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# $G$-COMPLETE REDUCIBILITY IN NON-CONNECTED GROUPS

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Abstract. In this paper we present an algorithm for determining whether a subgroup $H$ of a non-connected reductive group $G$ is $G$-completely reducible. The algorithm consists of a series of reductions; at each step, we perform operations involving connected groups, such as checking whether a certain subgroup of $G^0$ is $G^0$-cr. This essentially reduces the problem of determining $G$-complete reducibility to the connected case.

## 1. Introduction

Let $G$ be a connected reductive linear algebraic group over an algebraically closed field of characteristic $p \geq 0$. Following Serre [14], we say a subgroup $H$ of $G$ is *$G$-completely reducible* ($G$-cr) if whenever $H$ is contained in a parabolic subgroup $P$ of $G$, $H$ is contained in some Levi subgroup of $P$. The definition extends to non-connected reductive $G$ as well: one replaces parabolic and Levi subgroups with so-called Richardson parabolic and Richardson Levi subgroups, respectively (see [2], [13], and Section 2).

Even if one is interested mainly in connected reductive groups, one must sometimes consider non-connected groups. For instance, natural subgroups of a connected group, such as normalizers and centralizers, are often non-connected. The notion of $G$-complete reducibility is much better understood in the connected case; e.g., see [1], [10], and [11]. In this paper we present an algorithm for determining whether a subgroup $H$ of a non-connected reductive group $G$ is $G$-cr. The algorithm consists of a series of reductions; at each step, we perform operations involving connected groups, such as checking whether a certain subgroup of $G^0$ is $G^0$-cr. This essentially reduces the problem of determining $G$-complete reducibility to the connected case.

An important special case of the general problem described above is the following. Let $H$ be a subgroup of $G$. We say *$H$ acts on $G^0$ by outer automorphisms* if for each $1 \neq h \in H$, conjugation by $h$ gives a non-inner automorphism of $G^0$. In this case, we may identify $H$ with a subgroup of $\mathrm{Out}(G^0)$. Now suppose also that $G^0$ is simple; then $H$ is cyclic except for possibly when $G^0$ is of type $D_4$. It is convenient when studying conjugacy classes in $G^0$ to determine the fixed point set of a non-inner automorphism. See, e.g., [12, Lem. 2.9] when $H$ is cyclic and semisimple (that is, $H$ is of order coprime to $p$); note that if $H$ is generated by a semisimple element, then $H$ is $G$-cr by Theorem 2.6, as semisimple conjugacy classes are closed. On the

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other hand, if $H$ is cyclic and unipotent (that is, $H$ is a $p$-group), then $H$ can be $G$-cr or non-$G$-cr.

We prove the following result, which gives a criterion for $G$-complete reducibility of $H$. It is an ingredient in our algorithm. In case $H$ is cyclic, this follows from a recent result due to Guralnick and Malle, cf. Theorem 3.1 and Corollary 3.2, which is valid without the simplicity assumption on $G^0$.

**Theorem** (Corollary 4.5)**.** *Suppose $G^0$ is simple and $H$ acts on $G^0$ by outer automorphisms. Then $H$ is $G$-completely reducible if and only if $C_{G^0}(H)$ is reductive.*

Our work fits into a study begun in our earlier papers [2], [3]. It was shown in [2, Thm. 3.10] that if $H$ is a $G$-cr subgroup of $G$ and $N$ is a normal subgroup of $H$, then $N$ is also $G$-cr. In [3] we considered a complementary question: if $H$ is a subgroup of $G$, $N$ is a normal subgroup of $H$ and $N$ is $G$-cr, then under what hypotheses is $H$ also $G$-cr? We gave an example (due to Liebeck) with $H$ of the form $M \times N$, where $M$ and $N$ are both $G$-cr but $H$ is not [3, Ex. 5.3]. We also showed this kind of pathological behaviour does not happen when $G$ is connected and $p$ is good for $G$ [3, Thm. 1.3]. Here we study the above question in the case when $N$ is the normal subgroup $H \cap G^0$ of $H$ (see the algorithm in Theorem 5.3).

## 2. Preliminaries

2.1. **Notation.** Throughout, we work over an algebraically closed field $k$ of characteristic $p \geq 0$; we let $k^*$ denote the multiplicative group of $k$. Let $H$ be a linear algebraic group. By a subgroup of $H$ we mean a closed subgroup. In particular, the (topologically) cyclic subgroup generated by $h$ in $H$ is the closure of the subgroup generated by $h$. We let $Z(H)$ denote the centre of $H$ and $H^0$ the connected component of $H$ that contains 1. For $h \in H$, we let $\mathrm{Int}_h$ denote the automorphism of $H$ given by conjugation with $h$. Frequently, we abbreviate $\mathrm{Int}_h(g)$ by $h \cdot g$. If $S$ is a subset of $H$ and $K$ is a subgroup of $H$, then $C_K(S)$ denotes the centralizer of $S$ in $K$ and $N_K(S)$ the normalizer of $S$ in $K$. Likewise, if $S$ is a group of algebraic automorphisms of $H$, then we denote the fixed point subgroup of $S$ in $H$ by $C_H(S)$. For a subgroup $K$ of $H$, we denote the commutator subgroup of $K$ by $[K, K]$. If $H$ acts on a set $X$, then we also write $C_H(x)$ for the stabilizer of a point $x \in X$ in $H$.

For the set of cocharacters (one-parameter subgroups) of $H$ we write $Y(H)$; the elements of $Y(H)$ are the homomorphisms from $k^*$ to $H$.

The *unipotent radical* of $H$ is denoted $R_u(H)$; it is the maximal connected normal unipotent subgroup of $H$. The algebraic group $H$ is called *reductive* if $R_u(H) = \{1\}$; note that we do not insist that a reductive group is connected. In particular, $H$ is reductive if it is simple as an algebraic group. Here, $H$ is said to be *simple* if $H$ is connected and all proper normal subgroups of $H$ are finite. The algebraic group $H$ is called *linearly reductive* if all rational representations of $H$ are semisimple.

Throughout the paper $G$ denotes a reductive algebraic group, possibly non-connected.

**Definition 2.1.** Let $H \subseteq G$ be a subgroup. We say that $H$ *acts on $G^0$ by outer automorphisms* if for every $1 \neq h \in H$, the automorphism $\mathrm{Int}_h|_{G^0}$ of $G^0$ is non-inner, i.e., is not given by conjugation with an element of $G^0$. This is equivalent to the condition that $H$ maps bijectively onto its image under the natural map $G \to \mathrm{Aut}(G^0) \to \mathrm{Out}(G^0)$.

2.2. **$G$-complete reducibility.** In [2, §6], Serre's original notion of $G$-complete reducibility is extended to include the case when $G$ is reductive but not necessarily connected (so that $G^0$ is a connected reductive group). The crucial ingredient of this extension is the use of so-called *Richardson-parabolic subgroups* (*R-parabolic subgroups*) of $G$. We briefly recall the main definitions here; for more details on this formalism, see [2, §6].

