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CACCIOPPOLI ESTIMATES THROUGH AN ANISOTROPIC PICONE'S IDENTITY

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Abstract. Caccioppoli-type estimates for a class of nonlinear differential operators which include the p-Laplacian and the pseudo p-Laplacian as special cases are obtained by means of the differential identity involving an arbitrary norm in $\mathbb{R}^n$ which generalizes the well-known multidimensional Picone's formula.

1. Introduction

One of the alternative ways to establish the basic Caccioppoli inequality in which the L^2 norm of the gradient of a harmonic (or subharmonic) function v is estimated in terms of the L^2 norm of v itself is to derive it from the Picone identity (see [15] or [17])

$$\|\nabla u\|_2^2 - \big\langle \nabla\big(\frac{u^2}{v}\big), \nabla v\big\rangle = \|\nabla u - \frac{u}{v}\nabla v\|_2^2, \tag{1.1}$$

where u and v are differentiable functions on a given domain $\Omega \subset \mathbb{R}^n$ with $v(x) \neq 0$ in Ω and $\|.\|_2$, ∇ and $\langle , \rangle$ stand for the Euclidean norm, the usual gradient and the inner product in $\mathbb{R}^n$, respectively.

Indeed, if $v > 0$ is a weak (continuous) solution of $\operatorname{div}(\nabla v) = 0$ in Ω and $\eta \in C_c^\infty(\Omega)$ is a nonnegative test function, then from the integrated form of (1.1) with $u = \eta v$, using Young's and the Cauchy-Schwartz inequality, we easily obtain

$$0 = \int_\Omega \|\eta\nabla v\|_2^2 dx - 2\int_\Omega \big\langle \eta\nabla v, \nabla(\eta v)\big\rangle dx \leq -\frac{1}{2}\int_\Omega \|\eta\nabla v\|_2^2 dx + 2\int_\Omega \|v\nabla\eta\|_2^2 dx$$

which yields the desired estimate

$$\int_\Omega \|\eta\nabla v\|_2^2 dx \leq 4\int_\Omega \|v\nabla\eta\|_2^2 dx. \tag{1.2}$$

More generally, if $p > 1$ is fixed and $v > 0$ is a weak continuous solution (or subsolution) of the p-harmonic equation $\operatorname{div}(\|\nabla v\|_2^{p-2}\nabla v) = 0$, then the L^p-version of the Caccioppoli estimate ([11]; see also the inequality (5.27) in [16] and Corollary A.6 in [13])

$$\int_\Omega \|\eta\nabla v\|_2^p dx \leq p^p \int_\Omega \|v\nabla\eta\|_2^p dx \tag{1.3}$$

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can easily be obtained from the p-Laplacian generalization of Picone's identity (see [2] and [9])

$$\|\nabla u\|_2^p - \left\langle \nabla\left(\frac{|u|^p}{v^{p-1}}\right), \|\nabla v\|_2^{p-2}\nabla v\right\rangle = \Phi_p(u,v), \tag{1.4}$$

where

$$\Phi_p(u,v) := \|\nabla u\|_2^p + (p-1)\frac{|u|^p}{v^p}\|\nabla v\|_2^p - p\frac{|u|^{p-2}u}{v^{p-1}}\langle \nabla u, \|\nabla v\|_2^{p-2}\nabla v\rangle \geq 0, \tag{1.5}$$

by setting $u = \eta v$ and making use of Young's and the Hölder inequality.

The purpose of this paper is to extend the Caccioppoli inequality to a class of differential operators of the form

$$\Delta_{H,p} v := \operatorname{div}\big(H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\big) \tag{1.6}$$

where $p > 1$, $H : \mathbb{R}^n \to [0, +\infty)$, $n \geq 2$, is a convex function of the class $C^1(\mathbb{R}^n\backslash\{0\})$ which is positively homogeneous of degree 1 and ∇ and ∇_ξ stand for usual gradient operators with respect to the variables x and ξ, respectively. We refer to the operator $\Delta_{H,p}$ as the Finsler p-Laplacian (or the anisotropic p-Laplacian). A prototype of H satisfying the above conditions is the l_r-norm

