rangle = 0} \frac{\|\nabla \mathcal{E}_\tau \theta_0\|_\infty}{\|\nabla \theta_0\|_\infty} \right) = +\infty$$

by taking $\lambda \to \infty$ in Theorem 6.1. Then, to have the statement at time t, we only need to multiply θ_0 by a suitable number and use Remark 2 from the first section.

Vice versa, in Theorem 6.1 the combination $\lambda^{e^T - 1}$ can be replaced by an arbitrarily large number. In this formulation, the statement follows from the proposition. $\square$

As the statement of Theorem 6.1 holds with any λ, the double-exponential function is not relevant at all in the formulation itself. However, it is this very special hyperbolic scenario with double-exponential rate of contraction that ultimately provided the superlinear dependence on the initial data.

The interesting and important question is whether the vorticity gradient can grow in the same double exponential rate starting with initial value ~ 1. We do not know the answer to this question yet and the best known bound is (see, e.g., [5])

$$\max_{t \in [0,T]} \|\nabla \theta(\cdot, t)\|_\infty > e^{0.001 T}$$

for arbitrary T and for T–dependent θ_0 with $\|\theta_0\|_\infty \sim \|\nabla \theta_0\|_\infty \sim 1$.

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