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THE WEIGHTED SOBOLEV AND MEAN VALUE INEQUALITIES

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(Communicated by Lei Ni)

Dedicated to my son João Gabriel

ABSTRACT. In this paper we prove a Michael-Simon inequality in the weighted setting and using this inequality we obtain a diameter control depending of the f-mean curvature, which is based in the work of Topping.

1. INTRODUCTION

The classical Sobolev inequality, proved by S. L. Sobolev in the celebrated paper [13], states that for any smooth function on $\mathbb{R}^n$ with compact support $u \in C_0^\infty(\mathbb{R}^n)$ and $1 < p < n$ there exists a constant $S = S(p,n) > 0$ such that

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{np}} \leq S(p,n) \cdot \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{\frac{1}{p}}. \tag{1.1}$$

The Sobolev inequality (1.1) also holds for $p = 1$ by the Gagliardo-Nirenberg estimates, see [18]. Its numerous versions play a central role in mathematics, specially in the theory of partial differential equations, mathematical analysis, mathematical physics, and differential geometry as one can glimpse in the three excellent volumes, edited by Vladimir Maz'ya, [8]. The Sobolev inequality was extended to minimal hypersurfaces of $\mathbb{R}^n$ by M. Miranda [10] and refined in [2] to derive a priori gradient bounds of solutions to the minimal surface equation. The next step was taken by J. Michael and L. Simon in [9], where they proved a version of the Sobolev inequality valid for a general class of measures and subsets of $\mathbb{R}^n$ and of particular interest in the study of surfaces of prescribed mean curvature. The Sobolev inequality has been generalized to general Riemannian submanifolds by D. Hoffman and J. Spruck [5] in the following theorem.

Theorem 1.1 (Hoffman-Spruck, [5]). *Let $\varphi\colon M \to \overline{M}$ be an isometric immersion of a Riemannian m-manifold M into the Riemannian n-manifold $\overline{M}$ with mean curvature vector H and injectivity radius $\overline{R} = \inf_{p\in M} \operatorname{dist}_{\overline{M}}(p, \operatorname{cut}(\overline{M}))$. Suppose that the sectional curvature $K_{\overline{M}} \leq b$, $b \in \mathbb{R} \setminus \{0\}$. Let $h \in C^1(M)$ be a nonnegative function vanishing in ∂M. Let*

$$\rho_0 = \begin{cases} (\sqrt{b})^{-1} \sin^{-1}(\sqrt{b})(1-\alpha)^{-1/m} \left(\omega_m^{-1} \operatorname{vol}(\operatorname{supp} h)\right)^{1/m} & \text{if} \quad b > 0 \\ (1-\alpha)^{-1/m} \left(\omega_m^{-1} \operatorname{vol}(\operatorname{supp} h)\right)^{1/m} & \text{if} \quad b < 0. \end{cases}$$

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If $2\rho_0 \leq \overline{R}$ *and* $b(1-\alpha)^{-2/m}\left(\omega_m^{-1}\mathrm{vol}(\mathrm{supp}h)\right)^{2/m} \leq 1$, *then*

$$\left(\int_M h^{m/(m-1)}dv\right)^{(m-1)/m} \leq c(m)\int_M \left(|\nabla h| + h|H|\right)dv,$$

where $c(m)$ *is a constant depending on* m, ω_m *is the volume of the unit ball in* $\mathbb{R}^m$.

In this note we are concerned with establishing a weighted version of the Michael-Simon inequality. An m-dimensional weighted manifold is a triple $M_f = (M, g, dv_f)$, where (M, g) is a Riemannian manifold, $f : M \to \mathbb{R}$ is a selected smooth function on M and $dv_f = e^{-f}dv$ is the weighted measure, where dv denotes the Riemannian measure of (M, g). There has been an increasing interest in the geometric analysis on weighted manifolds and it is important to search for these results since one has the flexibility to change the measure without changing the underlying manifold. In this spirit our version of the Michael-Simon Sobolev inequality is stated in the following theorem.

Theorem 1.2 (Weighted Michael-Simon Inequality)**.** *Let* $\mathbb{R}^n_f = (\mathbb{R}^n, g, d_f v)$ *be a weighted Riemannian manifold modeled upon* $\mathbb{R}^n$ *with the Euclidean metric structure* g. *So, there exists* c_k *depending only on* k *so that if* $u > 0$ *is a Lipschitz function with compact support on a submanifold* $\Sigma^k \subset \mathbb{R}^n$ *and*

$$\int_\Sigma \langle x, \bar{\nabla} f\rangle d_f v \leq 0 \quad \text{on } \Sigma, \tag{1.2}$$

then

$$\left(\int_\Sigma u^{\frac{k}{k-1}} d_f v\right)^{\frac{k-1}{k}} \leq c_k \int_\Sigma \left[|\nabla^\Sigma u| + u|\mathbf{H}_f|\right] d_f v. \tag{1.3}$$

The proof of our result is based on the f-mean value inequality and a covering lemma also used by Michael and Simon.