$$H(\xi) = \|\xi\|_r = \left(\sum_{i=1}^n |\xi_i|^r\right)^{1/r}, \ r > 1, \tag{1.7}$$

for which the operator defined by (1.6) has the form

$$\Delta_{r,p} v := \operatorname{div}\big(\|\nabla v\|_r^{p-r}\nabla^r v\big) \tag{1.8}$$

where

$$\nabla^r v := \left(\left|\frac{\partial v}{\partial x_1}\right|^{r-2}\frac{\partial v}{\partial x_1}, ..., \left|\frac{\partial v}{\partial x_n}\right|^{r-2}\frac{\partial v}{\partial x_n}\right). \tag{1.9}$$

The class of operators of the form (1.8) includes the usual p-Laplacian and the so-called pseudo p-Laplace operator as the special cases corresponding to $r = 2$ and $p \in (1, \infty)$, and $r = p > 1$, respectively. Clearly, if $p = r = 2$, then (1.8) reduces to the standard Laplacian Δ.

Anisotropic elliptic problems involving this kind of operator and related questions have recently been studied in several papers including [1], [3]–[8], [10] and [18]–[20].

Our main result in this note is Theorem 3.1 below which states that if $v > 0$ is a weak subsolution in Ω of the equation

$$-\Delta_{H,p} v = g(x)|v|^{p-2}v,$$

where $0 \leq g \in L^\infty_{loc}(\Omega)$, then, for any fixed $q > p-1$ and $w = v^{q/p}$, the inequality

$$\int_\Omega H(\eta\nabla w)^p dx \leq \left(\frac{q}{q-p+1}\right)^p \int_\Omega H(w\nabla\eta)^p dx + \frac{q^p p^{1-p}}{q-p+1}\int_\Omega g(x) w^p \eta^p dx$$

holds for any nonnegative function $\eta \in C_c^\infty(\Omega)$. Analogous result is valid also for positive supersolutions of the above equation and $q < p-1$.

The paper is organized as follows. Section 2 contains a brief review of the basic properties of general norms in $\mathbb{R}^n$ that will be used in the sequel. Here we also present an extension of Picone's identity to the Finsler p-Laplacian. In Section 3 we establish Caccioppoli-type estimates for positive sub- and supersolutions of nonlinear equations involving anisotropic elliptic operators.

2. Background

We begin by recalling basic properties of general norms in $\mathbb{R}^n$ and their duals. For the proofs see [5] or [6].

Let H be an arbitrary norm in $\mathbb{R}^n$, i.e., a convex function $H : \mathbb{R}^n \to [0, \infty)$ satisfying $H(\xi) > 0$ for all $\xi \neq 0$ which is positively homogeneous of degree 1, so that

$$H(t\xi) = |t|H(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \quad \text{and } t \in \mathbb{R}. \tag{2.1}$$

If we assume that $H \in C^1(\mathbb{R}^n \backslash \{0\})$, then from (2.1) it follows that

$$\nabla_\xi H(t\xi) = \operatorname{sgn} t \; \nabla_\xi H(\xi) \quad \text{for all } \xi \neq 0 \quad \text{and } t \neq 0 \tag{2.2}$$

and

$$\langle \xi, \nabla_\xi H(\xi) \rangle = H(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \tag{2.3}$$

where the left-hand side is defined to be 0 if $\xi = 0$. Let $\langle , \rangle$ denote the usual inner product in $\mathbb{R}^n$ and define the dual norm H_0 of H by

$$H_0(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{H(\xi)} \quad \text{for } x \in \mathbb{R}^n. \tag{2.4}$$

The H_0-unit-ball, i.e., the set $\{x \in \mathbb{R}^n : H_0(x) \leq 1\}$ is sometimes called the *Wulff shape* (or *equilibrium crystal shape*) of H.

