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EARTHQUAKES IN THE LENGTH-SPECTRUM TEICHMÜLLER SPACES

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ABSTRACT. Let X_0 be a complete hyperbolic surface of infinite type that has a geodesic pants decomposition with cuff lengths bounded above. The length spectrum Teichmüller space $T_{ls}(X_0)$ consists of homotopy classes of hyperbolic metrics on X_0 such that the ratios of the corresponding simple closed geodesic for the hyperbolic metric on X_0 and for the other hyperbolic metric are bounded from below away from 0 and from above away from ∞. This paper studies earthquakes in the length spectrum Teichmüller space $T_{ls}(X_0)$. We find a necessary condition and several sufficient conditions on the earthquake measure μ such that the corresponding earthquake E^{μ} describes a hyperbolic metric on X_0 which is in the length spectrum Teichmüller space. Moreover, we give examples of earthquake paths $t \mapsto E^{t\mu}$, for $t \geq 0$, such that $E^{t\mu} \in T_{ls}(X_0)$ for $0 \leq t < t_0$, $E^{t_0\mu} \notin T_{ls}(X_0)$ and $E^{t\mu} \in T_{ls}(X_0)$ for $t > t_0$.

1. Introduction

The length spectrum Teichmüller space $T_{ls}(X_0)$ of a complete hyperbolic surface X_0 of infinite type consists of all homotopy classes of marked complete hyperbolic surfaces whose length spectrum is comparable to the length spectrum of the base point X_0 (cf. [1] and §2). A reasonable assumption on X_0 (cf. [1], [16]) is that there exists a geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}$ and a constant $L_0 > 0$ such that

$$l_{\alpha_n}(X_0) \leq L_0.$$

Such a pants decomposition $\mathcal{P}$ is said to be *upper bounded* and from now on we assume that an upper bounded pants decomposition always exists. An assignment of the lengths and the twists to the cuffs of $\mathcal{P}$ parametrizes the length spectrum Teichmüller space $T_{ls}(X_0)$ and the image of $T_{ls}(X_0)$ in these Fenchel-Nielsen coordinates is completely described (cf. [1]). Moreover, the Fenchel-Nielsen coordinates give a locally bi-Lipschitz homeomorphism between $T_{ls}(X_0)$ and l_∞ (cf. [15]).

The *quasiconformal Teichmüller space* $T_{qc}(X_0)$ of a hyperbolic surface X_0 consists of all quasiconformal deformation of the base surface X_0 up to post-compositions by hyperbolic isometries and up to homotopies. The earthquake deformations of $T_{qc}(X_0)$ when X_0 is an infinite surface were studied by several authors (cf. [17], [8], [9], [13], [14]). A measured lamination μ on a hyperbolic surface X_0 is *Thurston bounded* if

$$\|\mu\|_{Th} = \sup_I \mu(I) < \infty$$

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where the supremum is over all unit length arcs I on X_0. A well-known result is that an earthquake E^μ along a measured lamination μ induces a quasiconformal deformation of X_0 if and only if $\|\mu\|_{Th} < \infty$ (cf. [17], [8], [9], [7], [13], [14]). Our main result is an analogous statement in the context of the length spectrum. Namely, we describe which measured laminations induce earthquake deformations of X_0 into hyperbolic surfaces which are not necessarily quasiconformally equivalent to X_0 but rather length spectrum equivalent to X_0.

We study earthquakes in the length spectrum Teichmüller space $T_{ls}(X_0)$. Unlike many other deformations of hyperbolic structures on surfaces, earthquakes are completely described in terms of the initial hyperbolic structure (cf. Thurston [17]). Thus one could say that earthquakes are natural in the context of hyperbolic geometry. An earthquake is a deformation of a hyperbolic structure along the support of the earthquake (a geodesic lamination on the initial surface) by an amount given by the earthquake measure (a transverse measure to the earthquake support). A pair of hyperbolic structures on a fixed surface are said to be *continuous* if the lift to the universal coverings of the identity map extends to a continuous map of the boundary circles (cf. Thurston [17]). Thurston defined earthquakes and proved that any two continuous hyperbolic structures are related by a unique earthquake (cf. [17]). The upper bounded geodesic pants decomposition guarantees that the hyperbolic metric on X_0 and on an element of the corresponding length spectrum Teichmüller space are continuous and thus related by an earthquake. We study under which conditions on an earthquake measure μ the image hyperbolic structure $E^\mu(X_0)$ (under the corresponding earthquake E^μ) is in the length spectrum Teichmüller space $T_{ls}(X_0)$.

Let $E^\mu : X_0 \to X^\mu$ be an earthquake from the hyperbolic surface X_0 (which has an upper bounded pants decomposition) onto another hyperbolic surface X^μ, where the measured geodesic lamination μ is the earthquake measure of E^μ. Define the *length spectrum norm* of μ by

$$\|\mu\|_{ls} = \sup_{\beta \in \mathcal{S}} \frac{\mu(\beta)}{l_\beta(X_0)},$$

where $\mathcal{S}$ is the set of all simple closed geodesics on X_0 and $\mu(\beta)$ is the total μ-mass deposited on $\beta \in \mathcal{S}$. We note that if $|\mu|_{Th} < \infty$, then $\|\mu\|_{ls} < \infty$ which gives many examples of measured laminations on any hyperbolic surface with finite length spectrum norm. Moreover, there are examples of measured laminations μ such that $\|\mu\|_{ls} < \infty$ and $|\mu|_{Th} = \infty$ (cf. §3, [1]).