As we mentioned earlier, there are several applications of the Sobolev inequality. A direct consequence is the following weighted isoperimetric inequality.

Theorem 1.3. *Let* $\Omega \subset \Sigma$ *be open with Lipschitz boundary* $\partial\Omega$. *If* (1.2) *holds, then we have the inequality*

$$V_f(\Omega)^{\frac{k-1}{k}} \leq c_k \left[V_f(\partial\Omega) + \int_\Omega |\mathbf{H}_f|\right]. \tag{1.4}$$

Another application that we would like to highlight can be found in Topping [16], where the author obtains an intrinsic diameter control depending of the mean curvature. Here, we use the weighted Sobolev inequality for generalizing this result in the weighted setting.

Theorem 1.4. *Let* $\Sigma^k \hookrightarrow \mathbb{R}^n_f$ *be a closed connected submanifold. If* (1.2) *occurs, then we have the following estimate for the intrinsic diameter* d_{int}*:*

$$d_{int} \leq C(n, f)\int_\Sigma |\mathbf{H}_f|^{k-1} d_f v. \tag{1.5}$$

Remark. We would like to mention some related works by M. Rimoldi and D. Impera [6] and M. Batista and H. Mirandola [11] where similar inequalities had been proved using another techniques and substantially distinct applications from ours were presented.

2. Preliminaries

The f-Laplacian associated with the weighted manifold M_f is the operator

$$\Delta_f u = \mathrm{div}_f(\nabla u) \colon = e^f \mathrm{div}(e^{-f}\nabla u) = \Delta u - \langle \nabla u, \nabla f\rangle,$$

which is symmetric on $L^2(M, d_f v)$. Hereafter, $B_r(p)$ and $\partial B_r(p)$ will denote the geodesic ball and sphere of (M, g) of radius $r > 0$ centered at $p \in M$, and we also set

$$\mathrm{vol}_f(B_r(p)) = \int_{B_r(p)} e^{-f} dv, \quad \mathrm{vol}_f(\partial B_r(p)) = \int_{\partial B_r(p)} e^{-f} dS,$$

where dS stands for the $(m-1)$-Hausdorff measure.

Proposition 2.1. *Let $\phi\colon \Sigma^n \to N^{n+1}$ be a hypersurface and let $\Phi\colon (-\epsilon, \epsilon)\times\Sigma \to N$ be a proper variation of ϕ with normal variational vector field V. If $\mathbf{H}$ is the mean curvature vector, then*

$$\left.\frac{d}{dt}\right|_{t=0} vol_f(\Phi(t,\Sigma)) = -\int_\Sigma \langle \bar\nabla f + nHN, V\rangle e^{-f} dv. \tag{2.1}$$

Proof. It is well known that

$$\left.\frac{d}{dt}\right|_{t=0} \int_\Sigma dv_t = -\int_\Sigma n\langle H, V\rangle dv.$$

Furthermore,

$$\frac{d}{dt}\left[e^{-f(\Phi_t)}\right] = -e^{-f}\langle \bar\nabla f, V\rangle.$$

Therefore,

$$\left.\frac{d}{dt}\right|_{t=0} vol_f(\Phi(t,\Sigma)) = -\int_\Sigma \langle \bar\nabla f + nH, V\rangle e^{-f} dv.$$

This finishes the proof of the proposition. □

Motivated by the previous proposition, we define the f-mean curvature vector field by

$$\mathbf{H}_f = \mathbf{H} + (\bar\nabla f)^\perp, \tag{2.2}$$

where $\perp$ denotes the projection on the normal bundle. Consequently, we say that an immersed submanifold is f-minimal if $\mathbf{H}_f = 0$.

3. The mean value inequality

In this section, we prove the mean value inequality in the weighted setting.