For a cocharacter $\lambda \in Y(G)$, the *R-parabolic subgroup* corresponding to $\lambda$ is defined by

$$P_\lambda := \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}.$$

Here, for a morphism of algebraic varieties $\phi : k^* \to X$, we say that $\lim_{a\to 0} \phi(a)$ exists provided that $\phi$ extends to a morphism $\widehat{\phi} : k \to X$; in this case we set $\lim_{a\to 0} \phi(a) = \widehat{\phi}(0)$. Then $P_\lambda$ admits a Levi decomposition $P_\lambda = R_u(P_\lambda) \rtimes L_\lambda$, where

$$L_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} = g\} = C_G(\lambda(k^*)).$$

We call $L_\lambda$ an *R-Levi subgroup* of $P_\lambda$. For an R-parabolic subgroup $P$ of $G$, the different R-Levi subgroups of $P$ correspond in this way to different choices of $\lambda \in Y(G)$ such that $P = P_\lambda$; moreover, the R-Levi subgroups of $P$ are all conjugate under the action of $R_u(P)$. An R-parabolic subgroup $P$ is a parabolic subgroup in the sense that $G/P$ is a complete variety; the converse is true when $G$ is connected, but not in general ([13, Rem. 5.3]).

*Remark* 2.2. For a subgroup $H$ of $G$, there is a natural inclusion $Y(H) \subseteq Y(G)$. If $\lambda \in Y(H)$, and $H$ is reductive, we can therefore associate to $\lambda$ an R-parabolic subgroup of $H$ as well as an R-parabolic subgroup of $G$. To avoid confusion, we reserve the notation $P_\lambda$ for R-parabolic subgroups of $G$ and distinguish the R-parabolic subgroups of $H$ by writing $P_\lambda(H)$ for $\lambda \in Y(H)$. The notation $L_\lambda(H)$ has the obvious meaning. Note that $P_\lambda(H) = P_\lambda \cap H$ and $L_\lambda(H) = L_\lambda \cap H$ for $\lambda \in Y(H)$. In particular, $P_\lambda^0 = P_\lambda(G^0)$ and $L_\lambda^0 = L_\lambda(G^0)$. If $\lambda \in Y(H)$, then the R-Levi subgroups of $P_\lambda(H)$ are the $R_u(P_\lambda(H))$-conjugates of $L_\lambda(H)$; in particular, any R-Levi subgroup of $P_\lambda(H)$ is of the form $L \cap H$ for some R-Levi subgroup $L$ of $P_\lambda$.

For later use, we record the following way to construct R-Levi subgroups.

**Lemma 2.3.** *Let $P$ be an R-parabolic subgroup of $G$, and let $M$ be a Levi subgroup of $P^0$. Then $N_P(M)$ is an R-Levi subgroup of $P$.*

*Proof.* We may choose $\lambda \in Y(G)$ such that $P = P_\lambda$, $P^0 = P_\lambda(G^0)$ and $M = L_\lambda(G^0) = L_\lambda^0$. We have the Levi decomposition $P = R_u(P_\lambda) \rtimes L_\lambda = R_u(P_\lambda^0) \rtimes L_\lambda$. Since $L_\lambda \subseteq N_P(L_\lambda^0)$ and $R_u(P_\lambda^0) \cap N_P(L_\lambda^0) = 1$ (as $R_u(P_\lambda^0)$ acts simply transitively on the set of Levi subgroups of $P_\lambda^0$), we conclude that $N_P(M) = N_P(L_\lambda^0) = L_\lambda$. $\square$

**Definition 2.4.** Suppose $H$ is a subgroup of $G$. We say $H$ is *$G$-completely reducible* ($G$-cr for short) if whenever $H$ is contained in an R-parabolic subgroup $P$ of $G$, there exists an R-Levi subgroup $L$ of $P$ with $H \subseteq L$.

Since all parabolic subgroups (respectively all Levi subgroups of parabolic subgroups) of a connected reductive group are R-parabolic subgroups (respectively R-Levi subgroups of R-parabolic subgroups), Definition 2.4 coincides with Serre's original definition for connected groups [15].

The following consequence of Lemma 2.3 gives a converse of [4, Lem. 5.1] in the case of the normal subgroup $G^0$ of $G$.

**Corollary 2.5.** *Let $H$ be a subgroup of $G$. Suppose that whenever $H$ normalizes $P^0$ for an R-parabolic subgroup $P$ of $G$, then $H$ also normalizes a Levi subgroup of $P^0$. Then $H$ is $G$-completely reducible.*

*Proof.* Let $P$ be an R-parabolic subgroup of $G$ containing $H$. Then $H \subseteq N_G(P^0)$. Thus, by our assumption, there is a Levi subgroup $M$ of $P^0$ which is normalized by $H$. Consequently, $H \subseteq P \cap N_G(M) = N_P(M)$, which is an R-Levi subgroup of $P$, by Lemma 2.3. □

Let $H$ be a subgroup of $G$ and let $G \hookrightarrow \mathrm{GL}_m$ be an embedding of algebraic groups. Let $\mathbf{h} \in H^n$ be a tuple of generators of the associative subalgebra of $\mathrm{Mat}_m$ spanned by $H$ (such a tuple exists for $n$ sufficiently large). Then $\mathbf{h}$ is called a *generic tuple* of $H$; see [5, Def. 5.4]. We recall the following geometric criterion for $G$-complete reducibility [5, Thm. 5.8]; it provides a link between the theory of $G$-complete reducibility and the geometric invariant theory of reductive groups.

**Theorem 2.6.** *Let $H$ be a subgroup of $G$ and let $\mathbf{h} \in H^n$ be a generic tuple for $H$. Then $H$ is $G$-completely reducible if and only if the orbit $G \cdot \mathbf{h}$ under simultaneous conjugation is closed in $G^n$. In particular, if $H = \overline{\langle h \rangle}$ is a cyclic subgroup of $G$, then $H$ is $G$-completely reducible if and only if the conjugacy class $G \cdot h \subseteq G$ is closed.*

The following result has been proved with methods from geometric invariant theory (see [5, Def. 5.17]):

**Theorem 2.7.** *Assume that the subgroup $H$ of $G$ is not $G$-completely reducible. Then there exists an R-parabolic subgroup $P$ of $G$ with the following two properties:*

(i) *$H$ is not contained in any R-Levi subgroup of $P$,*
(ii) *$N_G(H) \subseteq P$.*

The geometric construction of $P$ in [5, §4] is roughly as follows: there is a class of so-called *optimal destabilizing cocharacters* $\Omega \subseteq Y(G)$ such that if $\lambda \in \Omega$, then $P := P_\lambda$ has properties (i) and (ii) as in Theorem 2.7. We call such an R-parabolic subgroup $P$ of $G$ an *optimal destabilizing R-parabolic subgroup for $H$*.

2.3. **Criteria for $G$-complete reducibility.** In this subsection we study criteria for $G$-complete reducibility in terms of some smaller group.

We first recall the following results about normal subgroups (cf. [3, Thm. 3.4, Cor. 3.7(ii)]).

**Theorem 2.8.** *Suppose that $N \subseteq H$ are subgroups of $G$ and $N$ is normal in $H$.*

(i) *If $N$ is also normal in $G$, then $H$ is $G$-completely reducible if and only if $H/N$ is $G/N$-completely reducible.*
(ii) *If $H/N$ is linearly reductive, then $H$ is $G$-completely reducible if and only if $N$ is $G$-completely reducible.*

A homomorphism $\pi : G_1 \to G_2$ is called *non-degenerate* provided that $\ker(\pi)^0$ is a torus. The next result is contained in [2, Lem. 2.12 and §6]:

**Lemma 2.9.** *Let $\pi : G_1 \to G_2$ be a non-degenerate epimorphism of reductive groups. Let $H \subseteq G_1$ be a subgroup. Then $H$ is $G_1$-completely reducible if and only if $\pi(H)$ is $G_2$-completely reducible.*

As an immediate consequence, we obtain the following result which allows us to focus on the part of $G$ that is effectively acting on $G^0$. Note that $C_G(G^0)^0 \subseteq Z(G^0)^0$ is a torus.