A necessary condition for $E^\mu : X_0 \to X^\mu$ to belong to the length spectrum Teichmüller space $T_{ls}(X_0)$ is that $\|\mu\|_{ls} < \infty$ (cf. §5). Such a measure μ is said to be *length spectrum bounded*. However, it turns out that this condition is not sufficient (cf. §8). In fact, an earthquake path $t \mapsto (E^{t\mu} : X_0 \to X^{t\mu})$, for $t \geq 0$, can start in $T_{ls}(X_0)$ leave $T_{ls}(X_0)$ at some finite time t_0 and return afterwards to $T_{ls}(X_0)$. Moreover, there exist length spectrum bounded earthquake measures μ such that the earthquake path $E^{t\mu}$ leaves and returns to $T_{ls}(X_0)$ infinitely many times (cf. §8).

The only reason that $\|\mu\|_{ls} < \infty$ is not a sufficient condition for an earthquake $E^\mu : X_0 \to X^\mu$ to belong to $T_{ls}(X_0)$ is that the lengths of simple closed curves might decrease too much under the earthquake thus making the ratio of lengths of the corresponding simple closed curves on X_0 and X^μ too small. We give sufficient conditions on μ in order to guarantee that E^μ is in $T_{ls}(X_0)$.

Let α_n be a cuff in the upper bounded geodesic pants decomposition of X_0 and let P_n^1, P_n^2 be the two pairs of pants in $\mathcal{P}$ with the common cuff α_n. For a leaf g of the support of μ that intersects α_n, denote by g_{comp} a component of $g \cap (P_n^1 \cup P_n^2)$. The *winding number* $w_{\alpha_n}(g_{comp})$ of g_{comp} around the curve α_n is defined in §4 and can be essentially thought of as the amount of winding of g_{comp} around α_n. The quantity $w_{\alpha_n}(g_{comp})$ becomes crucial when the angle between α_n and g is larger than $\frac{\pi}{2}$. We have the following (cf. Theorem 4.4 and §5)

Theorem 1. *Let X_0 be a complete hyperbolic surface which has a geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}_{n\in\mathbb{N}}$ such that*

$$l_{\alpha_n}(X_0) \leq L_0$$

for some fixed $L_0 > 0$. Let μ be a measured (geodesic) lamination on X_0 such that

$$\|\mu\|_{ls} < \infty.$$

If there exist $C_0 > 1$, $C_0' > C(L_0, \|\mu\|_{ls}) \geq 1$ for the constant $C(L_0, \|\mu\|_{ls})$ from Lemma 4.2 and $C_1 > 0$ such that (for each cuff α_n of $\mathcal{P}$) μ satisfies one of the following:

1. $\mu(\alpha_n) > C_0 l_{\alpha_n}(X_0)$,
2. $\mu(\alpha_n) < \frac{1}{C_0'} l_{\alpha_n}(X_0)$,
3. *the angle between α_n and a leaf g of μ is less than or equal to $\frac{\pi}{2}$,*
4. $\frac{1}{C_0'} l_{\alpha_n}(X_0) \leq \mu(\alpha_n) \leq C_0 l_{\alpha_n}(X_0)$, *the angle between α_n and a leaf g of μ is greater than $\frac{\pi}{2}$ and $w_{\alpha_n}(g_{comp}) < C_1 \frac{1}{l_{\alpha_n}(X_0)}$,*

then the earthquake

$$E^{\mu} : X_0 \to X^{\mu}$$

belongs to the length spectrum Teichmüller space $T_{ls}(X_0)$.

The above theorem facilitates finding sufficient conditions on μ such that the whole earthquake path $t \mapsto E^{t\mu}$, for $t \geq 0$, stays in the length spectrum Teichmüller space $T_{ls}(X_0)$. We have (cf. Theorems 8.1 and 8.2)

Theorem 2. *Let X_0 be a complete hyperbolic surface with an upper bounded geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}$ and let μ be a measured geodesic lamination on X_0 with*

$$\|\mu\|_{ls} < \infty.$$

Then $E^{t\mu}(X_0) = X^{t\mu} \in T_{ls}(X_0)$ for all $t \geq 0$ if there exists $C > 0$ such that for each α_n one of the following holds:

1. *the angle between α_n and a leaf of μ is less than $\frac{\pi}{2}$,*
2. *the angle between α_n and a leaf of μ is greater than $\frac{\pi}{2}$, and $w_{\alpha_n}(g_{comp}) \leq C \frac{1}{l_{\alpha_n}(X_0)}$.*

In addition, $E^{t\mu}(X_0) = X^{t\mu} \in T_{ls}(X_0)$ for all $t \geq 0$ if $\mathcal{P}$ can be partitioned into $\mathcal{P}'$ and $\mathcal{P}''$ such that each $\alpha_n \in \mathcal{P}'$ satisfies either (1) or (2) for a fixed $C > 0$, and that for $\alpha_n \in \mathcal{P}''$

$$l_{\alpha_n}(X_0) \to 0$$

and

$$\frac{\mu(\alpha_n)}{l_{\alpha_n}(X_0)} \to 0$$

as $n \to \infty$.

2. The Fenchel-Nielsen coordinates

Let X_0 be a complete hyperbolic surface without boundary of infinite type. Assume that there exists $L_0 > 0$ and a geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}$ of X_0 such that for each $\alpha_n \in \mathcal{P}$

$$l_{\alpha_n}(X_0) \leq L_0,$$

where $l_{\alpha_n}(X_0)$ is the length of the geodesic representative of the curve α_n in the hyperbolic metric on X_0.