Proposition 3.1 (The mean value inequality)**.** *Let $(\mathbb{R}^n, g, d_f v)$ a weighted Riemannian manifold. If $\Sigma^k \subset \mathbb{R}^n$ is an f-minimal submanifold and u is a function on Σ, then*

$$t^{-k}\int_{B_t\cap\Sigma} u d_f v - s^{-k}\int_{B_s\cap\Sigma} u d_f v = \int_{(B_t\setminus B_s)\cap\Sigma} u\frac{|x^\perp|^2}{|x^T|^{k+2}} d_f v - A, \tag{3.1}$$

where

$$A = \int_s^t \tau^{-k-1}\left[\int_{B_\tau\cap\Sigma}\left\{u\langle\bar\nabla f, x\rangle - (\tau^2 - |x|^2)\Delta_f^\Sigma u\right\} d_f v\right] d\tau.$$

Proof. We have

$$\langle \nabla^\Sigma |x|^2, w\rangle = \langle \bar{\nabla}|x|^2, w\rangle = 2\langle x, w\rangle.$$

Therefore,

$$\nabla^\Sigma |x|^2 = 2x^T. \tag{3.2}$$

We conclude that

$$\begin{aligned}\Delta_\Sigma |x|^2 = \langle \nabla^\Sigma_{e_i}\nabla^\Sigma |x|^2, e_i\rangle &= 2\langle \nabla^\Sigma_{e_i} x^T, e_i\rangle \\ &= 2\left[\langle \nabla^\Sigma_{e_i} x, e_i\rangle - \langle \nabla^\Sigma_{e_i} x^\perp, e_i\rangle\right] \\ &= 2k + 2|x^\perp|H \\ &= 2k + 2|x^\perp|(H_f - (\bar{\nabla} f)^\perp).\end{aligned}$$

Since Σ is an f-minimal submanifold, we have

$$\Delta_\Sigma |x|^2 = 2k - 2|x^\perp|(\bar{\nabla} f)^\perp.$$

In view of this, we have

$$\Delta^\Sigma_f |x|^2 = \Delta_\Sigma |x|^2 - \langle \nabla^\Sigma |x|^2, \nabla^\Sigma f\rangle = 2k - 2|x^\perp|(\bar{\nabla} f)^\perp - 2\langle x^T, \nabla^\Sigma f\rangle.$$

Using this we obtain

$$2k\int_{B_s\cap\Sigma} u d_f v = \int_{B_s\cap\Sigma} u\Delta^\Sigma_f |x|^2 d_f v + 2\int_{B_s\cap\Sigma} u(|x^\perp|(\bar{\nabla} f)^\perp + \langle \nabla^\Sigma f, x^T\rangle) d_f v. \tag{3.3}$$

Note that, using the Stokes theorem, we get

$$\begin{aligned}\int_{B_s\cap\Sigma} u\Delta^\Sigma_f |x|^2 d_f v &= \int_{B_s\cap\Sigma} div_f(u\nabla^\Sigma |x|^2) d_f v - 2\int_{B_s\cap\Sigma} \langle \nabla^\Sigma u, \nabla^\Sigma |x|^2\rangle d_f v \\ &= \int_{\partial B_s\cap\Sigma} \langle u\nabla^\Sigma |x|^2, \nu\rangle d_f S + 2\int_{B_s\cap\Sigma} |x|^2\Delta^\Sigma_f u d_f v - 2s^2\int_{\partial B_s\cap\Sigma} \langle \nabla^\Sigma u, \nu\rangle \\ &= 2\int_{\partial B_s\cap\Sigma} u|x^T| d_f S + 2\int_{B_s\cap\Sigma} (|x|^2 - s^2)\Delta^\Sigma_f u d_f v.\end{aligned} \tag{3.4}$$

Furthermore, it follows from the co-area formula that

$$\int_{B_s\cap\Sigma} ue^{-f} dv = \int_{\{r\le s\}} \frac{ue^{-f}|\nabla^\Sigma r|}{|\nabla^\Sigma r|} dv = \int_{-\infty}^s \int_{r=\tau} \frac{ue^{-f}}{|\nabla^\Sigma r|} d\tau = \int_{-\infty}^s \int_{r=\tau} \frac{ue^{-f}|x|}{|x^T|} d\tau,$$

which implies that

$$\frac{d}{ds}\left(\int_{B_s\cap\Sigma} u d_f v\right) = s\int_{\partial B_s} \frac{u}{|x^T|} d_f v. \tag{3.5}$$