Any norm H of the class C^1 for $\xi \neq 0$ and its dual H_0 satisfy

$$H_0(\nabla_\xi H(\xi)) = 1 \quad \text{for all } \xi \in \mathbb{R}^n \backslash \{0\}. \tag{2.5}$$

Similarly, if H_0 is continuously differentiable for $x \neq 0$, then

$$H(\nabla H_0(x)) = 1 \quad \text{for all } x \in \mathbb{R}^n \backslash \{0\}. \tag{2.6}$$

Also, the identities

$$H\big[H_0(x)\nabla H_0(x)\big]\nabla_\xi H\big[H_0(x)\nabla H_0(x)\big] = x \tag{2.7}$$

and

$$H_0\big[H(\xi)\nabla_\xi H(\xi)\big]\nabla H_0\big[H(\xi)\nabla_\xi H(\xi)\big] = \xi \tag{2.8}$$

hold for all $x, \xi \in \mathbb{R}^n$, where $H(0)\nabla_\xi H(0)$ and $H_0(0)\nabla H_0(0)$ are defined to be 0.

From the definition (2.4) we easily obtain the Hölder-type inequality

$$\langle x, \xi \rangle \leq H(\xi)H_0(x) \quad \text{for all } x, \xi \in \mathbb{R}^n \tag{2.9}$$

with equality holding if and only if

$$x = H(\xi)\nabla_\xi H(\xi) \quad (\text{or, equivalently, } H_0(x) = H(\xi)). \tag{2.10}$$

It is well known that a continuously differentiable function F defined in an open convex subset of $\mathbb{R}^n$ is strictly convex if and only if

$$F(y) - F(x) - \big\langle \nabla F(x), y - x \big\rangle > 0 \tag{2.11}$$

for all $x \neq y$ (see, for example, [14]). As an immediate consequence of this result we have the following simple lemma.

Lemma 2.1. *Let H be a norm in $\mathbb{R}^n$ such that $H \in C^1(\mathbb{R}^n \backslash \{0\})$ and $H^p, 1 < p < \infty$, is strictly convex. If*

$$H(\xi)^p + (p-1)H(\eta)^p - p\big\langle \xi, H(\eta)^{p-1}\nabla H(\eta)) \big\rangle = 0 \tag{2.12}$$

for some $\xi, \eta \in \mathbb{R}^n$, $\eta \neq 0$, and $H(\xi) = H(\eta)$, then $\xi = \eta$.

Proof. If $\xi, \eta \in \mathbb{R}^n$ with $\eta \neq 0$ satisfy $H(\xi) = H(\eta)$ and (2.12), then

$$0 = pH(\eta)^p - p\langle \eta, H(\eta)^{p-1}\nabla H(\eta)\rangle + p\langle \eta - \xi, H(\eta)^{p-1}\nabla H(\eta)\rangle \tag{2.13}$$
$$= p\langle \eta - \xi, H(\eta)^{p-1}\nabla H(\eta)\rangle.$$

We claim that $pH(\eta)^{p-1}\nabla H(\eta) = \nabla\big(H(\eta)\big)^p \neq 0$. Indeed, if $\nabla\big(H(\eta)\big)^p$ were the zero vector for some $\eta \in \mathbb{R}^n$, i.e., even strictly convex function $H(\eta)^p$ attained its global minimum at η, then η would necessarily be equal to 0, a contradiction. Therefore, by strict convexity of H^p, $\xi = \eta$, and the proof is complete. □

Next, we present a generalization of Picone's identity for the Finsler p-Laplacian $\Delta_{H,p}$.

Lemma 2.2. *Let H be an arbitrary norm in $\mathbb{R}^n$ which is of class C^1 for $x \neq 0$. Assume that $u, v \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$ with $v(x) \neq 0$ in Ω and denote*

$$\Phi(u,v) := H(\nabla u)^p + (p-1)\frac{|u|^p}{|v|^p}H(\nabla v)^p - p\frac{|u|^{p-2}u}{|v|^{p-2}v}\langle \nabla u, H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\rangle.$$

Then

$$H(\nabla u)^p - \Big\langle \nabla\Big(\frac{|u|^p}{|v|^{p-2}v}\Big), H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\Big\rangle = \Phi(u,v) \tag{2.14}$$

and $\Phi(u,v) \geq 0$ a.e. in Ω. If, in addition, $H(\xi)^p$ is strictly convex in $\mathbb{R}^n$, then $\Phi(u,v) = 0$ a.e. in Ω if and only if u is a constant multiple of v in Ω.