**Corollary 2.10.** *Let $H \subseteq G$ be a subgroup. Let $\pi : G \to G/C_G(G^0)$ be the natural projection. Then $H$ is $G$-completely reducible if and only if $\pi(H)$ is $\pi(G)$-completely reducible.*

The following lemma gives two necessary conditions for a subgroup of $G$ to be $G$-completely reducible, both of which can be checked in the connected group $G^0$.

**Lemma 2.11.** *Let $H$ be a $G$-completely reducible subgroup of $G$. Then the following hold:*

(i) *$H \cap G^0$ is $G^0$-completely reducible;*
(ii) *$C_{G^0}(H)$ is $G^0$-completely reducible.*

*Proof.* (i). This is the content of [2, Lem. 6.10 (ii)]. (ii). Since $H$ is $G$-cr, so is its centralizer $C_G(H)$, by [2, Cor. 3.17 and §6]. Now the result follows from part (i). □

Under the assumption that assertion (i) or (ii) of Lemma 2.11 holds, the next two lemmas allow us to replace the ambient group $G$ with a potentially smaller subgroup $M$.

**Lemma 2.12.** *Let $H$ be a subgroup of $G$ and suppose that $H \cap G^0$ is $G^0$-completely reducible. Then $M = HC_{G^0}(H \cap G^0)$ is reductive. Moreover, $H$ is $G$-completely reducible if and only if it is $M$-completely reducible.*

*Proof.* First note that $M$ is a subgroup of $G$, since $H$ normalizes $C_{G^0}(H \cap G^0)$. As $C_{G^0}(H \cap G^0)$ is a $G^0$-cr subgroup (Lemma 2.11(ii)), it is reductive, by [14, Property 4]. The same is true for $H \cap G^0$ by assumption. Hence $(H \cap G^0)C_{G^0}(H \cap G^0)$ is the product of two reductive groups and thus is reductive. As this group contains $M^0$ as a normal subgroup, the group $M$ is reductive as well.

Now first suppose that $H$ is not $G$-cr. Let $\lambda \in Y(G) = Y(G^0)$ be an optimal destabilizing cocharacter for $H$ in $G$. Then $P_\lambda(G^0)$ contains the subgroup $H \cap G^0$, which is $G^0$-cr by assumption. Hence after replacing $\lambda$ with an $R_u(P_\lambda)$-conjugate, we may assume that $\lambda$ centralizes $H \cap G^0$. This implies that $\lambda \in Y(M)$ and $H \subseteq P_\lambda(M) \subseteq P_\lambda$. Since $H$ is not contained in an R-Levi subgroup of $P_\lambda$ (cf. Theorem 2.7), it is not contained in an R-Levi subgroup of $P_\lambda(M)$. We conclude that $H$ is not $M$-cr.

Conversely, suppose that $H$ is $G$-cr. Let $\lambda \in Y(M)$ and suppose that $H \subseteq P_\lambda(M)$. To show that $H$ is $M$-cr we need to show that $H$ is contained in an R-Levi subgroup of $P_\lambda(M)$. Since $H$ is $G$-cr, we can find a cocharacter $\mu \in Y(G)$ such that $H \subseteq L_\mu \subseteq P_\mu = P_\lambda$. In particular, $\mu$ centralizes $H \cap G^0$, so that $\mu \in Y(M)$. Hence $H \subseteq L_\mu(M) \subseteq P_\mu(M) = P_\lambda(M)$, as required. □

*Remark* 2.13. In the situation of Lemma 2.12, the subgroup $N = H \cap G^0$ of $H$ is normal in $M$. Thus, by Theorem 2.8(i), we deduce that $H$ is $G$-completely reducible if and only if $H/(H \cap G^0)$ is $M/(H \cap G^0)$-completely reducible.

**Lemma 2.14.** *Let $H$ be a subgroup of $G$ and suppose that $C_{G^0}(H)$ is $G^0$-completely reducible. Then $M = HC_{G^0}(C_{G^0}(H))$ is reductive. Moreover, $H$ is $G$-completely reducible if and only if it is $M$-completely reducible.*

*Proof.* We proceed as in the proof of Lemma 2.12: Again, $M$ is a subgroup since $H$ normalizes $C_{G^0}(C_{G^0}(H))$. Moreover, $M^0 \subseteq (H \cap G^0)C_{G^0}(C_{G^0}(H)) = C_{G^0}(C_{G^0}(H)) \subseteq M$, yielding $M^0 = (C_{G^0}(C_{G^0}(H)))^0$. Since $C_{G^0}(H)$ is $G^0$-cr by assumption, as before we may conclude that its centralizer is reductive, so that $M$ is reductive.

Suppose that $H$ is not $G$-cr. Let $\lambda \in Y(G) = Y(G^0)$ be an optimal destabilizing cocharacter for $H$ in $G$. By Theorem 2.7(ii), $P_\lambda$ contains $C_{G^0}(H)$, which is $G^0$-cr. Thus we may again assume that $\lambda$ centralizes $C_{G^0}(H)$, so that $\lambda \in Y(M)$. As before, we conclude that $H$ is not $M$-cr.

Conversely, suppose that $H$ is $G$-cr. Let $\lambda \in Y(M)$ such that $H \subseteq P_\lambda(M)$. As $\lambda$ evaluates in $M^0$, it centralizes $C_{G^0}(H)$, so that $C_{G^0}(H)$ is contained in $P_\lambda$. On the other hand, since $H$ is $G$-cr, we may find $\mu \in Y(G)$ such that $P_\mu = P_\lambda$ and such that $\mu \in Y(C_{G^0}(H))$. But then $P_\mu(C_{G^0}(H)) = C_{G^0}(H)$, which forces $\mu$ to centralize $C_{G^0}(H)$ [2, Lem. 2.4]. So $\mu \in Y(M)$, and $H \subseteq L_\mu(M) \subseteq P_\mu(M) = P_\lambda(M)$. As before, this shows that $H$ is $M$-cr. □

*Remark* 2.15. Let $H$ be a subgroup of $G$ and let $\pi\colon G \to G'$ be an isogeny. Then $\pi(C_{G^0}(H)^0) = C_{G'^0}(\pi(H))^0$ (see the proof of [4, Lem. 3.1]), so $C_{G'^0}(\pi(H))^0$ is reductive if and only if $C_{G^0}(H)^0$ is.

We may write the connected reductive group $G^0$ in the form

$$G^0 = SG_1 \cdots G_n, \tag{2.16}$$

where $S$ is the radical of $G^0$ and $G_1, \ldots, G_n$ are the simple components of the derived group of $G^0$. Any subgroup $H$ of $G$ acts via conjugation on the derived subgroup of $G^0$ and hence permutes the simple components. We obtain an induced action of $H$ on the set of indices $\{1, \ldots, n\}$. For $1 \le i \le n$, we use the shorthand

$$\widehat{G}_i = S \prod_{j \neq i} G_j$$

for the product of all factors in $G^0$ above with the exception of $G_i$.

Our next lemma allows us to replace $G$ with a collection of reductive groups whose identity components are simple.

**Lemma 2.17.** *Let $H$ be a subgroup of $G$. For $1 \le i \le n$, let $H_i := N_H(G_i)$ and let $\pi_i : H_iG^0 \to H_iG^0/\widehat{G}_i$ be the natural projection. Let $I \subseteq \{1, \ldots, n\}$ be a subset meeting each $H$-orbit. Then $H$ is $G$-completely reducible if and only if $\pi_i(H_i)$ is $\pi_i(H_iG^0)$-completely reducible for each $i \in I$.*

*Proof.* First note that, by construction, $H_i$ and $G^0$ both normalize the group $\widehat{G}_i$. Hence the map $\pi_i$ is well-defined. Since $H_iG^0$ is reductive, so is its image under $\pi_i$.

