

## A COUNTEREXAMPLE TO A THEOREM OF BREMERMANN ON SHILOV BOUNDARIES

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ABSTRACT. We give a counterexample to the following theorem of Bremermann on Shilov boundaries: if  $D$  is a bounded domain in  $\mathbb{C}^n$  having a univalent envelope of holomorphy, say  $\tilde{D}$ , then the Shilov boundary of  $D$  with respect to the algebra  $\mathcal{A}(D)$ , call it  $\partial_S D$ , coincides with the corresponding one for  $\tilde{D}$ , called  $\partial_S \tilde{D}$ .

Let us first repeat some basic notions: let  $D$  be a bounded domain in  $\mathbb{C}^n$ . Put  $\mathcal{A}(D) := \mathcal{C}(\overline{D}) \cap \mathcal{O}(D)$ . Then  $\mathcal{A}(D)$  together with the supremum norm  $\|\cdot\|_{\overline{D}}$  is a Banach algebra of functions on  $\overline{D}$ . Moreover, we set  $\mathcal{B}(D)$  as the closure of  $\mathcal{O}(\overline{D})$  in  $\mathcal{C}(\overline{D})$  with the above norm. Again this is a Banach algebra. Then both of these algebras have a Shilov boundary, called  $\partial_S D$ , respectively,  $\partial_B D$ . Obviously, we have  $\partial_B D \subset \partial_S D$ .

Now we assume in addition that  $D$  has a univalent envelope of holomorphy  $\tilde{D}$ . Then, we have

$$\partial_S \tilde{D} \subset \partial_S D \subset \partial D \text{ and } \partial_B \tilde{D} \subset \partial_B D \subset \partial D.$$

In [Bre 1959] (see Theorem in section 6.6), Bremermann claimed that equality  $\partial_S \tilde{D} = \partial_S D$  is also true. While he proved  $\partial_S \tilde{D} \subset \partial_S D$  in a detailed way he was saying that the inverse inclusion is an obvious fact. But as we will show (more than fifty years later) this claim is not true, already if  $D$  is a Hartogs domain over an annulus. For positive results regarding Reinhardt domains see [Kos-Zwo 2013].

Following the construction of some Hartogs domain with non univalent envelope of holomorphy (see [Jar-Pfl 2000], pages 1-2) we get the following result.

**Theorem.** *There exists a bounded Hartogs domain  $D \subset \mathbb{C}^2$  with a univalent envelope of holomorphy  $\tilde{D}$  such that*

- $\partial_S D \neq \partial_S \tilde{D}$ ,
- $\partial_B D \neq \partial_B \tilde{D}$ ,
- *there exists a function  $f \in \mathcal{O}(\overline{D})$  such that the holomorphic extension  $\tilde{f}$  of  $f|_D$  to  $\tilde{D}$  has no continuous extension to  $\overline{\tilde{D}}$ .*

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*Proof.* Let  $A := \{z \in \mathbb{C} : 1/2 < |z| < 1\}$ . Then we introduce

$$\begin{aligned}
 D &:= \{z \in A \times \mathbb{C} : \operatorname{Re} z_1 < 0, |z_2| < 3\} \\
 &\cup \{z \in A \times \mathbb{C} : 0 \leq \operatorname{Re} z_1, \operatorname{Im} z_1 > 0, |z_2| < 1\} \\
 &\cup \{z \in A \times \mathbb{C} : 0 \leq \operatorname{Re} z_1, \operatorname{Im} z_1 < 0, 2 < |z_2| < 3\}.
 \end{aligned}$$

Note that  $D$  is a Hartogs domain over the annulus  $A$  which cuts the base  $A \times \{0\}$ . Then Corollary 3.1.10(b) in [Jar-Pfl 2000] implies that  $D$  has a univalent envelope of holomorphy  $\tilde{D}$ . Moreover, using the Cauchy integral formula shows that  $\tilde{D}$  contains the following domain

$$\{z \in A \times \mathbb{C} : \operatorname{Re} z_1 < 0 \text{ or (if } \operatorname{Re} z_1 \geq 0, \text{ then } \operatorname{Im} z_1 < 0), |z_2| < 3\}.$$

In particular, all the discs  $\mathbb{D}_{z_1} := \{z_1\} \times 3\mathbb{D}$ ,  $1/2 \leq \operatorname{Re} z_1 = z_1 \leq 1$ , belong to  $\overline{\tilde{D}}$  (here  $\mathbb{D}$  means the open unit disc in  $\mathbb{C}$ ).

Therefore, if  $f \in \mathcal{A}(\tilde{D})$ , then  $f(z_1, \cdot) \in \mathcal{A}(3\mathbb{D})$  for all the above  $z_1$  (use Weierstrass' theorem here). Hence, by the maximum principle, these discs don't contain any point of  $\partial_S \tilde{D}$ .