Proof. First, verify (2.14) directly by expanding the left-hand side of (2.14). Next, notice that $\Phi(u,v) = \Phi_1(u,v) + \Phi_2(u,v)$, where

$$\Phi_1(u,v) := H(\nabla u)^p - pH(\nabla u)H\big(\frac{u}{v}\nabla v\big)^{p-1} + (p-1)H\big(\frac{u}{v}\nabla v\big)^p$$

and

$$\Phi_2(u,v) := p\big[H(\nabla u)H\big(\frac{u}{v}\nabla v\big)^{p-1} - \big\langle \nabla u, H\big(\frac{u}{v}\nabla v\big)^{p-1}\nabla_\xi H\big(\frac{u}{v}\nabla v\big)\big\rangle\big],$$

and apply the Young inequality to Φ_1 and the generalized Hölder inequality (2.9) to Φ_2 to conclude that both $\Phi_1(u,v) \geq 0$ and $\Phi_2(u,v) \geq 0$.

Finally, $\Phi(u,v) = 0$ means that equality case simultaneously occurs in the Young inequality and the Hölder inequality (2.9) a.e. in Ω. A necessary and sufficient condition for the first equality is that

$$H(\nabla u) = H\big(\frac{u}{v}\nabla v\big) \quad \text{a.e. in } \Omega. \tag{2.15}$$

If $\big(u\nabla v/v\big)(x_0) \neq 0$ for some $x_0 \in S := \{x \in \Omega : \Phi(u,v) = 0\}$, then by Lemma 2.1 we have $\nabla u = u\nabla v/v$ at x_0, or, equivalently, $\nabla(u/v)(x_0) = 0$. On the other hand, if $u\nabla v/v = 0$ on some subset S_0 of S, then $\nabla u = 0$ a.e. in S_0 which implies $\nabla(u/v) = 0$ a.e. in S_0. Summarizing the above facts we get $\nabla(u/v) = 0$ a.e. in Ω which forces u/v to be constant in Ω. □

Remark 2.1. In the special case where H is an l_r-norm, $r \in (1,\infty)$, the differential identity (2.14) reduces to

$$\|\nabla u\|_r^p - \left\langle \nabla\left(\frac{|u|^p}{|v|^{p-2}v}, \|\nabla v\|_r^{p-r}\nabla^r v\right)\right\rangle = \Phi_{p,r}(u,v), \tag{2.16}$$

where $\nabla^r v$ is defined in (1.9) and

$$\Phi_{p,r}(u,v) := \|\nabla u\|_r^p + (p-1)\frac{|u|^p}{|v|^p}\|\nabla v\|_r^p - p\frac{|u|^{p-2}u}{|v|^{p-2}v}\big\langle \nabla u, \|\nabla v\|_r^{p-r}\nabla^r v\big\rangle \geq 0. \tag{2.17}$$

3. Caccioppoli-type estimates

Consider the equation

$$-\Delta_{H,p} v = g(x)|v|^{p-2}v, \tag{3.1}$$

where $0 \leq g \in L^\infty_{loc}(\Omega)$. Following the classical terminology, we will say that a continuous function $v \in W^{1,p}_{loc}(\Omega)$ is a *weak solution* of (3.1), if for any function $\eta \in W^{1,p}_c(\Omega) \cap C(\Omega)$ we have

$$\int_\Omega H(\nabla v)^{p-1}\big\langle \nabla_\xi H(\nabla v), \nabla\eta\big\rangle dx = \int_\Omega g|v|^{p-2}v\eta dx. \tag{3.2}$$

Here $W^{1,p}_c(\Omega)$ is used to denote the set of all functions $u \in W^{1,p}_{loc}(\Omega)$ which are compactly supported in Ω.

Weak subsolutions and supersolutions of (3.1) are defined analogously with test functions $\eta \in W^{1,p}_c(\Omega) \cap C(\Omega)$ satisfying $\eta \geq 0$ and "=" in (3.2) replaced by "$\leq$" and "$\geq$", respectively.