Consider the set of all homeomorphisms

$$h : X_0 \to X$$

such that there exists $L > 0$ with

$$\frac{1}{L} \leq \frac{l_\beta(X)}{l_\beta(X_0)} \leq L$$

for all simple closed geodesics $\beta \in \mathcal{S}$, where $l_\beta(X)$ is the length of the geodesic representative of $h(\beta)$ on X. The *length spectrum Teichmüller space* $T_{ls}(X_0)$ of the surface X_0 consists of all equivalence classes of the above homeomorphisms, where (h, X) is equivalent to (h', X') if there exists an isometry $I : X \to X'$ such that $(h')^{-1} \circ I \circ h : X_0 \to X_0$ is homotopic to the identity with a homotopy map that moves points on X_0 by a bounded amount (cf. [1]). We require the boundedness of the homotopy map in order to guarantee that the lift of the map $(h')^{-1} \circ I \circ h$ to the universal covering continuously extends to the identity on the boundary.

The Fenchel-Nielsen coordinates on infinite type surfaces are defined in the same fashion as on the finite type surfaces (cf. [1]). We recall a characterization of the length spectrum Teichmüller space $T_{ls}(X_0)$ in terms of the Fenchel-Nielsen coordinates (cf. [1], [15]):

Theorem 2.1. *Let X_0 be an infinite type complete hyperbolic surface equipped with an upper bounded geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}_{n\in\mathbb{N}}$. The normalized Fenchel-Nielsen coordinates*

$$F(X) = \Big\{ \Big(\log \frac{l_{\alpha_n}(X)}{l_{\alpha_n}(X_0)}, \frac{t_{\alpha_n}(X) - t_{\alpha_n}(X_0)}{\max\{1, |\log l_{\alpha_n}(X_0)|\}} \Big) \Big\}_{n\in\mathbb{N}} \tag{2.1}$$

associated to each $X \in T_{ls}(X_0)$ induce a locally bi-Lipschitz surjective homeomorphism

$$F : T_{ls}(X_0) \to l^\infty.$$

3. The length spectrum bound on earthquake measures

Let X_0 be a complete hyperbolic surface (without boundary) that has upper bounded geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}$. Let μ be a measured lamination on X_0. Recall a definition of Thurston [17].

Definition 3.1. A measured lamination μ on X_0 is *Thurston bounded* if

$$\|\mu\|_{Th} := \sup_I \mu(I) < \infty$$

where the supremum is over all geodesic arcs I on X_0 of length 1.

We introduce a new notion of boundedness of measured laminations which is better suited for the length spectrum Teichmüller spaces.

Definition 3.2. A measured lamination μ is *length spectrum bounded* if

$$\|\mu\|_{ls} := \sup_{\beta} \frac{\mu(\beta)}{l_\beta(X_0)} < \infty$$

where the supremum is over all simple closed geodesics β on X_0.

Remark 3.3. If $\|\mu\|_{Th} < \infty$, then $\|\mu\|_{ls} < \infty$. Since every hyperbolic surface supports Thurston bounded measured lamination [13], it follows that every hyperbolic metric supports length spectrum bounded measured laminations.

Moreover, if a hyperbolic surface X_0 has a sequence $\{\alpha_n\}$ of simple closed curves with $l_{\alpha_n}(X_0) \to 0$ as $n \to \infty$, then $\mu := \sum_n |\log l_{\alpha_n}(X_0)|\delta_{\alpha_n}$ is not Thurston bounded but it is length spectrum bounded (cf. [1]).

Consider a marking homeomorphism $h : X_0 \to X$ for any $X \in T_{ls}(X_0)$. Then h induces a homeomorphism $\tilde{h}$ of the boundaries of the universal coverings of X_0 and X. Note that X_0 is complete by the assumption. The surface X has an upper bounded geodesic pants decomposition because $X \in T_{ls}(X_0)$ which implies that X is a complete hyperbolic surface. Since X_0 and X are complete hyperbolic surfaces without the boundary the set of fixed points of the covering groups G_0 and G are dense in the unit circle S^1. The map $\tilde{h}$ is first defined as a one-to-one correspondence between the fixed points of elements of G_0 and the fixed points of elements of G which preserves the order on S^1. Therefore $\tilde{h}$ extends to a homeomorphism of S^1 which conjugates the action of G_0 onto the action of G.

Thurston [17] proved that each homeomorphism of the circle is realized as the continuous extension of an earthquake of the hyperbolic plane $\mathbb{H}^2$. Moreover if the homeomorphism of the circle is induced from a homeomorphism of X_0 to X, then the earthquake descends onto the surface X_0, namely the earthquake measure is invariant under the covering group of X_0 [17]. An *earthquake measure* is a measured lamination μ, and the earthquake $E^\mu : X_0 \to X$ is a piecewise isometry on each stratum of μ (cf. [17]).

Note that $t\mu$ for $t > 0$ is also a measured lamination on X_0 which is simply obtained by scaling μ by a factor t. However, $E^{t\mu}$ is in general not an earthquake map even for $0 < t < 1$ [8]. In fact, a map $E^{t\mu} : X_0 \to E^{t\mu}(X_0)$ can be defined to be piecewise isometry on the strata of $t\mu$ and it turns out that its image $E^{t\mu}(X_0)$ does not have to be a complete surface. In this case, the lift of $E^{t\mu}$ to the universal coverings does not extend to a homeomorphism of the boundary circles.