To prove the forward implication, suppose the assertion fails for some $i \in I$. Up to reordering the indices, we may assume that $\pi_1(H_1)$ is not $\pi_1(H_1G^0)$-cr and that $H$ acts transitively on the set $\{1, \ldots, r\}$ for some $r \ge 1$. Let $Q$ be an optimal destabilizing R-parabolic subgroup of $\pi_1(H_1G^0)$ for $\pi_1(H_1)$. To obtain a contradiction, we show that $\pi_1(H_1)$ is contained in an R-Levi subgroup of $Q$. Consider the group $Q^0$. This is a parabolic subgroup of $\pi_1(H_1G^0)^0 = \pi_1(G^0) = \pi_1(G_1)$; hence it is of the form $Q^0 = \pi_1(P_1)$, where $P_1$ is a parabolic subgroup of $G_1$.

Since $P_1$ contains the centre of $G_1$ and $\pi_1(P_1) = Q^0$ is normalized by $\pi_1(H_1) \subseteq Q$, it follows that $P_1$ is normalized by $H_1$. Indeed, let $h \in H_1$. Then $h \cdot P_1 \subseteq G_1$. On the other hand, $\pi_1(h \cdot P_1) = \pi_1(h) \cdot \pi_1(P_1) = \pi_1(P_1)$. Since $\ker(\pi_1) = \widehat{G}_1$, this

implies that $h \cdot P_1 \subseteq P_1\widehat{G}_1$. We conclude that $h \cdot P_1 \subseteq P_1(G_1 \cap \widehat{G}_1) = P_1$, where we have used that the last intersection is central in $G_1$.

For $2 \leq j \leq r$, let $h_j \in H$ be an element satisfying $h_j \cdot G_1 = G_j$. Let $P_j = h_j \cdot P_1$, which is a parabolic subgroup of $G_j$. Since we have just verified that $H_1$ normalizes $P_1$, the definition of $P_j$ does not depend on the choice of $h_j$ that transports $G_1$ to $G_j$.

We now consider the parabolic subgroup $P = SP_1 \cdots P_n$ of $G^0$, where we take $P_j = G_j$ for $j > r$. By construction, $P$ is normalized by $H$. Indeed, any $h \in H$ fixes $S$ under conjugation and permutes the groups $G_1, \ldots, G_n$. If $h$ maps $G_i$ to $G_j$ with $i, j \in \{1, \ldots, r\}$, then $(hh_i) \cdot G_1 = G_j$, and hence $h \cdot P_i = (hh_i) \cdot P_1 = P_j$. So $h$ also permutes the groups $P_1, \ldots, P_r$, and thus normalizes $P$.

The group $N_G(P)$ is thus an R-parabolic subgroup of $G$ containing $H$ with $N_G(P)^0 = P$ (see [2, Prop. 6.1]). Since $H$ is $G$-cr, it is contained in an R-Levi subgroup $L$ of $N_G(P)$; hence it normalizes the Levi subgroup $L^0$ of $P$. We may write $L^0 = SL_1 \cdots L_n$ for certain Levi subgroups $L_j$ of $P_j$. Then $H_1$ normalizes $L_1$, since $L_1 = L^0 \cap G_1$. This forces $\pi_1(H_1)$ to normalize a Levi subgroup of $Q^0 = \pi_1(P_1)$. By Lemma 2.3, $\pi_1(H_1)$ is contained in an R-Levi subgroup of $Q$, yielding a contradiction.

To prove the reverse implication, we again assume after reordering the indices that $1 \in I$ and that $H$ permutes the set $\{1, \ldots, r\}$ transitively for some $r \geq 1$. Assume that $H$ is not $G$-cr and that $Q \subseteq G$ is an optimal destabilizing R-parabolic subgroup of $G$ containing $H$. Again we want to deduce that $H$ is contained in an R-Levi subgroup of $Q$, contradicting our assumption.

Write $Q^0 = SP_1 \cdots P_n$, where the $P_i$ are parabolic subgroups of $G_i$. Since $H$ normalizes $Q^0$, $H_1$ normalizes $P_1 = Q^0 \cap G_1$. This means that $\pi_1(H_1)$ is contained in $N_{\pi_1(H_1G^0)}(\pi_1(P_1))$, and the latter is an R-parabolic subgroup of $\pi_1(H_1G^0)$. Since $1 \in I$, the subgroup $\pi_1(H_1)$ is contained in an R-Levi subgroup, say $M$, of this normalizer. Thus $\pi_1(H_1)$ normalizes the Levi subgroup $M^0$ of $\pi_1(P_1)$, which is hence of the form $\pi_1(L_1)$ for some Levi subgroup $L_1$ of $P_1$.

Since $L_1$ contains the centre of $G_1$, as in the proof of the forward implication (where we have proved that $H_1$ normalizes $P_1$), we may conclude that $H_1$ normalizes $L_1$. Choosing again elements $h_j \in H$ with $h_j \cdot G_1 = G_j$ for $2 \leq j \leq r$, we obtain well-defined Levi subgroups $L_j := h_j \cdot L_1$ of $h_j \cdot P_1 = P_j$, where the latter equality follows from $P_j = Q^0 \cap G_j$. Proceeding similarly for the other $H$-orbits on $\{1, \ldots, n\}$ (each of which contains an element of $I$ by assumption), we construct an $H$-stable Levi subgroup $L = SL_1 \cdots L_n$ of $Q^0$. As before, by Lemma 2.3, $H$ is contained in an R-Levi subgroup of $Q$, which gives the desired contradiction. This finishes the proof. □

## 3. Cyclic subgroups

Following Steinberg [17, §9], we say that $g$ in $G$ is *quasi-semisimple* provided $g$ normalizes a pair $(B, T)$ consisting of a Borel subgroup $B$ of $G$ and a maximal torus $T \subseteq B$. The next result is contained in recent work of Guralnick and Malle (see [8, Thm. 2.3]).

**Theorem 3.1.** *For $g \in G$, the following properties are equivalent:*

(i) *the conjugacy class $G \cdot g$ is closed;*
(ii) *the centralizer $C_G(g)$ is reductive;*
(iii) *$g$ is quasi-semisimple.*

According to Theorem 2.6, a cyclic subgroup of $G$ is $G$-completely reducible if and only if the conjugacy class of a generator is closed. With this characterization we may reformulate the equivalence of (i) and (ii) from Theorem 3.1 as follows.

**Corollary 3.2.** *Let $H$ be a cyclic subgroup of $G$. Then $H$ is $G$-completely reducible if and only if $C_{G^0}(H)$ is reductive.*

By Lemma 2.11(ii), we deduce a criterion for the complete reducibility of the fixed-point set.

**Corollary 3.3.** *Let $H$ be a cyclic subgroup of $G$. Then $C_{G^0}(H)$ is $G^0$-completely reducible if and only if it is reductive.*

Combining some of our previous reductions, we give a different proof of Corollary 3.2. We believe this is of independent interest, as our arguments allow us to avoid the case-by-case considerations that are needed for the proof of Theorem 3.1. Here is the first ingredient of the proof.

**Proposition 3.4.** *Let $U$ be a (not necessarily connected) unipotent algebraic group with $\dim U > 0$. Then the centre $Z(U)$ satisfies $\dim Z(U) > 0$.*

*Proof.* Let $V = U^0$. As $V$ is nilpotent, connected and of positive dimension, we have $\dim Z(V) > 0$ (see, e.g., [9, Prop. 17.4]). Hence it is enough to prove the result when $V$ is abelian, so we assume this. The finite $p$-group $H = U/V$ acts on $V$, and it suffices to show that $H$ has infinitely many fixed points on $V$. Up to passing to a characteristic subgroup (the subgroup of elements of order dividing $p$), we may assume that $V$ has exponent $p$. Thus $V$ has the structure of an $\mathbb{F}_p$-vector space of infinite dimension with an $\mathbb{F}_p$-linear $H$-action.