On the other side we discuss the following domain

$$\begin{aligned}
 D' &:= \{z \in A' \times \mathbb{C} : \operatorname{Re} z_1 < \varepsilon |\operatorname{Im} z_1|, |z_2| < 3 + \varepsilon\} \\
 &\cup \{z \in A' \times \mathbb{C} : 0 \leq \operatorname{Re} z_1, \operatorname{Im} z_1 > -\varepsilon \operatorname{Re} z_1, |z_2| < 1 + \varepsilon\} \\
 &\cup \{z \in A' \times \mathbb{C} : 0 \leq \operatorname{Re} z_1, \operatorname{Im} z_1 < \varepsilon \operatorname{Re} z_1, 2 - \varepsilon < |z_2| < 3 + \varepsilon\},
 \end{aligned}$$

where  $0 < \varepsilon \ll 1/4$  and  $A' := \{z \in \mathbb{C} : 1/2 - \varepsilon < |z| < 1 + \varepsilon\}$ . Observe that  $D \subset \subset D'$ .

Now we define the following concrete holomorphic function  $g$  on  $D'$ :

$$g(z) := \begin{cases} \log_1 z_1, & \text{if } z \in D', |z_2| > 1.4 \\ \log_2 z_1, & \text{if } z \in D', |z_2| < 1.6 \end{cases},$$

where  $\log_1$ , respectively  $\log_2$ , is the branch of the logarithm function on  $\mathbb{C} \setminus \{w \in \mathbb{C} : \operatorname{Re} w \geq 0, \operatorname{Im} w = \operatorname{Re} w\}$ , respectively on  $\mathbb{C} \setminus \{w \in \mathbb{C} : \operatorname{Re} w \geq 0, \operatorname{Im} w = -\operatorname{Re} w\}$ , with  $\log_1(-1/2) = \log_2(-1/2) = \log 1/2 + i\pi$ . Observe that  $g$  is well defined on  $D'$  and  $g \in \mathcal{O}(D')$ .

Define  $f := g|_{\overline{D}}$ . Then  $f \in \mathcal{B}(D)$  and if  $z \in \overline{D}$  with  $z_1 = \operatorname{Re} z_1 > 0$ , then

$$f(z) = \begin{cases} \log \operatorname{Re} z_1, & \text{if } |z_2| \leq 1 \\ \log \operatorname{Re} z_1 + 2\pi i, & \text{if } |z_2| \geq 2 \end{cases}.$$

Finally, we observe that the function  $h$  defined as  $h(z) := \exp(if(z) + 2\pi)$ ,  $z \in \overline{D}$ , belongs to  $\mathcal{B}(D)$  and

$$|h(z)| = \begin{cases} 1, & \text{if } z \in \overline{D}, \operatorname{Re} z_1 > 0, |z_2| \geq 2 \\ e^{2\pi}, & \text{if } z \in \overline{D}, \operatorname{Re} z_1 > 0, |z_2| \leq 1 \end{cases}$$

and  $|h(z)| < e^{2\pi}$  on the remaining part of  $\overline{D}$ . Therefore,  $\partial_B D$  contains points  $z \in \overline{D}$  with  $0 < \operatorname{Re} z_1 = z_1$  and  $|z_2| \leq 1$ .

Combining this concrete information with the general one from the former discussion on the Shilov boundaries for  $\tilde{D}$  we conclude that  $\partial_S \tilde{D}$  and  $\partial_B \tilde{D}$  are strictly contained in  $\partial_B D$ . In particular, this shows that the claimed equality in Bremermann's paper does not hold.

Moreover, the function  $f$  is the one whose existence was claimed in the third claim in the theorem.  $\square$

*Remark.* (a) Recall that the equality  $\partial_S D = \partial_S \tilde{D}$  is true for a Reinhardt domain  $D$ .

(b) Let  $D$  be a bounded balanced domain. Obviously,  $\bar{D}$  has a neighborhood basis of balanced domains  $G$ . Moreover,  $D$  and each  $G$  have univalent envelopes of holomorphy with  $\tilde{D} \subset\subset \tilde{G}$ . Hence the equality  $\partial_B D = \partial_B \tilde{D}$  holds in an obvious way.

(c) What remains is to discuss the equality of the Shilov boundaries for  $\mathcal{A}(D)$  and  $\mathcal{A}(\tilde{D})$  in the case when  $D$  is a balanced domain.

#### REFERENCES

- [Bre 1959] H. J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains. Characterization of Šilov boundaries*, Trans. Amer. Math. Soc. **91** (1959), 246–276. MR0136766 (25 #227)
- [Kos-Zwo 2013] Lukasz Kosiński and Włodzimierz Zwonek, *Proper holomorphic mappings vs. peak points and Shilov boundary*, Ann. Polon. Math. **107** (2013), no. 1, 97–108, DOI 10.4064/ap107-1-7. MR3001625
- [Jar-Pfl 2000] Marek Jarnicki and Peter Pflug, *Extension of holomorphic functions*, de Gruyter Expositions in Mathematics, vol. 34, Walter de Gruyter & Co., Berlin, 2000. MR1797263 (2001k:32017)

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