We associate with (3.1) the homogeneous (of degree p) functional

$$J_H(u;\Omega) := \int_\Omega \big[H(\nabla u)^p - g(x)|u|^p\big]dx, \quad u \in W^{1,p}_c(\Omega) \cap C(\Omega). \tag{3.3}$$

If $v(x) > 0$ is a (continuous) weak subsolution of (3.1) in Ω and $u \in W^{1,p}_c(\Omega) \cap C(\Omega)$, then we can choose

$$\eta = \frac{|u|^p}{v^{p-1}}$$

as a test function in (3.2) and conclude, by Picone's identity (2.14), that

$$\begin{aligned} J_H(u;\Omega) &\leq \int_\Omega \left[H(\nabla u)^p - H(\nabla v)^{p-1}\left\langle \nabla_\xi H(\nabla v), \nabla\left(\frac{|u|^p}{v^{p-1}}\right)\right\rangle\right]dx \\ &= \int_\Omega \Phi(u,v)dx. \end{aligned} \tag{3.4}$$

Clearly, for positive supersolutions v of (3.1) and any $u \in W^{1,p}_c(\Omega) \cap C(\Omega)$, the reversed inequality

$$J_H(u;\Omega) \geq \int_\Omega \Phi(u,v)dx \tag{3.5}$$

holds true. For solutions (3.4) becomes an equality.

The following estimate is the main result of this paper.

Theorem 3.1 (Caccioppoli inequality)**.** *Let $v > 0$ be a weak subsolution of* (3.1) *in Ω. Then, for any fixed $q > p-1$ and $w = v^{q/p}$, the inequality*

$$\int_\Omega H(\eta\nabla w)^p dx \leq \left(\frac{q}{q-p+1}\right)^p \int_\Omega H(w\nabla\eta)^p dx + \frac{q^p p^{1-p}}{q-p+1}\int_\Omega g(x)w^p\eta^p dx \tag{3.6}$$

holds for all $0 \leq \eta \in C^\infty_c(\Omega)$.

Proof. Let v be a positive subsolution of (3.1) in Ω. Fix a nonnegative function $\eta \in C_c^\infty(\Omega)$. Then $u := v^{q/p}\eta$ belongs to $W_c^{1,p} \cap C(\Omega)$ and we can use it as a test function in (3.4) to get

$$\begin{aligned} J_H(v^{q/p}\eta;\Omega) &\le \int_\Omega H\big(\nabla(v^{q/p}\eta)\big)^p dx + (p-1)\int_\Omega v^{q-p}H(\eta\nabla v)^p dx \\ &-p\int_\Omega \big(v^{(q-p)/p}\eta\big)^{p-1}\big\langle H(\nabla v)^{p-1}\nabla_\xi H(\nabla v), \nabla(v^{q/p}\eta)\big\rangle dx \\ &= \int_\Omega H\big(\nabla(v^{q/p}\eta)\big)^p dx - (q-p+1)\int_\Omega v^{q-p}H(\eta\nabla v)^p dx \\ &-p\int_\Omega \big\langle H(v^{(q-p)/p}\eta\nabla v)^{p-1}\nabla_\xi H(v^{(q-p)/p}\eta\nabla v), v^{q/p}\nabla\eta\big\rangle dx, \end{aligned}$$

where we have used $\nabla(v^{q/p}\eta) = (q/p)v^{(q-p)/p}\eta\nabla v + v^{q/p}\nabla\eta$ and the property (2.5) of the norm H. To estimate the latter integral, we use the generalized Hölder inequality (2.9), (2.5) and Young's inequality in the form

$$ab^{p-1} \le \frac{1}{p\tau^{p-1}}a^p + \frac{p-1}{p}\tau b^p, \quad a,b \ge 0,\ \tau > 0,$$

applied to $a = H(v^{q/p}\nabla\eta)$ and $b = H(v^{(q-p)/p}\eta\nabla v)$. As a result we obtain

$$\begin{aligned} J_H(v^{q/p}\eta;\Omega) \le \int_\Omega H\big(\nabla(v^{q/p}\eta)\big)^p dx - [q-p+1-\tau(p-1)]\int_\Omega H(v^{(q-p)/p}\eta\nabla v)^p dx \\ +\frac{1}{\tau^{p-1}}\int_\Omega H(v^{q/p}\nabla\eta)^p dx. \end{aligned}$$