When μ is Thurston bounded the map $E^\mu : X_0 \to E^\mu(X_0)$ is surjective and the image $E^\mu(X_0) = X_\mu$ is a complete hyperbolic surface that is quasiconformally equivalent to X_0 (cf. [17], [8], [7], [13]). Since $\|t\mu\|_{Th} = t\|\mu\|_{Th} < \infty$ if $\|\mu\|_{Th} < \infty$, it follows that the earthquake path $E^{t\mu} : X_0 \to X^\mu$ is well defined for all $t \geq 0$.

The *quasiconformal Teichmüller space* $T_{qc}(X_0)$ consists of all quasiconformal maps $f : X_0 \to X$ from the base surface X_0 onto arbitrary surfaces with the following equivalence relation. Two quasiconformal maps $f_1 : X_0 \to X_1$ and $f_2 : X_0 \to X_2$ are equivalent if there exists an isometry $i : X_1 \to X_2$ such that $f_2^{-1} \circ i \circ f_1$ is homotopic to the identity with a homotopy map moving points of X_0 by a bounded amount.

The above says that in $T_{qc}(X_0)$ every point is connected by an earthquake path to the base point X_0. Moreover the topology of $T_{qc}(X_0)$ can be recovered from the uniform weak* topology on the space of earthquake measures on X_0 [12]. We consider the question whether in $T_{ls}(X_0)$ every point is connected to the base point by an earthquake path.

4. A lower bound on the lengths of the cuffs

In this section we estimate the ratio $l_{\alpha_n}(X_0)/l_{\alpha_n}(X_\mu)$. To achieve this, we prove a few preparatory lemmas. Let $L_0 > 0$ be such that

$$l_{\alpha_n}(X_0) \leq L_0$$

holds for every n.

Let $C > 0$ such that $\|\mu\|_{ls} < C$. Then

$$\mu(\alpha_n) \leq C l_{\alpha_n}(X_0)$$

by the definition of the length spectrum norm on measured lamination space.

Let P_1 and P_2 be two geodesic pairs of pants with cuff lengths bounded by $L_0 > 0$ that are glued along a common cuff α. Let g be either a simple closed geodesic in $P_1 \cup P_2$ or a simple geodesic arc in $P_1 \cup P_2$ which joins two boundary cuffs of $P_1 \cup P_2$ and transversely intersecting α. We define the *winding number* $w_\alpha(g)$ of the arc g around the curve α as follows. Let γ_i, for $i = 1, 2$, denote the unique simple geodesic arc in P_i which is orthogonal to α at both endpoints. Divide each P_i into two right-angled hexagons Σ_i^j for $j = 1, 2$ by cutting along three geodesic arcs orthogonal to pairs of cuffs of P_i. Let

$$\gamma_i^j := \gamma_i \cap \Sigma_i^j$$

for $i, j = 1, 2$. Note that the length of γ_i^j is half the length of γ_i.

Let g_s, for $s = 1, 2, \ldots, k$, be the components of the intersections $g \cap P_i$ for $i = 1, 2$. If the angle from α_n to g_s is greater than $\frac{\pi}{2}$, we define the *winding number* $w_\alpha(g)$ by

$$w_\alpha(g) = \max_{1 \leq s \leq k;\ i,j=1,2} \#(g_s, \gamma_i^j)$$

where $\#(g_s, \gamma_i^j)$ is the number of intersection points of g_s and $\gamma_i^j = \gamma_i \cap \Sigma_i^j$. Note that $w_\alpha(g) - \#(g_s, \gamma_i^j) \leq 2$ for all $s = 1, 2, \ldots, k$.

In order to use the winding number, we need the following lemma.

Lemma 4.1. *Consider the universal covering $\pi : \mathbb{H}^2 \to X_0$ such that one lift $\tilde{\alpha}_n$ is the positive y-axis. Let $\tilde{g}$ be a lift to $\mathbb{H}^2$ of a leaf g of the measured lamination μ that intersects the positive y-axis between i and $e^{l_{\alpha_n}(X_0)}i$. Denote by $k_1 < 0$ and $k_2 > 0$ the endpoints of $\tilde{g}$. Then*

$$1 \leq -k_1 k_2 \leq e^{2L_0}.$$

If the angle between α_n and g_{comp} is greater than $\frac{\pi}{2}$, then

$$-k_1 \geq C(L_0) e^{-l_{\alpha_n}(X_0) w_{\alpha_n}(g_{comp})},$$

where g_{comp} is a component of $g \cap (P_1 \cup P_2)$ and P_1, P_2 are two pairs of pants in $\mathcal{P}$ with a common cuff α_n.

Proof. Since iy with $1 \leq y \leq e^{l_{\alpha_n}(X_0)} \leq e^{L_0}$ belongs to the geodesic with endpoints $k_1 < 0$ and $k_2 > 0$, we have

$$|iy - \frac{k_2 + k_1}{2}| = \frac{k_2 - k_1}{2}.$$

This gives the first inequality in the above lemma.