Now, using (3.3),(3.4) and (3.5), we obtain

$$\begin{aligned}\frac{d}{ds}\left(s^{-k}\int_{B_s\cap\Sigma} u d_f v\right) &= -\frac{k}{s^{k+1}}\int_{B_s\cap\Sigma} u d_f v + \frac{1}{s^k}\frac{d}{ds}\left(\int_{B_s\cap\Sigma} u d_f v\right) \\ &= -\frac{1}{s^{k+1}}\left[\int_{\partial B_s\cap\Sigma} u|x^T| d_f S + \int_{B_s\cap\Sigma} u\langle \bar{\nabla} f, x\rangle d_f v\right] \\ &\quad - \frac{1}{s^{k+1}}\int_{B_s\cap\Sigma} (|x|^2 - s^2)\Delta^\Sigma_f u d_f v + \frac{1}{s^k}\int_{\partial B_s\cap\Sigma} \frac{|x|}{|x^T|} u d_f S \\ &= \frac{1}{s^{k+1}}\int_{\partial B_s\cap\Sigma} \frac{|x^\perp|^2}{|x^T|^2} u d_f S - \frac{1}{s^{k+1}}\int_{B_s\cap\Sigma} u\langle \bar{\nabla} f, x\rangle d_f v \\ &\quad - \int_{B_s\cap\Sigma} (|x|^2 - s^2)\Delta^\Sigma_f u d_f v.\end{aligned}$$

Integrating and using the co-area formula prove the statement. □

The next lemma is an important tool that we will use in the proof of the weighted Michael-Simon Inequality.

Lemma 3.2. *Let $\Sigma^k \subset \mathbb{R}^n_f$ be a submanifold with the f-mean curvature H_f and let u be a nonnegative function on Σ. Then for $s < t$,*

(3.6)
$$t^{-k}\int_{B_t\cap\Sigma} u d_f v - s^{-k}\int_{B_s\cap\Sigma} d_f v \geq \int_s^t \tau^{-k-1}\left[\int_{B_\tau\cap\Sigma}\langle x, \nabla^\Sigma u + u(\mathbf{H_f} - \bar{\nabla} f)\rangle d_f v\right] d\tau.$$

Proof. As in the previous proof, we have

$$2k\int_{B_s\cap\Sigma} u d_f v = \int_{B_s\cap\Sigma} u\Delta_f^\Sigma |x|^2 d_f v + 2\int_{B_s\cap\Sigma} u\left[|x^\perp|((\bar{\nabla} f)^\perp - H_f) + \langle \nabla^\Sigma f, x^T\rangle\right] d_f v.$$

Therefore,

$$\begin{aligned}\frac{d}{ds}\left(s^{-k}\int_{B_s\cap\Sigma} u d_f v\right) &= -\frac{k}{s^{k+1}}\int_{B_s\cap\Sigma} u d_f v + \frac{1}{s^k}\frac{d}{ds}\left(\int_{B_s\cap\Sigma} u d_f v\right)\\ &= -\frac{1}{s^{k+1}}\int_{B_s\cap\Sigma} u\left[|x^\perp|((\bar{\nabla} f)^\perp - H_f) + \langle \nabla^\Sigma f, x^T\rangle\right] d_f v\\ &\quad + \frac{1}{s^k}\int_{\partial B_s\cap\Sigma} u\frac{|x|}{|x^T|} - \frac{1}{2s^{k+1}}\int_{B_s\cap\Sigma} u\Delta_f^\Sigma |x|^2 d_f v\end{aligned}$$

Now, using Green's Formula, we get

$$\int_{B_s\cap\Sigma} u\Delta_f^\Sigma |x|^2 d_f v = \int_{\partial B_s\cap\Sigma} u\langle \nabla^\Sigma |x|^2, \nu\rangle d_f S - \int_{B_s\cap\Sigma}\langle \nabla^\Sigma u, \nabla^\Sigma |x|^2\rangle d_f v.$$

By the previous calculus, we have

$$\frac{1}{s^k}\int_{\partial B_s\cap\Sigma} u\frac{|x|}{|x^T|} - \frac{1}{2s^{k+1}}\int_{\partial B_s\cap\Sigma} u\langle \nabla^\Sigma |x|^2, \nu\rangle d_f S \geq 0.$$

Applying this in the inequality above, we obtain

$$\frac{d}{ds}\left(s^{-k}\int_{B_s\cap\Sigma} u d_f v\right) \geq \frac{1}{s^{k+1}}\int_{B_s\cap\Sigma}\langle x, \nabla^\Sigma u + u(\mathbf{H_f} - \bar{\nabla} f)\rangle d_f v.$$

Integrating, we prove the statement. □

Corollary 3.3. *On the assumptions of the previous lemma, if*

$$\int_\Sigma \langle x, \bar{\nabla} f\rangle \leq 0 \quad \text{on } \Sigma,$$

then

$$t^{-k}\int_{B_t\cap\Sigma} u d_f v - s^{-k}\int_{B_s\cap\Sigma} d_f v \geq \int_s^t \tau^{-k-1}\left[\int_{B_\tau\cap\Sigma}\langle x, \nabla^\Sigma u + u\mathbf{H_f}\rangle d_f v\right] d\tau. \tag{3.7}$$

4. Weighted Michael-Simon Inequality

In order to prove the weighted Michael-Simon Inequality we need three lemmas. The proof of these results may be found in [3]. The first result is a covering lemma.