If $H \neq 1$, let $h \in H$ be a central element of order $p$. Then $h$ fixes infinitely many points on $V$. Indeed, on any $h$-stable finite dimensional subspace $W$ of $V$ the automorphism induced by $h$ may be brought into Jordan normal form with block sizes bounded by $p$ (the Jordan normal form exists as the only eigenvalue of $h$ is $1 \in \mathbb{F}_p$). As each block contributes at least $p-1$ fixed points, $h$ has at least $(p-1)(\dim W)/p$ fixed points on $W$, and we can make $\dim_{\mathbb{F}_p} W$ arbitrarily large. Hence the fixed-point set $V^h$ is positive-dimensional.

Finally consider the exact sequence $1 \to \langle h \rangle \to H \to H' \to 1$, with $H' = H/\langle h \rangle$. The group $H'$ acts on $V^h$. Using induction on the order of $H$, we may conclude that $V^H = (V^h)^{H'}$ is infinite. □

*Proof of Corollary* 3.2. The forward implication is clear, by Lemma 2.11(ii).

Conversely, assume that $C_{G^0}(H)$ is reductive. Let $H = \overline{\langle g \rangle}$ and let $g = g_s g_u$ be the Jordan decomposition of $g$ and let $H_s$ and $H_u$ be the closed subgroups of $G$ generated by $g_s$ and $g_u$, respectively. Then $H_s$ is linearly reductive, $H$ is isomorphic to $H_s \times H_u$ and $C_{G^0}(H) = C_{C_{G^0}(H_s)}(H_u)$. By Lemma 2.11(ii), $C_G(H_s)$ is reductive (note that $H_s$ is $G$-cr by [2, Lem. 2.6]). By [3, Prop. 3.9], $H$ is $G$-cr if and only if $H_u$ is $C_G(H_s)$-cr. So we can replace the pair $(G, H)$ by $(C_G(H_s), H_u)$, and hence we can assume that $H = H_u$ is finite and unipotent.

We first show that $C_{G^0}(H)$ is $G^0$-cr. Suppose this fails, and let $P \subseteq G^0$ be an optimal destabilizing parabolic subgroup for $C_{G^0}(H)$ in $G^0$. Then $H$ normalizes $P$, by Theorem 2.7(ii). Let $U$ be the unipotent radical of $P$. We may apply Proposition 3.4 to the unipotent group $HU$ to obtain a positive-dimensional centre $Z = Z(HU)$. Since $H$ is finite, the identity component $Z^0$ of $Z$ lies in $U$. Thus

$C_{G^0}(H)$ normalizes $U$ and so $Z^0$ yields a non-trivial, connected, normal, unipotent subgroup of $C_{G^0}(H)$, contradicting the reductivity assumption. We thus conclude that $C_{G^0}(H)$ is $G^0$-cr.

By Lemma 2.14, it therefore suffices to show that $H$ is $M$-cr, where $M = HC_{G^0}(C_{G^0}(H))$. Suppose $M^0$ is not a torus. Let $M_1, \ldots, M_r$ be the simple components of $M^0$ and set $H_1 := N_H(M_1)$. Then by a result of Steinberg (cf. [17, Thm. 10.13]), the fixed-point set $C_1 := C_{M_1}(H_1)$ of the cyclic group $H_1$ is positive-dimensional. Without loss of generality, we can assume $H$ acts transitively on the set $M_1, \ldots, M_s$ for some $1 \leq s \leq r$. For each $1 \leq i \leq s$, choose $h_i \in H$ such that $h_i M_1 h_i^{-1} = M_i$. Set $C := \{\prod_{i=1}^{s} h_i c h_i^{-1} \mid c \in C_1\}$. Then $C$ is a positive-dimensional subgroup of $C_{[M^0,M^0]}(H)$. But $C_{M^0}(H)$ is contained in $Z(M^0)$, so $C_{[M^0,M^0]}(H)$ is finite, a contradiction. We deduce that $M^0$ is a torus, and the result follows. □

Finally, using Corollary 3.2, we can give a short alternative proof of the implication (ii) ⇒ (iii) in Theorem 3.1 which is free of any case-by-case considerations.

*Remark* 3.5. Fix $g \in G$ and let $H = \overline{\langle g \rangle}$. Thanks to [17, Thm. 7.2], $g$ normalizes a Borel subgroup $B$ of $G^0$. Suppose that $C_{G^0}(g)$ is reductive. Then $H$ is $G$-cr, by Corollary 3.2. Thus, since $H$ normalizes $B$, it normalizes a Levi subgroup $T$ of $B$, thanks to [4, Lem. 5.1]. Thus $g$ is quasi-semisimple.

## 4. Outer automorphisms for $D_4$

In this section, let $D_4$ denote an adjoint simple group of type $D_4$. Amongst the simple groups $D_4$ has the largest outer automorphism group, in that $\mathrm{Out}(D_4) \cong S_3$, the symmetric group on 3 letters. We may identify $\mathrm{Out}(D_4)$ with the set of graph automorphisms in $\mathrm{Aut}(D_4)$ induced by the symmetries of the Dynkin diagram. However, there are other subgroups isomorphic to $S_3$ in $\mathrm{Aut}(D_4)$ that act via outer automorphisms. As this is the only situation where outer automorphisms of a simple group arise that is not covered by Corollary 3.2, we treat this case separately in this section.

Let $T$ be a maximal torus of $D_4$ with associated root system $\Phi$. Let $\Delta = \{\alpha, \beta, \gamma, \delta\}$ be a set of simple roots for $\Phi$, where $\delta$ is the unique simple root that is non-orthogonal to every other simple root. Let $\lambda = \omega_\delta^\vee \in Y(T)$ be the fundamental dominant coweight determined by $\langle \alpha, \lambda \rangle = \langle \beta, \lambda \rangle = \langle \gamma, \lambda \rangle = 0$, $\langle \delta, \lambda \rangle = 1$. For $\epsilon \in \Phi$ we denote by $u_\epsilon : \mathbb{G}_a \to U_\epsilon$ a fixed root homomorphism onto the corresponding root subgroup of $G$.

Let $\sigma \in \mathrm{Aut}(D_4)$ be the triality graph automorphism determined by requiring that for all $c \in k$,

$$\sigma(u_\alpha(c)) = u_\beta(c), \quad \sigma(u_\beta(c)) = u_\gamma(c), \quad \sigma(u_\gamma(c)) = u_\alpha(c), \quad \sigma(u_\delta(c)) = u_\delta(c).$$

Then $C_{D_4}(\sigma)$ is a simple group of type $G_2$. In fact, $\tilde{T} = C_T(\sigma)$ is a maximal torus of $C_{D_4}(\sigma)$, and $\tilde{\alpha} = \alpha|_{\tilde{T}} = \beta|_{\tilde{T}} = \gamma|_{\tilde{T}}$ and $\tilde{\beta} = \delta|_{\tilde{T}}$ form a pair of simple roots with respect to $\tilde{T}$, with corresponding root groups given by $u_{\tilde{\alpha}}(c) = u_\alpha(c)u_\beta(c)u_\gamma(c)$, $u_{\tilde{\beta}}(c) = u_\delta(c)$. Since $\lambda$ evaluates in $\tilde{T}$, we may regard it as an element of $Y(\tilde{T})$; we denote this element by $\tilde{\lambda}$. We have $\langle \tilde{\alpha}, \tilde{\lambda} \rangle = 0$, $\langle \tilde{\beta}, \tilde{\lambda} \rangle = 1$.