Now, making use of the definition of J_H, we get

$$\begin{aligned} \int_\Omega v^{q-p}H(\eta\nabla v)^p dx \le \frac{\tau^{1-p}}{q-p+1-\tau(p-1)}\int_\Omega H(v^{q/p}\nabla\eta)^p dx \\ +\frac{1}{q-p+1-\tau(p-1)}\int_\Omega g(x)v^q\eta^p dx, \end{aligned} \tag{3.7}$$

which after choosing the constant τ appropriately leads to

$$\int_\Omega v^{q-p}H(\eta\nabla v)^p dx \le \left(\frac{p}{q-p+1}\right)^p \int_\Omega H(v^{q/p}\nabla\eta)^p dx + \frac{p}{q-p+1}\int_\Omega g(x)v^q\eta^p dx. \tag{3.8}$$

Finally, the substitution $w = v^{q/p}$ yields (3.6) as claimed. □

Corollary 3.1. *Let $q = p$ and $g(x) \equiv 0$ in Ω. If $v > 0$ is a weak subsolution of* (3.1) *in Ω, then*

$$\int_\Omega H(\eta\nabla v)^p dx \le p^p \int_\Omega H(v\nabla\eta)^p dx \tag{3.9}$$

for any nonnegative $\eta \in C_c^\infty(\Omega)$.

For the next corollary, let $B(x_0;r)$ be the Wulff ball with the radius $r > 0$ centered at x_0, i.e., the set $\{x \in \mathbb{R}^n : H_0(x-x_0) \le r\}$, and denote by B_r and B_{2r} two concentric H_0-balls with $x_0 \in \Omega$.

Corollary 3.2. *If the conditions of Corollary* 3.1 *are satisfied and $B_{2r} \subset \Omega$, then*

$$\int_{B_r} H(\nabla v)^p dx \le (p/r)^p \int_{B_{2r}\setminus B_r} v^p dx. \tag{3.10}$$

Proof. To get the estimate (3.10), choose in (3.9) an H_0-radial test function η which satisfies $\eta \equiv 1$ in B_r, $H_0(\nabla\eta) \leq 1/r$ in $B_{2r}\backslash B_r$ and $\eta \equiv 0$ in $\Omega\backslash B_{2r}$. □

An analogous result as Theorem 3.1 above holds for positive supersolutions of (3.1) and a range of q smaller than $p-1$. The proof is left to the reader. (If necessary, use $v(x)+\epsilon, \epsilon > 0$ instead of $v(x)$.)

Theorem 3.2. *Let $v > 0$ be a supersolution of* (3.1) *in Ω. Then, for any fixed $q < p-1$, the inequality*

(3.11)
$$\int_\Omega v^{q-p} H(\eta\nabla v)^p dx \leq \left(\frac{p}{p-q-1}\right)^p \int_\Omega H(v^{q/p}\nabla\eta)^p dx - \frac{p}{p-q-1}\int_\Omega g(x) v^q \eta^p dx$$

holds for all $\eta \in C_c^\infty(\Omega)$ with $\eta \geq 0$.

The particular case $q = 0$ of the above theorem is interesting in the sense that the right-hand side of (3.11) does not contain v. The result specializes as follows.

Corollary 3.3 (Logarithmic Caccioppoli inequality)**.** *Let $v > 0$ be a weak supersolution of* (3.1) *in Ω. Then*

$$\int_\Omega H(\eta\nabla \log v)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_\Omega H(\nabla\eta)^p dx - \frac{p}{p-1}\int_\Omega g(x)|\eta|^p dx, \tag{3.12}$$

whenever $0 \leq \eta \in C_c^\infty(\Omega)$.

If $H(\xi) = \|\xi\|_2, \xi \in \mathbb{R}^n$, and $g(x) \equiv 0$ in Ω, then (3.12) reduces to the well-known logarithmic Caccioppoli inequality for positive p-superharmonic functions (cf. [12]).

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