Lemma 4.1. *If $\mathcal{B}$ is a family of closed balls in a metric space with*

$$\sup\{\mathrm{diam}(B)|B \in \mathcal{B}\} < \infty,$$

then there is a pairwise disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ so that

$$\bigcup_{B\in\mathcal{B}} \subset \bigcup_{B\in\mathcal{B}'} 5B,$$

where $5B$ is the ball with the same center as B but with 5 times the radius.

Lemma 4.2. *Suppose $f, g \geq 0$ are bounded, nondecreasing, with*

$$1 \leq \limsup_{\sigma\to 0} \frac{f(\sigma)}{\sigma^k}, \tag{4.1}$$

and for any $0 < \sigma < \rho$,

$$\frac{f(\sigma)}{\sigma^k} \leq \frac{f(\rho)}{\rho^k} + \int_0^\rho t^{-k} g(t)dt. \tag{4.2}$$

Set $f_\infty = \lim_{t\to\infty} f(t)$ and $\rho_0 = 2(f_\infty)^{1/k}$. Then there exists $\rho \in (0, \rho_0)$ with

$$f(5\rho) \leq \frac{5^k}{2}\rho_0 g(\rho). \tag{4.3}$$

Lemma 4.3. *Suppose that ν is a measure on a space X, $f \geq 0$ is in $L^1(\nu)$, $A_t = \{x|f(x) > t\}$, and $\alpha > 0$. Then*

$$\frac{1}{\alpha}\int f^\alpha d\nu = \int_0^\infty t^{\alpha-1}\nu(A_t)dt. \tag{4.4}$$

It follows that if $t_0 \geq 0$, then

$$\frac{1}{\alpha}\int_{A_{t_0}} (f^\alpha - t_0^\alpha)d\nu = \int_{t_0}^\infty t^{\alpha-1}\nu(A_t)dt. \tag{4.5}$$

Now, we are able to prove the weighted Michael-Simon inequality. In fact, we prove the following slightly more general inequality than (1.3).

Theorem 4.4. *Let $\Sigma^k \subset \mathbb{R}^n_f$ be a submanifold. If $u > 0$ is a Lipschitz function with compact support on Σ, then there exists a constant c_k depending only on k satisfying*

$$\left(\int_\Sigma u^{\frac{k}{k-1}} d_f v\right)^{\frac{k-1}{k}} \leq c_k \int_\Sigma \left[|\nabla^\Sigma u| + u|\mathbf{H_f} - \bar{\nabla} f|\right] d_f v. \tag{4.6}$$

Remark 4.5. The particular form stated in Theorem 1.3 may be proved by minor modification in the proof of the general case using hypothesis (1.2). In this case, combining (4.6) and Corollary 3.3 yield (1.3).

Proof. Let $\Sigma \subset \mathbb{R}^n$ be a submanifold and define $J(r)$ by

$$J(r) = \frac{1}{\omega_k r^k}\int_{B_r\cap\Sigma} h d_f v, \tag{4.7}$$

where $\omega_k = \int_{B_1} d_f v$ and h is a nonnegative function with compact support. Therefore, using the mean value inequality in Lemma 3.2 and the Cauchy-Schwarz inequality, we obtain

$$r^{k+1}J'(r) \geq \frac{1}{\omega_k}\int_{B_r\cap\Sigma} \langle x, \nabla^\Sigma h + h(\mathbf{H_f} - \bar{\nabla} f)\rangle d_f v \geq -\frac{r}{\omega_k}\int_{B_r\cap\Sigma} |\nabla^\Sigma h| + h|\mathbf{H_f} - \bar{\nabla} f| d_f v. \tag{4.8}$$

Integrating on $0 < \sigma < \rho$, we get

$$J(\sigma) \leq J(\rho) + \int_\sigma^\rho \left(\frac{r^{-k}}{\omega_k} \int_{B_r \cap \Sigma} |\nabla^\Sigma h| + h|\mathbf{H_f} - \bar{\nabla} f| d_f v \right) dr. \tag{4.9}$$