We begin with a detailed description of triality in the particular case where the ground field has characteristic three, using the results of [6] and [7].

**Proposition 4.1.** *Assume that $p = 3$. In $\mathrm{Aut}(D_4)$ there are exactly two conjugacy classes of cyclic groups of order three generated by outer automorphisms. Let $\langle\sigma_1\rangle$, $\langle\sigma_2\rangle$ be representatives of the respective classes, and let $M_i = C_{D_4}(\sigma_i)$ $(i = 1, 2)$. Then we may choose the labelling such that the following holds:*

(i) *$M_1$ is a simple group of type $G_2$; moreover $\mathrm{Aut}(D_4) \cdot \sigma_1$, the orbit of $\sigma_1$ under conjugation, is closed in $\mathrm{Aut}(D_4)$.*
(ii) *$M_2$ is an 8-dimensional group with 5-dimensional unipotent radical and corresponding reductive quotient isomorphic to $\mathrm{SL}_2$; the orbit $\mathrm{Aut}(D_4) \cdot \sigma_2$ is not closed and contains $\sigma_1$ in its closure.*
(iii) *We may take $\sigma_1 = \sigma$.*
(iv) *Let $u = u_{\alpha+\beta+\gamma+2\delta}(1)$. Then we may take $\sigma_2 = \sigma u$.*
(v) *With the choices in* (iii) *and* (iv)*, we have $M_2 = C_{M_1}(u) = \langle U_{\tilde\alpha}, U_{-\tilde\alpha}\rangle \ltimes R_u(P_{\tilde\lambda}) \subseteq P_{\tilde\lambda}$, where $P_{\tilde\lambda}$ denotes $P_\lambda(M_1)$.*

*Proof.* By [6, Cor. 6.5, Thm. 9.1], there are precisely two conjugacy classes of cyclic groups of order three generated by outer automorphisms, which are denoted by type I and type II, respectively. They are distinguished by the structure of the corresponding fixed point groups, where type I yields a group of type $G_2$, whereas type II in characteristic 3 gives a group with the structure described in (ii) (see [6, §9] together with [7, Thm. 7]). This implies the first statements of (i) and (ii), as well as (iii).

Working in the algebraic group $\mathrm{Aut}(D_4)$, using $\sigma(\lambda) = \lambda$ and $\langle\alpha+\beta+\gamma+2\delta, \lambda\rangle = 2 > 0$ we compute that

$$\lim_{a\to 0} \lambda(a)\sigma u\lambda(a)^{-1} = \sigma \lim_{a\to 0} \lambda(a)u\lambda(a)^{-1} = \sigma,$$

which proves that $\sigma$ is contained in the closure of the orbit through $\sigma u$. By [5, Thm. 3.3], to show that $\sigma$ and $\sigma u$ are not conjugate it is enough to show that they are not $D_4$-conjugate. This non-conjugacy follows from [16, §I, Prop. 3.2], as $\sigma$ and $\sigma u$ are given there as class representatives for distinct unipotent classes in $\sigma D_4$. Moreover, by [16, §II, Lem. 1.15], the orbit through $\sigma$ is closed. This proves (iv) and the remaining assertions of (i) and (ii).

To prove (v), we first note that $R_u(P_{\tilde\lambda})$ consists of the root groups for the roots $\tilde\beta, \tilde\alpha + \tilde\beta,\ 2\tilde\alpha + \tilde\beta,\ 3\tilde\alpha + \tilde\beta$ and $3\tilde\alpha + 2\tilde\beta$. In particular, the semi-direct product $\langle U_{\tilde\alpha}, U_{-\tilde\alpha}\rangle \ltimes R_u(P_{\tilde\lambda})$ has dimension 8 and is contained in $C_{M_1}(u) = C_{M_1}(u_{3\tilde\alpha+2\tilde\beta}(1))$. As $\tilde\lambda$ centralizes $\pm\tilde\alpha$, the semi-direct product is also contained in $P_{\tilde\lambda}$. Since clearly $C_{M_1}(u) \subseteq M_2$, the assertion (v) follows by comparing dimensions. This finishes the proof. □

We can now characterize $G$-complete reducibility in the case where $G^0 = D_4$ and $H$ maps isomorphically onto the full group of outer automorphisms of $D_4$. The following result is the analogue of Corollary 3.2.

**Theorem 4.2.** *Let $H$ be a subgroup of $G$. Assume that $G^0 = D_4$ and $H \cong S_3$ acts by outer automorphisms on $G^0$. Then $H$ is $G$-completely reducible if and only if $C_{G^0}(H)$ is reductive.*

*Proof.* The forward implication is clear by Lemma 2.11. Conversely, assume that $H$ is not $G$-cr. Let $h \in H$ be an element of order 3, so that $K = \langle h\rangle$ is a normal subgroup of index 2 in $H$. By the assumption on $H$, the map $\pi : G \to \mathrm{Aut}(D_4), g \mapsto \mathrm{Int}(g)|_{G^0}$ is surjective. Since $\ker(\pi) = C_G(G^0)$, $\pi$ is an isogeny. Hence $\pi(H)$ is not

$\mathrm{Aut}(D_4)$-cr, by Lemma 2.9. It now follows from Remark 2.15 that we can take $G$ to be $\mathrm{Aut}(D_4)$.

First assume that $p = 3$. Let $M_1 = C_{D_4}(\sigma)$ and $M_2 = C_{D_4}(\sigma u)$ with notation as in Proposition 4.1. Then $K$ is a normal subgroup of order 3 and index 2 in $H$. Since $p = 3$ is coprime to 2, we have by Theorem 2.8(ii) that $K$ is not $\mathrm{Aut}(D_4)$-cr. This implies (by Theorem 2.6) that the orbit $\mathrm{Aut}(D_4) \cdot h$ is not closed; whence by Proposition 4.1 there exists $g \in G$ with $ghg^{-1} = \sigma u$. Replacing $H$ with $gHg^{-1}$, we may assume that $h = \sigma u$. Let $s \in H$ be an element of order 2 such that $h$ and $s$ generate $H$. Let $\tau \in \mathrm{Aut}(D_4)$ be the graph automorphism of order 2 determined by $s$, i.e., the graph automorphism that induces the same element as $s$ in $\mathrm{Out}(D_4)$. Let $t = \tilde{\beta}^\vee(-1) \in \tilde{T} \subseteq M_1$. Since $\tau$ and $\sigma$ fix $M_1$, both elements commute with $t$ and $u$. Moreover, by construction $tut = u^{-1}$. This implies that $\tau t$ has order 2 and $(\tau t)(\sigma u)(\tau t) = \tau\sigma\tau u^{-1} = \sigma^{-1}u^{-1} = (\sigma u)^{-1}$. As $\tau t$ induces the same element as $s$ in $\mathrm{Out}(D_4)$, we can find $x \in D_4$ with $s = \tau t x$. We conclude that both pairs of elements $h = \sigma u, s = \tau t x$ as well as $\sigma u, \tau t$ generate a group isomorphic to $S_3$. In particular, $x(\sigma u)x^{-1} = (\tau t)(\tau t x)(\sigma u)(\tau t x)^{-1}(\tau t) = (\tau t)(\sigma u)^{-1}(\tau t) = \sigma u$. Thus $x \in M_2 \subseteq M_1$ (cf. Proposition 4.1(v)), so that $s = \tau t x$ normalizes $M_1$. As $M_1$ is simple of type $G_2$, it has no outer automorphisms. Therefore we may find $s' \in M_1$ with $\mathrm{Int}(s)|_{M_1} = \mathrm{Int}(s')|_{M_1}$. Since $M_1$ is adjoint, $s'$ is of order 2. Now