Let $d > 0$ be the length of the common orthogonal to α_n and the side of the hexagon Σ_i^j opposite α_n, for fixed i, j. Let $\varphi > 0$ be such that the distance d between the positive y-axis and the Euclidean half-line through the origin which subtends the angle φ with the x-axis satisfies

$$\sin\varphi = \frac{1}{\cosh d}.$$

Let $r > 0$ be such that the geodesic with endpoints $-k_1e^{-r}$ and k_2e^{-r} goes through the point $e^{i\varphi}$ of the unit circle centered at 0. It follows that

$$\frac{r}{l_{\alpha_n}(X_0)} \leq w_{\alpha_n}(g_{comp}) + 2$$

because $\frac{r}{l_{\alpha_n}(X_0)}$ is the number of translates of $\tilde{g}$ (under the covering transformation corresponding to α_n) between $\tilde{g}$ and the geodesic with endpoints $-k_1e^{-r}$ and k_2e^{-r}, each translate intersects the lift of γ_i^j (adjacent to the y-axis) exactly once, and these translates glued together form a single component covering g_{comp} in P_i.

Using Euclidean geometry, we have

$$|e^{i\varphi} - \frac{k_1+k_2}{2}e^{-r}|^2 = (\frac{k_2-k_1}{2})^2e^{-2r}$$

which implies

$$k_2 \leq C'(L_0)e^r \leq C''(L_0)e^{l_{\alpha_n}(X_0)(w_{\alpha_n}(g_{comp})+2)}.$$

Then the first inequality in the theorem implies the second inequality. $\square$

Let $\tilde{\mu}$ be the lift of μ to the universal covering $\mathbb{H}^2$ and let $E^{\tilde{\mu}} : \mathbb{H}^2 \to \mathbb{H}^2$ be the corresponding earthquake. Let O be the stratum of $\tilde{\mu}$ that contains $e^{l_{\alpha_n}(X_0)}i$ and normalize the earthquake such that $E^{\tilde{\mu}}|_O = id$. Let O_1 be the stratum of $\tilde{\mu}$ which contains i that is the image of O under the covering map $B \in PSL_2(\mathbb{R})$ of the geodesic α_n. Note that $B(z) = e^{-l_{\alpha_n}(X_0)}z$. Let $B^{\tilde{\mu}}$ be the covering map for α_n on the surface $E^{\mu}(X_0) = X^{\mu}$. Then by [6] we have

$$B^{\tilde{\mu}} = E^{\tilde{\mu}}|_{O_1} \circ B.$$

Since $E^{\tilde{\mu}}$ is a left earthquake, it follows that $E^{\tilde{\mu}}|_{O_1}$ is a hyperbolic translation whose axis separates O from O_1. Let $k_1 < 0$ and $k_2 > 0$ be the repelling and the attracting fixed points of $E^{\tilde{\mu}}|_{O_1}$ and let $m \geq 0$ be its translation length. Then we have

$$\begin{aligned} trace(B^{\tilde{\mu}}) &= \frac{e^m k_2 - k_1}{e^{m/2}(k_2-k_1)}e^{-\frac{l}{2}} + \frac{k_2 - e^m k_1}{e^{m/2}(k_2-k_1)}e^{\frac{l}{2}} \\ &= 2\cosh\frac{m-l}{2} - \frac{2k_1}{k_2-k_1}[\cosh\frac{m+l}{2} - cosh\frac{m-l}{2}] \end{aligned} \tag{4.1}$$

where for short $l = l_{\alpha_n}(X_0)$. The above equation gives the inequality

$$trace(B^{\tilde{\mu}}) \geq 2 - \frac{2k_1}{k_2-k_1}ml. \tag{4.2}$$

We relate the translation length m to the earthquake measure $\mu(\alpha_n)$ using the following lemma.

Lemma 4.2. *Let $d > 0$ be the length of a geodesic arc I in $\mathbb{H}^2$ that transversely intersects a geodesic lamination μ. Denote by E^{μ} the left earthquake of $\mathbb{H}^2$ which is normalized to be the identity on the stratum O of μ that contains an endpoint* *of I and denote by O_1 another stratum of μ that contains the other endpoint of I. Then there exists $C(d,\mu(I)) \geq 1$ such that*

$$\mu(I) \leq m \leq C(d,\mu(I))\mu(I),$$

where m is the translation length of $E^{\mu}|_{O_1}$.

Proof. Let S and T be two hyperbolic translations whose axes are disjoint and both intersect a closed arc of length d. Denote by $\tau(S)$ the translation length of S. Then (cf. [17], [8])

$$\tau(S)+\tau(T) \leq \tau(S\circ T) \leq \tau(S)+\tau(T)+C'(d)\min\{\tau(S),\tau(T)\}d^2$$

for some constant $C'(d)>0$.

Assume that only finitely many leaves $\{g_1,\dots,g_n\}$ of μ intersect the geodesic arc I. Let $T_1,\dots,T_n$ be the hyperbolic translations whose axes are $g_1,\dots,g_n$ and whose translation lengths are $\mu(g_1),\dots,\mu(g_n)$. The above inequality gives

$$\sum_{i=1}^{n}\tau(T_i) \leq \tau(T_1\circ\cdots\circ T_n) \leq \sum_{i=1}^{n}\tau(T_i)+C'(d)\sum_{i=1}^{n}\tau(T_i)d^2.$$

Since $\tau(T_i)=\mu(g_i)$, $\sum_{i=1}^{n}\tau(g_i)=\mu(I)$ and $m=\tau(T_1\circ\cdots\circ T_n)$ the lemma is proved in this case. For a general μ, note that E^{μ} is approximated by finite earthquakes and the earthquake measure μ is approximated by the measure of finite earthquakes. The lemma follows by continuity. $\square$

Assume that there exists $C_0>1$ such that

$$\mu(\alpha_n) \geq C_0 l,$$

where for short $l=l_{\alpha_n}(X_0)$. By Lemma 4.2 we have that

$$m \geq C_0 l.$$

Then equation (4.1) gives

$$trace(B^{\tilde{\mu}}) \geq 2+\Big(\frac{C_0-1}{2}\Big)^2 l^2$$

which implies that

$$l_{\alpha_n}(X^{\mu}) \geq C'(L_0)l_{\alpha_n}(X_0) \tag{4.3}$$

for some $C'(L_0)>0$.