Now, set

$$\phi(\rho) = \frac{1}{\omega_k} \int_{B_\rho \cap \Sigma} h d_f v \quad \text{and} \quad \psi(\rho) = \frac{1}{\omega_k} \int_{B_\rho \cap \Sigma} \left[|\nabla^\Sigma h| + h|\mathbf{H_f} - \bar{\nabla} f| \right] d_f v.$$

If $y \in \Sigma$ is a point with $h(y) \geq 1$, then using (4.9) we have that our choice of ϕ and ψ implies that (4.1) and (4.2) hold. Furthermore,

$$\rho_0 = 2\phi_\infty^{1/k},$$

where

$$\phi_\infty = \lim_{\rho \to \infty} \phi(\rho) = \frac{1}{\omega_k} \int_\Sigma h d_f v < \infty,$$

because h has compact support. So, applying Lemma 4.2 we have that there exists $\rho_y \in (0, \rho_0)$ such that

$$\phi(5\rho_y) \leq \frac{5^k}{2} \rho_0 \psi(\rho_y),$$

i.e,

$$\int_{B_{5\rho_y} \cap \Sigma} h d_f v \leq \frac{5^k}{w_k^{1/k}} \left(\int_\Sigma h d_f v \right)^{1/k} \int_{B_\rho \cap \Sigma} \left[|\nabla^\Sigma h| + h|\mathbf{H_f} - \bar{\nabla} f| \right] d_f v. \tag{4.10}$$

Let $S \subset \Sigma$ be the set where $h \geq 1$, that is,

$$S = \{ y \in \Sigma | h(y) \geq 1 \}.$$

Using Covering Lemma 4.1, we can choose disjoint balls B^j satisfying (4.10) so that

$$S \subset \bigcup_j 5(B^j). \tag{4.11}$$

It follows that

$$\int_S h d_f v \leq \frac{5^k}{w_k^{1/k}} \left(\int_\Sigma h d_f v \right)^{1/k} \int_\Sigma \left[|\nabla^\Sigma h| + h|\mathbf{H_f} - \bar{\nabla} f| \right] d_f v. \tag{4.12}$$

Fix some $\epsilon > 0$ and a monotone function $\gamma(t)$ with

$$\gamma(t) = \begin{cases} 1 & \text{if } t \geq \epsilon, \\ 0 & \text{if } t \leq 0. \end{cases}$$

Given any $t > 0$, we set

$$A_t = \{ x \in \Sigma | u(x) > t \}. \tag{4.13}$$

Let μ be the volume measure on Σ. Using $h(x) = \gamma(u(x) - t)$ in (4.12), we have (4.14)

$$\begin{aligned} \mu(A_{t+\epsilon}) &\leq \frac{5^k}{w_k^{1/k}} [(t+\epsilon)^{\frac{k}{k-1}} \mu(A_t)]^{1/k} \int_\Sigma \left[|\nabla^\Sigma u| \gamma'(u-t) + \gamma(u-t)|\mathbf{H_f} - \bar{\nabla} f| \right] \\ &\leq \frac{5^k}{w_k^{1/k}} \left(\int_\Sigma (u+\epsilon)^{\frac{k}{k-1}} \right)^{1/k} \int_\Sigma |\nabla^\Sigma u| \gamma'(u-t) + \int_{A_t} |\mathbf{H_f} - \bar{\nabla} f|. \end{aligned}$$

Multiplying by $(t+\epsilon)^{\frac{1}{k-1}}$ gives

$$(t+\epsilon)^{\frac{1}{k-1}}\mu(A_{t+\epsilon}) \leq \frac{5^k}{w_k^{1/k}}(\mu(A_t))^{1/k}\int_\Sigma \left[|\nabla^\Sigma u|\gamma'(u-t)+\gamma(u-t)|\mathbf{H_f}-\bar{\nabla} f|\right].$$