$$C_{G^0}(H) = C_{M_2}(s'). \tag{4.3}$$

Since $s = \tau t x$ normalizes $M_2$, $s$ normalizes $N_{M_1}(M_2) = P_{\tilde{\lambda}}$ (see Proposition 4.1(v)). Hence $s' \in P_{\tilde{\lambda}}$ and $C_{M_2}(s') \subseteq P_{\tilde{\lambda}}$. Up to conjugation in $P_{\tilde{\lambda}}$ we may thus assume $s' \in \tilde{T}$. As $\tilde{T}$ is generated by the images of $\tilde{\alpha}^\vee$ and $\tilde{\beta}^\vee$, this reduces the possibilities to $s' \in \{\tilde{\alpha}^\vee(-1), \tilde{\beta}^\vee(-1), \tilde{\alpha}^\vee(-1)\tilde{\beta}^\vee(-1)\}$. But then $s'$ centralizes $U_{3\tilde{\alpha}+2\tilde{\beta}}$, or $U_{\tilde{\beta}}$, or $U_{\tilde{\alpha}+\tilde{\beta}}$ respectively. We deduce that $C_{M_2}(s') \cap R_u(P_{\tilde{\lambda}})$ is positive-dimensional in each case. Thus $C_{M_2}(s')$ is not reductive, as required.

Now let $p \neq 3$. Then the subgroup $K$ of $H$ of order 3 is linearly reductive; in particular it is $G$-cr and $C_{G^0}(K)$ is reductive. Moreover, the group $C_{G^0}(K)$ is connected being the fixed point group under a triality automorphism (cf. [6, §9]). Let $M = HC_{G^0}(K)$. By [3, Thm. 3.1(b)(ii)] applied to $K \subseteq H \subseteq M$, we deduce that $H$ is not $M$-cr. Since $K$ is normal in $M$, by Theorem 2.8(i), $H/K$ is not $M/K$-cr. But $H/K$ is cyclic of order 2, so we may apply Corollary 3.2 to conclude that $C_{(M/K)^0}(H/K)$ is not reductive. By construction, $(M/K)^0 \cong C_{G^0}(K)$ and

$$C_{(M/K)^0}(H/K) \cong C_{G^0}(H). \tag{4.4}$$

This finishes the proof. □

Having settled the case of $D_4$, we can combine Corollary 3.2 and Theorem 4.2 to characterize $G$-complete reducibility in case $G^0$ is simple and the subgroup $H$ acts by outer automorphisms.

**Corollary 4.5.** *Let $H \subseteq G$ be a subgroup acting on $G^0$ by outer automorphisms. Assume that $G^0$ is simple. Then $H$ is $G$-completely reducible if and only if $C_{G^0}(H)$ is reductive.*

*Proof.* We may assume that $G^0$ is adjoint (cf. the first paragraph of the proof of Theorem 4.2). Since $H$ acts via outer automorphisms, we may identify it as an abstract group with a subgroup of $\mathrm{Out}(G^0)$, the finite group of outer automorphisms

of $G^0$. As $G^0$ is simple, $\mathrm{Out}(G^0)$ is either simple of prime order or $G^0$ is of type $D_4$ and $\mathrm{Out}(G^0) \cong S_3$. The result now follows from Corollary 3.2 and Theorem 4.2. $\square$

*Remark* 4.6. In the situation of Corollary 4.5, we always have $\dim C_{G^0}(H) > 0$. This follows again from the theorem of Steinberg ([17, Thm. 10.13]) in case $H$ is cyclic. The general case follows from the identities for $C_{G^0}(H)$ in (4.3) and (4.4).

## 5. The algorithm

We return to the general situation where $H \subseteq G$ is a subgroup of a possibly non-connected reductive group. In this section, we are going to establish an algorithm that reduces the question of whether $H$ is $G$-cr to the question of whether certain subgroups of certain *connected* reductive groups are $G$-completely reducible.

We start with a proposition that recasts some of our earlier results as operations for a potential algorithm. We say that the pair $(H, G)$ is *completely reducible* provided that $H$ is $G$-completely reducible.

**Proposition 5.1.** *Let $(H, G)$ be a pair consisting of a reductive group $G$ and a subgroup $H \subseteq G$. Then each of the following operations replaces $(H, G)$ with pairs of the same form (i.e., consisting of a group and a reductive group containing it as a subgroup):*

(O1) *Let $\pi : G \to G/C_G(G^0)$ be the canonical projection. Replace $(H, G)$ with $(\pi(H), \pi(G))$.*

(O2) *Let $G^0 = SG_1 \cdots G_n$ be the decomposition as in (2.16), and let $H_i$, $\pi_i$ for $1 \leq i \leq n$ be defined as in Lemma 2.17. Replace $(H, G)$ with the pairs*

$$(\pi_1(H_1), \pi_1(H_1G^0)), \ldots, (\pi_n(H_n), \pi_n(H_nG^0)).$$

(O3) *If $H \cap G^0$ is $G^0$-completely reducible, replace $(H, G)$ with*

$$(H/(H \cap G^0), HC_{G^0}(H \cap G^0)/(H \cap G^0)).$$

*Moreover, $H$ is $G$-completely reducible if and only if each of the pairs obtained through one of these operations is completely reducible.*

*Proof.* The results follow from Corollary 2.10, Lemma 2.17 and Remark 2.13. $\square$

*Remark* 5.2. In the situation of (O2), suppose we are given a set $I$ as in Lemma 2.17. Then it is enough to replace $(H, G)$ with the pairs $(\pi_i(H_i), \pi_i(H_iG^0))_{i \in I}$ in (O2).

We are now in a position to give an algorithm that determines whether $H$ is $G$-completely reducible.

**Theorem 5.3.** *Let $H$ be a subgroup of $G$. The following algorithm, starting with the pair $(H, G)$, reduces the question of whether $H$ is $G$-completely reducible in a finite number of steps to questions of complete reducibility in connected groups:*

**Algorithm.** *Input: a pair $(H', G')$, where $H'$ is a subgroup of a reductive group $G'$.*

Step 1. *If $G'^0$ is not simple or $C_{G'}(G'^0) \neq 1$, apply (O2) and then (O1) to each of the newly obtained pairs. Restart instances of the algorithm for each of the new pairs. Then $H'$ is $G'$-cr if and only if each of these pairs turns out to be completely reducible.*

Step 2. *Identify $H'/(H' \cap G'^0)$ with a subgroup of* $\mathrm{Out}(G'^0)$. *Let $n \in \{1, 2, 3, 6\}$ be the order of $H'/(H' \cap G'^0)$. If $p$ does not divide $n$, $H'$ is $G'$-cr if and only if $H' \cap G'^0$ is $G'^0$-cr, and the algorithm stops.*

Step 3. *If $H' \cap G'^0$ is not $G'^0$-cr, the algorithm stops with the conclusion that $H'$ is not $G'$-cr.*

Step 4. *If $H' \cap G'^0 = 1$, $H'$ is $G'$-cr if and only if $C_{G'^0}(H')$ is reductive. The algorithm stops.*

Step 5. *If $1 \neq H'/(H' \cap G'^0) \ncong S_3$, let $M = H'C_{G'^0}(H' \cap G'^0)/(H' \cap G'^0)$. Then $H'$ is $G'$-cr if and only if $C_{M^0}(H'/(H' \cap G'^0))$ is reductive. The algorithm stops.*

Step 6. *If $H'/(H' \cap G'^0) \cong S_3$, apply* (O3) *and restart the algorithm with the new pair.*

*Proof.* Step 1 is covered by Proposition 5.1. Moreover, each of the new pairs $(H'', G'')$ produced in this step satisfies $G''^0$ simple and $C_{G''}(G''^0) = 1$, so that the algorithm moves on to Step 2 after a possible application of Step 1.