If there exists $C_0'>C(L_0,\|\mu\|_{ls})\geq 1$ for the constant $C(L_0,\|\mu\|_{ls})$ from Lemma 4.2 such that

$$\mu(\alpha_n) \leq \frac{1}{C_0'}l_{\alpha_n}(X_0),$$

then (by Lemma 4.2 again) we have

$$m \leq C_1(L_0,\|\mu\|_{ls})l_{\alpha_n}(X_0)$$

where $C_1(L_0,\|\mu\|_{ls})=\frac{C(L_0,\|\mu\|_{ls})}{C_0'}<1$. Then equation (4.1) gives

$$trace(B^{\tilde{\mu}}) \geq 2+\Big(\frac{1-1/C_1}{2}\Big)^2 l^2$$

which implies that

$$l_{\alpha_n}(X^{\mu}) \geq C''(L_0,\|\mu\|_{ls})l_{\alpha_n}(X_0) \tag{4.4}$$

for some $C''(L_0,\|\mu\|_{ls})>0$.

Assume that

$$\frac{1}{C_0'}l_{\alpha_n}(X_0) < \mu(\alpha_n) < C_0 l_{\alpha_n}(X_0) \tag{4.5}$$

which implies

$$\frac{1}{C_0'}l_{\alpha_n}(X_0) < m < C_0 C(L_0, \|\mu\|_{ls}) l_{\alpha_n}(X_0).$$

Then inequality (4.2) gives

$$trace(B^{\tilde{\mu}}) \geq 2 - \frac{1}{C_2(L_0, \|\mu\|_{ls})} \frac{2k_1}{k_2 - k_1} l^2 \tag{4.6}$$

which implies that $\frac{l_{\alpha_n}(X_0)}{l_{\alpha_n}(X^\mu)} \leq C'''$ when the angle between α_n and g_{comp} is less than $\frac{\pi}{2}$. If the angle between α_n and g_{comp} is greater than $\frac{\pi}{2}$, then (4.6) together with Lemma 4.1 implies

$$l_{\alpha_n}(X^\mu) \geq C_2(L_0, \|\mu\|_{ls}) e^{-l_{\alpha_n}(X_0) w_{\alpha_n}(g_{comp})} l_{\alpha_n}(X_0) \tag{4.7}$$

where g is a leaf of μ which intersects α_n and g_{comp} is a component of $g \cap P_i$ for either $i = 1$ or $i = 2$. To estimate $\frac{l_{\alpha_n}(X_0)}{l_{\alpha_n}(X^\mu)}$ we need to estimate the right hand side of (4.7).

Lemma 4.3. *Let α_n be a simple closed geodesic on X_0 from the fixed geodesic pants decomposition $\mathcal{P}$ and let P_1, P_2 be the two (possibly equal) pairs of pants in the decomposition $\mathcal{P}$ with a common cuff α_n. Let $w_{\alpha_n}(g_{comp})$ be a twisting number around α_n of a component g_{comp} of $g \cap P_i$ for a leaf g of the measured lamination μ. Then*

$$w_{\alpha_n}(g_{comp}) \leq C \frac{\max\{1, |\log l_{\alpha_n}(X_0)|\}}{\mu(\alpha_n)},$$

where $C = C(\|\mu\|_{ls}) > 0$ depends on the length spectrum norm $\|\mu\|_{ls}$ of μ.

Proof. We consider the leaves of μ that intersect α_n. For each such leaf g, we divide it into components of $g \cap P_i$. Observe that $w_{\alpha_n}(g_{comp})$ and $w_{\alpha_n}(g'_{comp})$ for any two components g_{comp} and g'_{comp} differ by at most an additive constant which can be taken to be 2. Then it follows that the number of intersections between each component and the arc γ_n^i is up to an additive constant equal to $2w_{\alpha_n}(g_{comp})$ for any component g_{comp}. Then we have

$$\mu(\alpha_n) w_{\alpha_n}(g_{comp}) \leq \mu(\beta_n) \leq \|\mu\|_{ls} l_{\beta_n}(X_0),$$

and since

$$l_{\beta_n}(X_0) \leq C' \max\{1, l_{\alpha_n}(X_0)\}$$

we obtain the desired conclusion. □

Thus if μ satisfies (4.5) and if the angle between α_n and g_{comp} is greater than $\frac{\pi}{2}$, Lemma 4.3 gives

$$w_{\alpha_n}(g_{comp}) \leq C(L_0, \|\mu\|_{ls}) \frac{|\log l_{\alpha_n}(X_0)|}{l_{\alpha_n}(X_0)}.$$

If the angle between α_n and g_{comp} is greater than $\frac{\pi}{2}$ and if there exists $C > 0$ such that

$$w_{\alpha_n}(g_{comp}) \leq C \frac{1}{l_{\alpha_n}(X_0)},$$

then

$$\frac{l_{\alpha_n}(X_0)}{l_{\alpha_n}(X^\mu)} \leq C'$$

for some C'.