Integrating the inequality above in t and using Lemma 4.3 with $\nu=\mu$, $t_0=\epsilon$, and $\alpha-1=\frac{1}{k-1}$, so that $\alpha=\frac{k}{k-1}$ in the first term gives

$$\begin{aligned}\frac{k-1}{k}\int_{A_\epsilon}(u^{\frac{k}{k-1}}-\epsilon^{\frac{k}{k-1}}) &\leq \frac{5^k}{w_k^{1/k}}\left(\int_\Sigma (u+\epsilon)^{\frac{k}{k-1}}\right)^{1/k}\\ &\quad\cdot\int_0^\infty\left(\int_\Sigma |\nabla^\Sigma u|\gamma'(u-t)+\int_{A_t}|\mathbf{H_f}-\bar{\nabla} f|\right)\\ &= \frac{5^k}{w_k^{1/k}}\left(\int_\Sigma (u+\epsilon)^{\frac{k}{k-1}}\right)^{1/k}\int_\Sigma(|\nabla^\Sigma u|+u|\mathbf{H_f}-\bar{\nabla} f|),\end{aligned}$$

where the equality used the fundamental theorem of calculus to evaluate the γ' term and Lemma 4.3 with $\nu=|\mathbf{H_f}-\bar{\nabla} f|\mu$, $t_0=0$ and $\alpha=1$ to evaluate the last term. Letting $\epsilon\to 0$, we obtain the estimate with

$$c_k=\frac{k}{k-1}\frac{5^k}{w_k^{1/k}}. \qquad \square$$

The other associated inequality comes from Theorem 1.2 and Hölder's inequality as we will see next.

Corollary 4.6. *There exists c_k depending only on k so that if $u\geq 0$ is a Lipschitz function with compact support on a submanifold $\Sigma^k\subset\mathbb{R}^n_f$ and $p\in[1,k)$, then*

$$\|u\|_{L^{p^*}}\leq c_k\left(\frac{(k-1)p}{k-p}\|\nabla^\Sigma u\|_{L^p}+\|u(\mathbf{H_f}-\bar{\nabla} f)\|_{L^p}\right), \tag{4.15}$$

where $p^*=\frac{kp}{k-p}$.

The proof follows as in Corollary 3.12 in [3].

As an immediate consequence of the Sobolev inequality we have the following general form of the isoperimetric inequality.

Proposition 4.7. *Let $\Omega\subset\Sigma$ be open with Lipschitz boundary $\partial\Omega$. Then we have the inequality*

$$V_f(\Omega)^{\frac{k-1}{k}}\leq c_k\left[V_f(\partial\Omega)+\int_\Omega|\mathbf{H_f}-\bar{\nabla} f|\right]. \tag{4.16}$$

To see that the isoperimetric inequality follows the weighted Michael-Simon Inequality, we take a suitable sequence of nonnegative C^1_c-functions on Σ which approximates the characteristic function of an open set $\Omega\subset\Sigma$ and apply the Sobolev Inequality for them.

5. Diameter control

To finish this work, we present one application of the weighted Michael-Simon inequality. We prove an estimate for the diameter depending on the f-mean curvature. This section is based on the work of Topping [16].

Lemma 5.1. *Let $\Sigma^k \hookrightarrow \mathbb{R}^n_f$ be a submanifold. If (1.2) occurs, then there exists a constant $\delta(n,f) > 0$ such that for any $x \in \Sigma$ and $R > 0$, at least one of the following two alternatives occurs:*

(1) $M_f(x,R) > \delta$;
(2) $\kappa_f(x,R) > \delta$,

where

$$M_f(x,R) := \sup_{r\in(0,R]} r^{-\frac{1}{k-1}}[V_f(x,r)]^{-\frac{k-2}{k-1}} \int_{B_r(x)} |\mathbf{H_f}| d_f v$$

is the maximal function, and

$$\kappa_f(x,R) := \inf_{(0,R]} \frac{V_f(x,r)}{r^k}.$$

is the measure collapsedness rate.

Proof. Given a $\delta > 0$ to be chosen later, suppose that $M_f(x,R) \le \delta$. Then, for all $r \in (0,R]$, we have

$$\int_{B_r(x)} |\mathbf{H_f}| d_f v \le \delta r^{\frac{1}{n-1}}[V_f(x,r)]^{\frac{n-2}{n-1}}. \tag{5.1}$$

Given a fixed x, let $V_f(r) := V_f(x,r)$. So, as the volume without weight is differentiable, whe have that $V_f(v)$ is differenciable a.e.w. $r > 0$. For such $r \in (0,R]$, and any $\lambda > 0$, define a Lipschitz cut-off function $u\colon \Sigma \to \mathbb{R}$ by

$$u(y) = \begin{cases} 1, & y \in B_r(x), \\ 1 - \frac{1}{\lambda}(dist_\Sigma(x,y) - r), & y \in B_{r+\lambda}(x) \setminus B_r(x), \\ 0 & y \notin B_{r+\lambda}(x). \end{cases}$$