From Step 2 on, we may assume that $G'^0$ is simple and $C_{G'}(G'^0) = 1$. This allows us to identify $H'/(H' \cap G'^0)$ with a subgroup of $\mathrm{Out}(G'^0)$ and yields the constraints on its order. If $p$ does not divide $n$, the quotient $H'/(H' \cap G'^0)$ is linearly reductive. By Theorem 2.8(ii), $H'$ is indeed $G'$-cr if and only if $H' \cap G'^0$ is $G'^0$-cr.

From Step 3 on, we may assume in addition that $p \in \{2, 3\}$ and that $H$ is not contained in $G^0$. The conclusion of Step 3 is correct by Lemma 2.11(i).

Step 4 is an application of Corollary 4.5.

Since we have passed Step 3, we may assume that $H' \cap G'^0$ is $G'^0$-cr. Under the condition of Step 5, $H'/(H' \cap G'^0)$ is cyclic. The conclusion of Step 5 thus follows from Remark 2.13 and Corollary 3.2.

Finally, Step 6 is again covered by Proposition 5.1. Moreover, this step is only applicable for $G'^0$ simple of type $D_4$. As we may assume $H' \cap G'^0 \neq 1$ and $Z(G'^0) = 1$, the group $M = H'C_{G'^0}(H' \cap G'^0)/(H' \cap G'^0)$ featured in (O3) satisfies $\dim M < \dim G'$.

It remains to show that the algorithm terminates. Step 1 may restart finitely many instances of the algorithm. In each instance the algorithm terminates in Step 2–Step 5 if Step 6 is not reached. If Step 6 is applicable, it replaces $G'$—which is simple of type $D_4$—with a group of smaller dimension. This implies that after Step 1 is applied again, Step 6 cannot be reached a second time, and the algorithm terminates. □

*Remark* 5.4. (i) It follows from the proof of Theorem 5.3 that Step 1, the only step that replaces a pair with several new pairs, need only be done at most twice along a path through the algorithm. Also, Step 6 only occurs at most once.

(ii) There are some situations where shortcuts may be applied to reduce to a connected group. First of all, if $H^0$ is not reductive, then $H$ cannot be $G$-cr. On the other hand, if $H$ is cyclic, then we may apply Corollary 3.2 to deduce that $H$ is $G$-cr if and only if $C_{G^0}(H)$ is $G^0$-cr. Finally, if $H/(H \cap G^0)$ is linearly reductive, we can apply Theorem 2.8(ii) to deduce that $H$ is $G$-cr if and only if $H \cap G^0$ is $G^0$-cr. However, the proposed algorithm gives a systematic approach that deals with all possible cases.

(iii) If $p = 0$, then a subgroup $H$ is $G$-cr if and only if it is reductive ([15, Prop. 4.2]). Of course, $H$ is reductive if and only if $H^0$ is reductive, which in turn is equivalent to $H^0$ being $G^0$-completely reducible.

## 6. Examples

We conclude with some examples of the algorithm outlined in Theorem 5.3.

**Example 6.1.** Let $p = 3$, $G = \mathrm{Aut}(D_4)$. Let $\sigma$ be the triality graph automorphism as in Section 4. Let $H = \langle\sigma\rangle K$, where $K = C_{D_4}(\sigma)$ is the fixed point subgroup of type $G_2$. We follow through the algorithm to deduce that $H$ is $G$-cr:

Step 1 is not applicable, as $G^0 = D_4$ is simple and $C_G(G^0) = 1$. In Step 2 we obtain $n = 3 = p$ as the order of $\langle\sigma\rangle \cong H/(H \cap G^0)$. Now $H \cap G^0 = K$ is $G$-cr (see Corollary 3.3); hence Steps 3 and 4 are not applicable. Step 5 applies and leads us to consider the group $M = HC_{D_4}(K)/K \cong \langle\sigma\rangle C_{D_4}(K)$. As $K$ is adjoint, we obtain $C_{M^0}(\sigma) = 1$ and thus this group is clearly reductive. The algorithm stops with the conclusion that $H$ is $G$-cr.

Here we have two commuting $G$-cr subgroups $\langle\sigma\rangle$ and $K$ of $G$ and their product is also $G$-cr. This is not always the case; see [3, Ex. 5.1].

**Example 6.2.** Let $\Gamma$ be a finite group acting transitively on a finite set $I$. Let $i_0 \in I$. Let $\rho : \Gamma \to M$ be a homomorphism to a simple group $M$ such that $\rho(C_\Gamma(i_0))$ is not $M$-cr. We set

$$G = \Gamma \ltimes \prod_{i\in I} M,$$

where $\Gamma$ acts on the product by permuting the indices. Clearly, $G^0 = \prod_i M$. Let $d : M \to G^0$ be the diagonal embedding. As $\Gamma$ commutes with the image of $d$, we may form the subgroup

$$H = \{\gamma d(\rho(\gamma)) \mid \gamma \in \Gamma\},$$

a finite subgroup of $G$. We claim that $H$ is not $G$-cr and use our algorithm to prove it. Here we use the notation from Subsection 2.3.

To apply Step 1, we compute that for $i \in I$, $N_G(G_i) = C_\Gamma(i) \ltimes G^0$. In particular, $H_i = \{\gamma d(\rho(\gamma)) \mid \gamma \in C_\Gamma(i)\}$. We obtain

$$\pi_i(H_iG^0) = H_iG^0/\prod_{j\neq i} G_j \cong C_\Gamma(i) \times M =: G'_i,$$

and correspondingly

$$\pi_i(H_i) \cong \{\gamma\rho(\gamma) \mid \gamma \in C_\Gamma(i)\} =: H'_i.$$

Since the action of $\Gamma$ on $I$ is transitive, we may by Remark 5.2 replace $(H, G)$ with $(H', G')$, where $G' = G'_{i_0}, H' = H'_{i_0}$.

Now $C_{G'}(G'^0) = C_\Gamma(i_0)$, so Step 1 is again applicable and replaces $(H', G')$ with the pair $(H'', M)$, where

$$H'' = \rho(C_\Gamma(i_0)).$$

By assumption, $H''$ is not $M$-cr and hence $H$ is not $G$-cr, by Step 2.

As a concrete realization of this example, take $\Gamma = \mathrm{PGL}_2(q)$ for $q$ a sufficiently large power of $p$, $I = \mathrm{PGL}_2(q)/B(q)$ where $B$ is a Borel subgroup of $\mathrm{PGL}_2$, and consider $I$ as a transitive $\Gamma$-set by left translation. Let $M = \mathrm{PGL}_2$, and let $\rho : \Gamma \hookrightarrow M$ be the canonical embedding. If we take $i_0 = B(q)$, then $C_\Gamma(i_0) = B(q)$ is not $M$-cr, for $q$ large enough, as $B$ is not $M$-cr. In this example, we have $H \cap G^0 = 1$ and $C_{G^0}(H) = 1$ (as $C_M(B(q)) = 1$ for $q$ sufficiently large). In particular, this example shows that $H$ may fail to be $G$-cr even if $C_{G^0}(H)$ and $H \cap G^0$ both are $G^0$-cr.

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