To summarize, we have

Theorem 4.4. *Let X_0 be a complete hyperbolic surface which has a geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}_{n\in\mathbb{N}}$ such that*

$$l_{\alpha_n}(X_0) \leq L_0$$

for some fixed $L_0 > 0$. Let μ be a measured (geodesic) lamination on X_0 such that

$$\|\mu\|_{ls} < \infty.$$

If there exist $C_0 > 1$, $C_0' > C(L_0, \|\mu\|_{ls}) \geq 1$ for the constant $C(L_0, \|\mu\|_{ls})$ from Lemma 4.2 and $C_1 > 0$ such that (for each cuff α_n of $\mathcal{P}$) μ satisfies one of the following:

(1) $\mu(\alpha_n) > C_0 l_{\alpha_n}(X_0)$,
(2) $\mu(\alpha_n) < \frac{1}{C_0'} l_{\alpha_n}(X_0)$,
(3) *the angle between α_n and a leaf g of μ is less than or equal to $\frac{\pi}{2}$,*
(4) $\frac{1}{C_0'} l_{\alpha_n}(X_0) \leq \mu(\alpha_n) \leq C_0 l_{\alpha_n}(X_0)$, *the angle between α_n and a leaf g of μ is greater than $\frac{\pi}{2}$ and $w_{\alpha_n}(g_{comp}) < C_1 \frac{1}{l_{\alpha_n}(X_0)}$,*

then there exists $C^* = C^*(L_0, \|\mu\|_{ls}, C_1) > 0$ such that

$$\frac{l_{\alpha_n}(X_0)}{l_{\alpha_n}(E^\mu(X_0))} \leq C^*$$

for all $\alpha_n \in \mathcal{P}$.

5. Upper bounds on lengths

We briefly describe the upper bounds on the lengths of simple closed geodesics under the earthquake map. Namely, if the support of an earthquake E^μ which intersects α_n consists of finitely many closed geodesics, then it is standard that

$$l_{\alpha_n}(X^\mu) \leq l_{\alpha_n}(X_0) + \mu(\alpha_n).$$

Indeed, the proof is by lifting the earthquake to the universal covering and noting that the shear is always to the left (cf. [10]). Since the cocycle map for E^μ is obtained by approximations with finitely many leaves whose total measure is $\mu(\alpha_n)$ (cf. [6]), we obtain the above inequality for arbitrary earthquakes.

Since $\mu(\alpha_n) \leq \|\mu\|_{ls} l_{\alpha_n}(X_0)$, the above inequality implies

$$\log \frac{l_{\alpha_n}(X^\mu)}{l_{\alpha_n}(X_0)} \leq \log(1 + \frac{\mu(\alpha_n)}{l_{\alpha_n}(X_0)}) \leq \frac{\mu(\alpha_n)}{l_{\alpha_n}(X_0)} \leq \|\mu\|_{ls}.$$

6. Bounding the twists of the cuffs

In this section we bound the twists $|t_{\alpha_n}(E^\mu(X_0)) - t_{\alpha_n}(X_0)|$. We recall that $t_{\alpha_n}(X_0)$ is chosen such that $0 \leq t_{\alpha_n}(X_0) < l_{\alpha_n}(X_0)$. By the proof in [15, Theorem 2.1, Step I, Case 1], we have that (cf. Figure 2 in [15])

$$|t_{\alpha_n}(X^\mu)| \leq \max\{1, |\log l_{\alpha_n}(X^\mu)|\} + l_{\beta_n}(X^\mu).$$

First of all $l_{\alpha_n}(X^\mu)$ is proportional to $l_{\alpha_n}(X_0)$ with universal constants depending on $\|\mu\|_{ls}$ by the previous two sections. Moreover, we have that

$$l_{\beta_n}(X^\mu) \leq l_{\beta_n}(X_0) + \mu(\beta_n). \tag{6.1}$$

Since $l_{\beta_n}(X_0) \leq C \max\{1, |\log l_{\alpha_n}(X_0)|\}$ and $\mu(\beta_n) \leq \|\mu\|_{ls} l_{\beta_n}(X_0)$, we obtain

$$l_{\beta_n}(X^\mu) \leq C(\|\mu\|_{ls}) \max\{1, |\log l_{\alpha_n}(X_0)|\}$$

which proves the desired bound on $|t_{\alpha_n}(E^\mu(X_0)) - t_{\alpha_n}(X_0)|$.

To see that (6.1) holds, it is enough to note that it holds when the support of μ intersects β_n in finitely many leaves and the general case follows by approximations of earthquakes cocycles with cocycles supported on finitely many leaves (cf. [6]).

7. Necessity of the condition $\|\mu\|_{ls} < \infty$

We show that

$$\|\mu\|_{ls} < \infty$$

is necessary for $E^\mu(X_0) = X^\mu$ to satisfy

$$d_{ls}(X_0, X^\mu) < \infty.$$

Assume on the contrary that there exists a sequence β_n of the simple closed geodesics on X_0 such that

$$\frac{\mu(\beta_n)}{l_{\beta_n}(X_0)} \to \infty$$

as $n \to \infty$. Then after normalizing the earthquake $E^{\tilde{\mu}} : \mathbb{H}^2 \to \mathbb{H}^2$ as in §4, we get by (4.1) that

$$trace(B^\mu) \geq 2 \cosh \frac{\mu(\beta_n) - l_{\beta_n}(X_0)}{2}$$

which implies

$$l_{\beta_n}(X^\mu) \geq C(\mu(\beta_n) - l_{\beta_n}(X_0)).$$

In conclusion

$$\frac{l_{\beta_n}(X^\mu)}{l_{\beta_n}(X_0)} \geq C \frac{\mu(\beta_n)}{l_{\beta_n}(X_0)} - C \to \infty$$

as $n \to \infty$ which contradicts $d_{ls}(X_0, X^\mu) < \infty$. Thus $\|\mu\|_{ls} < \infty$ is a necessary condition.