Now, applying the weighted Michel-Simon inequality, we get

$$\begin{aligned} V_f(r)^{\frac{k-1}{k}} &\le \left(\int_{B_{r+\lambda}} u^{\frac{k}{k-1}} d_f v \right)^{\frac{k-1}{k}} \\ &\le \frac{c_k}{\lambda}(V_f(r+\lambda) - V_f(r)) + c_k \|\mathbf{H_f}\|_{L^1(B_{r+\lambda}(x))}. \end{aligned}$$

Letting $\lambda \to 0$, we have

$$V_f(r)^{\frac{k-1}{k}} \le c_k \frac{dV_f}{dr} + c_k \|\mathbf{H_f}\|_{L^1(B_r(x))}.$$

Using (5.1), we obtain

$$c_k \frac{dV_f}{dr} + c_k \delta r^{\frac{1}{k-1}}[V_f(x,r)]^{\frac{k-2}{k-1}} - V_f(r)^{\frac{k-1}{k}} \ge 0. \tag{5.2}$$

Now, we choose $\delta > 0$ that to be sufficiently small so that $\delta < Cw_k$, where $C = \inf_{B_r}\{e^{-f}\}$ and w_k is the volume of the unit ball in $\mathbb{R}^k$, and define the function $v(r) := \delta r^k$. Computing as in Topping [16], for δ sufficiently small, we have

$$c_k \frac{dv_f}{dr} + c_k \delta r^{\frac{1}{k-1}}[v(r)]^{\frac{k-2}{k-1}} - v(r)^{\frac{k-1}{k}} \le 0. \tag{5.3}$$

Then, using (5.1), (5.3) and the fact that

$$\frac{V_f(r)}{r^k} \ge \frac{Cw_k V_f(r)}{V_f(B_r)} \to Cw_k$$

as $r \to 0$, while $v(r)/r^k = \delta < Cw_k$, we deduce that $V_f(r) > v(r)$ for all $r \in (0, R]$ and hence

$$\kappa_f(x, R) = \inf_{r \in (0,R]} \frac{V_f(x, r)}{r^k} > \delta,$$

as desired. □

In possession of the previous lemma we are able to prove Theorem 1.4.

Proof of Theorem 1.4. Choose $R > 0$ such that $V(\Sigma) < \delta R^k$, where $V(\Sigma)$ is the total volume of Σ and δ is given by Lemma 5.1. In particular, for all $z \in \Sigma$, we have

$$\kappa_f(z, R) \leq \frac{V_f(z, R)}{R^k} \leq \delta.$$

So, by Lemma 5.1, we must have that $M_f(z, R) > \delta$. Therefore, by definition of the maximal function and the Holder inequality, we have that there exists $r = r(z)$ such that

$$\begin{aligned} \delta &< r^{-\frac{1}{k-1}} V_f(z, r)^{-\frac{n-2}{n-1}} \int_{B_r(z)} |\mathbf{H_f}| d_f v \\ &\leq r^{-\frac{1}{k-1}} \left(\int_{B_r(z)} |\mathbf{H_f}|^{k-1} d_f v \right)^{\frac{1}{k-1}}, \end{aligned} \tag{5.4}$$

and hence

$$r(z) \leq \delta^{1-k} \int_{B_{r(z)}(z)} |\mathbf{H_f}|^{k-1} d_f v. \tag{5.5}$$

Now let $x, y \in \Sigma$ be any two extremal points, by which we mean that $d_{int} = dist_\Sigma(x, y)$, and let $\gamma \subset \Sigma$ be any shortest geodesic connecting x and y. The set of balls $B_{r(z)}(z)$ with $z \in \gamma$ is clearly a covering of γ. Then, using the same covering argument as in Topping [16], for any $\lambda \in (0, \frac{1}{2})$, we can find a sequence of points $\{z_i\} \subset \gamma$ such that the balls $\{B_{r(z_i)}(z_i)\}$ are disjoint and cover at least a fraction λ of the whole of Σ:

$$\lambda d_{int} \leq \sum_i 2r(z_i).$$

Joining this with (5.5), we obtain

$$\begin{aligned} d_{int} &\leq \frac{2}{\lambda} \sum_i r(z_i) \leq \frac{2}{\lambda} \delta^{1-k} \sum_i \int_{B_{r(z_i)}(z_i)} |\mathbf{H_f}|^{k-1} d_f v \\ &\leq \frac{2}{\lambda} \delta^{1-k} \int_\Sigma |\mathbf{H_f}|^{k-1} d_f v. \end{aligned} \tag{5.6}$$

This finishes the proof of Theorem 1.4. □

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