8. Conclusions

We established that the condition $\|\mu\|_{ls} < \infty$ is necessary for $E^\mu(X_0) = X^\mu$ to satisfy $d_{ls}(X_0, X^\mu) < \infty$. From Theorem 4.4, we immediately conclude that the following earthquake paths are inside $T_{ls}(X_0)$.

Theorem 8.1. *Let X_0 be a complete hyperbolic surface with an upper bounded geodesic pants decomposition $\mathcal{P} = \{\alpha_n\}$ and let μ be a measured geodesic lamination on X_0 with*

$$\|\mu\|_{ls} < \infty.$$

Then $E^{t\mu}(X_0) = X^{t\mu} \in T_{ls}(X_0)$ for all $t \geq 0$ if there exists $C > 0$ such that for each α_n one of the following holds:

1. *the angle between α_n and a leaf of μ is less than $\frac{\pi}{2}$,*
2. *the angle between α_n and a leaf of μ is greater than $\frac{\pi}{2}$, and $w_{\alpha_n}(g_{comp}) \leq C \frac{1}{l_{\alpha_n}(X_0)}$.*

If there is no $C>0$ such that $w_{\alpha_n}(g_{comp})\leq C\frac{1}{l_{\alpha_n}(X_0)}$, it is still possible that the whole earthquake path remains in $T_{ls}(X_0)$.

Theorem 8.2. *Assume that $\mathcal{P}$ is partitioned into $\mathcal{P}'$ and $\mathcal{P}''$ such that there exists $C>0$ with*

$$w_{\alpha_n}(g_{comp})\leq \frac{C}{l_{\alpha_n}(X_0)}$$

for all $\alpha_n\in\mathcal{P}'$, and that for $\alpha_n\in\mathcal{P}''$

$$l_{\alpha_n}(X_0)\to 0$$

and

$$\mu(\alpha_n)=o(l_{\alpha_n}(X_0))$$

as $n\to\infty$. Then the earthquake path $t\mapsto E^{t\mu}(X_0)=X^{t\mu}$ is contained in $T_{ls}(X_0)$ for all $t\geq 0$.

Proof. Since $\mu(\alpha_n)=o(l_{\alpha_n}(X_0))$, it follows that there exists $n_0=n_0(t)$ such that $t\mu(\alpha_n)<\frac{1}{2}l_{\alpha_n}(X_0)$ for $n\geq n_0$. The conclusion follows by the proof of Theorem 4.4. □

Finally, we establish that not all earthquake paths $t\mapsto E^{t\mu}(X_0)$ whose earthquake measure satisfies $\|\mu\|_{ls}<\infty$ remain in $T_{ls}(X_0)$ for all $t>0$. We define μ by choosing the support to be a family of simple closed curves $\{\beta_k\}_{k=1}^{\infty}$ such that each β_k is contained in the union $P_k^1\cup P_k^2$ of two pairs of pants $P_k^1,P_k^2\in\mathcal{P}$, that β_k intersects the common cuff α_k of P_k^1,P_k^2 in a single point, that the angle between α_k and β_k is greater than $\frac{\pi}{2}$, and that

$$w_{\alpha_k}(\beta_k)l_{\alpha_k}(X_0)\to\infty$$

as $k\to\infty$. We choose the measure μ such that

$$\mu(\alpha_k)=\frac{1}{2}l_{\alpha_k}(X_0)$$

or equivalently μ is the Dirac measure on $\{\beta_k\}_{k=1}^{\infty}$ multiplied by $\frac{1}{2}l_{\alpha_k}(X_0)$.

Note that $t\mu(\alpha_k)=\frac{t}{2}l_{\alpha_k}(X_0)$ which implies that $X^{t\mu}\in T_{ls}(X_0)$ for $t\geq 0$, $t\neq 2$. By (4.1), we have that

$$l_{\alpha_k}(X^{2\mu})\leq Ce^{-w_{\alpha_k}(\beta_k)l_{\alpha_k}(X_0)}l_{\alpha_k}(X_0)$$

which implies that $X^{2\mu}=E^{2\mu}(X_0)\notin T_{ls}(X_0)$. We note that $X^{2\mu}$ is a hyperbolic surface homeomorphic to X_0.

Remark 8.3. We can hypothetically think of $X^{2\mu}$ as belonging to some "augmentation" of $T_{ls}(X_0)$. Note the difference from the augmented Teichmüller spaces of finite surfaces where only pinched surfaces appear. This idea will be pursued elsewhere.

Note that this behavior that an earthquake path leaves $T_{ls}(X_0)$ at some time t and returns in $T_{ls}(X_0)$ afterwards can be repeated for infinitely many values of t by simply choosing different multiples of the Dirac measures along different subsequences of β_k. In particular, an earthquake path $t\mapsto E^{t\mu}(X_0)$ can leave and return to $T_{ls}(X_0)$ at infinitely many points t.

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